HILBERT SPACE REPRESENTATIONS OF DECOHERENCE FUNCTIONALS AND QUANTUM MEASURES

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Abstract

We show that any decoherence functional D can be represented by a spanning vector-valued measure on a complex Hilbert space. Moreover, this representation is unique up to an isomorphism when the system is finite. We consider the natural map U from the history Hilbert space K to the standard Hilbert space H of the usual quantum formulation. We show that U is an isomorphism from K onto a closed subspace of H and that U is an isomorphism from K onto H if and only if the representation is spanning. We then apply this work to show that a quantum measure has a Hilbert space representation if and only if it is strongly positive. We also discuss classical decoherence functionals, operator-valued measures and quantum operator measures.

1 Introduction

In the usual quantum description of a physical system, we begin with a complex Hilbert space H. The states of the system are represented by density operators, the observables by self-adjoint operators and the dynamics by unitary operators on H. In the history approach to quantum mechanics and in applications such as quantum gravity and cosmology, one defines a useful concept called a decoherence functional D [3, 7, 8, 11]. It is believed by researchers in these fields that D encodes important information about the system. For example, D can be employed to find the interference between quantum objects and can also be used to find a quantum measure that quantifies the propensity that quantum events occur [2, 5, 6, 11].

Because of the fundamental importance of D, it appears to be useful to reverse this formalism. We propose to begin with a decoherence functional D with natural properties and to then reconstruct the usual quantum formulation. We consider two types of reconstruction that we call vector and operator representations of D. We show that there always exists a spanning vector representation of D and when the system is finite, this representation is unique up to an isomorphism. For a finite system, cyclic operator representations always exist but for infinite systems, their existence is unknown.

Besides the standard Hilbert space H of the usual quantum formulation, there exists a history Hilbert space K that is directly associated with D [3]. Moreover, we can define a natural map $U\colon K \to H$ [3]. We show that U is an isomorphism from K onto a closed subspace of H and that U is an isomorphism from K onto H if and only if the vector representation is spanning.

We also present several characterizations of classical decoherence functionals. We show that a quantum measure has a Hilbert space representation if and only if it is strongly positive. We briefly consider quantum operator measures generated by decoherence operators.

2 Vector Representations

Let $(\Omega, \mathcal{A})$ be a measurable space. The elements of Ω represent outcomes and the sets in the σ-algebra $\mathcal{A}$ represent events for a physical system or process. A *decoherence functional* $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ from the Cartesian product of $\mathcal{A}$ with itself into the complex numbers satisfies the following conditions [3, 8, 12]:

(D1) $D(\Omega, \Omega) = 1$,

(D2) $A \mapsto D(A, B)$ is a complex measure for all $B \in \mathcal{A}$.

(D3) If $A_1, \ldots, A_n \in \mathcal{A}$, then $D(A_i, A_j)$ is a positive semi-definite $n \times n$ matrix.

Condition (D1) is an inessential normalization property that does not affect any of the results in this paper. Notice that (D3) implies $D(A,A) \geq 0$ and $D(A,B) = \overline{D(B,A)}$.

We now give two examples of decoherence functionals. If $\nu\colon \mathcal{A} \to \mathbb{C}$ is a complex measure satisfying $\nu(\Omega) = 1$, we can view ν as an amplitude measure for a physical system. It is easy to check that $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ given by $D(A,B) = \nu(A)\overline{\nu(B)}$ is a decoherence functional. The map $\mu\colon \mathcal{A} \to \mathbb{R}^+$ given by

$$\mu(A) = D(A,A) = |\nu(A)|^2 \tag{2.1}$$

is an example of a quantum measure [2, 5, 6, 11, 12] and these will be treated in Section 6. This is an example of a vector representation of D.

The second example is more general and illustrates an operator representation of D. Let H be a complex Hilbert and denote the set of bounded linear operators from H to H by $B(H)$. We say that $\mathcal{E}\colon \mathcal{A} \to B(H)$ is an *operator-valued measure* if for every sequence of mutually disjoint sets $A_i \in \mathcal{A}$ and every $\phi, \phi' \in H$ we have

$$\langle \mathcal{E}(\cup A_i)\phi, \phi' \rangle = \sum \langle \mathcal{E}(A_i)\phi, \phi' \rangle$$

where the summation converges absolutely. If $\mathcal{E}\colon \mathcal{A} \to B(H)$ is an operator-valued measure and $\psi \in H$ is a unit vector, we define $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$D(A,B) = \langle \mathcal{E}(A)\psi, \mathcal{E}(B)\psi \rangle \tag{2.2}$$

If $D(\Omega,\Omega) = 1$, then it is easy to check that D is a decoherence functional. If the closed span

$$\overline{\text{span}}\,\{\mathcal{E}(A)\psi\colon A \in \mathcal{A}\} = H$$

we say that ψ is a *cyclic vector* for $\mathcal{E}$. Again, the map $\mu\colon \mathcal{A} \to \mathbb{R}^+$ defined by

$$\mu(A) = D(A,A) = \|\mathcal{E}(A)\psi\|^2 \tag{2.3}$$

is an example of a quantum measure.

Lemma 2.1. *If D is an $n \times n$ positive semi-definite matrix, then there exists a complex Hilbert space H and a spanning set of vectors $e_i \in H$, $i = 1, \ldots, n$, such that $D_{ij} = \langle e_i, e_j \rangle$. Also, if $D_{ij} = \langle f_i, f_j \rangle$ for a spanning set of vectors f_i in a complex Hilbert space K, then there is a unitary operator $U\colon H \to K$ such that $Ue_i = f_i$, $i = 1, \ldots, n$.*

Proof. Since D is positive semi-definite, the map

$$\langle f, g\rangle = \sum D_{ij} f(i)\overline{g(j)}$$

becomes an indefinite inner product on the vector space $\mathbb{C}^n$. Defining $\|f\| = \langle f, f\rangle^{1/2}$, let $N \subseteq \mathbb{C}^n$ be the subspace

$$N = \{f \in \mathbb{C}^n \colon \ \|f\| = 0\}$$

Letting H be the quotient space $H = \mathbb{C}^n/N$, the elements of H become $[f] = f + N$, $f \in \mathbb{C}^n$. Then H is a finite-dimensional complex Hilbert space with inner product $\langle [f], [g]\rangle = \langle f, g\rangle$. Letting $e_1, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$ we have that

$$\langle [e_i], [e_j]\rangle = \sum D_{rs} e_i(r)\overline{e_j(s)} = D_{ij}$$

Since $\{e_1, \ldots, e_n\}$ spans $\mathbb{C}^n$, $\{[e_1], \ldots, [e_n]\}$ spans H. We can assume without loss of generality that $\{[e_1], \ldots, [e_m]\}$ forms a basis for H, $m \leq n$. Then

$$\dim H = m = n - \dim N = \operatorname{rank}(D)$$

Now suppose that $D_{ij} = \langle f_i, f_j\rangle$ for a spanning set of vectors $f_i \in K$, $i = 1, \ldots, n$. It is well-known that $\operatorname{rank}(D)$ is the number of linearly independent rows of D. Since $\{[e_1], \ldots, [e_m]\}$ are linearly independent we have that the first m rows of D are linearly independent. We now show that $f_1, \ldots, f_m$ are linearly independent. Suppose that $\sum_{i=1}^m \alpha_i f_i = 0$ for $\alpha_i \in \mathbb{C}$. Then $\sum_{i=1}^m \alpha_i \langle f_i, f_j\rangle = 0$ for $j = 1, \ldots, n$, and hence,

$$\alpha_1 \left(\langle f_1, f_1\rangle, \ldots, \langle f_1, f_n\rangle\right) + \cdots + \alpha_m \left(f_m, f_1\right), \ldots, \langle f_m, f_n\rangle = 0$$

We conclude that $\alpha_1, \ldots, \alpha_m = 0$ so $f_1, \ldots, f_m$ are linearly independent. It follows that $f_1, \ldots, f_m$ form a basis for K. Define the operator $U \colon H \to K$ by $U[e_i] = f_i$, $i = 1, \ldots, m$, and extend by linearity. We then have that

$$\langle U[e_i], U[e_j]\rangle = \langle f_i, f_j\rangle = D_{ij} = \langle [e_i], [e_j]\rangle$$

$i = 1, \ldots, m$. Since any $[f] \in H$ has a unique representation

$$[f] = \sum_{i=1}^m \alpha_i [e_i]$$

we have that

$$\|U[f]\|^2 = \left\langle \sum \alpha_i U[e_i], \sum \alpha_j U[e_j] \right\rangle = \sum \alpha_i \overline{\alpha_j} \langle U[e_i], U[e_j] \rangle$$
$$= \sum \alpha_i \overline{\alpha_j} \langle [e_i], [e_j] \rangle = \langle [f], [f] \rangle = \|f\|^2$$

Since U is surjective, U is unitary. □

A map $\mathcal{E}\colon \mathcal{A} \to H$ is a *vector-valued measure* on H if for any sequence of mutually disjoint sets $A_i \in \mathcal{A}$ we have that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathcal{E}(A_i) = \mathcal{E}(\cup A_i)$$

in the norm topology. A *vector representation* for a decoherence functional $D\colon \mathcal{A}\times\mathcal{A} \to \mathbb{C}$ is a pair $(H, \mathcal{E})$ where $\mathcal{E}\colon \mathcal{A} \to H$ is a vector-valued measure satisfying

$$D(A, B) = \langle \mathcal{E}(A), \mathcal{E}(B) \rangle \tag{2.4}$$

for all $A, B \in \mathcal{A}$. If $\overline{\mathrm{span}}\,\{\mathcal{E}(A)\colon A \in \mathcal{A}\} = H$, then $(H, \mathcal{E})$ is called a *spanning vector representation* for D. If $\Omega = \{\omega_1, \ldots, \omega_n\}$, then we let $\mathcal{A} = 2^\Omega$ and call $(\Omega, \mathcal{A})$ a *finite measurable space*. It is clear that any map $D\colon \mathcal{A}\times\mathcal{A} \to \mathbb{C}$ satisfying $D(\Omega, \Omega) = 1$ and (2.4) is a decoherence functional. The next two results show that the converse holds.

Theorem 2.2. *If $(\Omega, \mathcal{A})$ is a finite measurable space and $D\colon \mathcal{A}\times\mathcal{A} \to \mathbb{C}$ is a decoherence functional, then there exists a spanning vector representation $(H, \mathcal{E})$ for D. Moreover, if $(K, \mathcal{F})$ is a spanning vector representation for D, then there is a unitary operator $U\colon H \to K$ such that $U\mathcal{E}(A) = \mathcal{F}(A)$ for every $A \in \mathcal{A}$.*

Proof. Since D is a decoherence functional, we have that $D_{ij} = D(\omega_i, \omega_j)$ is positive semi-definite. By Lemma 2.1, there exists a spanning set $e_1, \ldots, e_n$ in a Hilbert space H such that $D_{ij} = \langle e_i, e_j \rangle$. For $A \in \mathcal{A}$, define $\mathcal{E}\colon \mathcal{A} \to H$ by

$$\mathcal{E}(A) = \sum \{e_i\colon \omega_i \in A\}$$

Then $\mathcal{E}$ is a vector-valued measure and we have

$$D(A, B) = \sum \{D(\omega_i, \omega_j)\colon \omega_i \in A, \omega_j \in B\} = \sum_{ij} \{\langle e_i, e_j \rangle\colon \omega_i \in A, \omega_j \in B\}$$
$$= \left\langle \sum \{e_i\colon \omega_i \in A\}, \sum \{e_j\colon \omega_j \in B\} \right\rangle = \langle \mathcal{E}(A), \mathcal{E}(B) \rangle$$

Hence $(H, \mathcal{E})$ is a spanning vector representation of D. For the second statement of the theorem, let $e_i = \mathcal{E}(\omega_i)$, $f_i = \mathcal{F}(\omega_i)$, $i = 1, \ldots, n$. It is clear that $\operatorname{span}\{e_1, \ldots, e_n\} = H$ and similarly $\operatorname{span}\{f_1, \ldots, f_n\} = K$. By Lemma 2.1, there is a unitary operator $U\colon H \to K$ such that $Ue_i = f_i$. Therefore,

$$\begin{aligned} U\mathcal{E}(A) &= U\left[\sum \{e_i \colon \omega_i \in A\}\right] = \sum \{Ue_i \colon \omega_i \in A\} \\ &= \sum \{f_i \colon \omega_i \in A\} = \mathcal{F}(A) \end{aligned}$$

for all $A \in \mathcal{A}$. □

For an arbitrary measurable space $(\Omega, \mathcal{A})$, we cannot use the method in the proof of Theorem 2.2. Moreover, we do not know whether the uniqueness result in Theorem 2.2 holds in general.

Theorem 2.3. *If $(\Omega, \mathcal{A})$ is a measurable space and $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a decoherence functional, then there exists a spanning vector representation $(H, \mathcal{E})$ for D.*

Proof. Let S be the set of all complex-valued measurable functions on Ω with a finite number of values (simple functions). Any $f \in S$ has a canonical representation $f = \sum a_i \chi_{A_i}$ where $a_i \neq a_j$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $a_i \neq 0$, $i, j = 1, \ldots, n$. If $f = \sum a_i \chi_{A_i}$, $g = \sum b_j \chi_{B_j}$ are canonical representations, we define

$$\langle f, g \rangle = \sum_{i,j} a_i \overline{b_j} D(A_i, B_j) \tag{2.5}$$

It is straightforward to show that (2.5) holds even if the representations of f and g are not canonical. It is also easy to verify that $\langle \cdot\,, \cdot \rangle$ is an indefinite inner product. As in Lemma 2.1, we let N be the subspace of S given by

$$N = \{f \in S \colon \|f\| = 0\}$$

Letting $H_0 = S/N$, the elements of H_0 are the equivalence classes $[f] = f + N$, $f \in S$. We define the inner product $\langle \cdot\,, \cdot \rangle$ on H_0 by $\langle [f], [g] \rangle = \langle f, g \rangle$. Letting H be the completion of H_0 we have that H_0 is a dense subspace of the HIlbert space H. Defining $\mathcal{E}\colon \mathcal{A} \to H$ by $\mathcal{E}(A) = [\chi_A]$ we have that

$$\overline{\operatorname{span}}\{\mathcal{E}(A) \colon A \in \mathcal{A}\} = H$$

and

$$\langle \mathcal{E}(A), \mathcal{E}(B) \rangle = \langle \chi_A, \chi_B \rangle = D(A, B)$$

To show that $\mathcal{E}$ is a vector-valued measure, let $A_i \in \mathcal{A}$ be mutually disjoint, $i = 1, 2, \ldots$. We then have that

$$\begin{aligned} &\left\| \mathcal{E}\left(\cup A_i\right) - \sum_{i=1}^{n} \mathcal{E}(A_i) \right\|^2 \\ &\quad = \left\| \mathcal{E}\left(\cup A_i\right) \right\|^2 + \left\| \sum_{i=1}^{n} \mathcal{E}(A_i) \right\|^2 - 2\mathrm{Re} \left\langle \mathcal{E}\left(\cup A_i\right), \sum_{i=1}^{n} \mathcal{E}(A_i) \right\rangle \\ &\quad = D(\cup A_i, \cup A_i) + \sum_{i,j=1}^{n} D(A_i, A_j) - 2\mathrm{Re} \sum_{i=1}^{n} D(\cup A_i, A_i) \end{aligned}$$

Applying Condition (D2) we conclude that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathcal{E}(A_i) = \mathcal{E}(\cup A_i)$$

in the norm topology. □

Results similar to Theorems 2.2 and 2.3 have appeared in [1].

3 Operator Representations

An *operator representation* for a decoherence functional $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a triple $(H, \mathcal{E}, \psi)$ where H is a complex Hilbert space, $\psi \in H$ is a unit vector and $\mathcal{E}\colon \mathcal{A} \to B(H)$ is an operator-valued measure such that (2.2) holds for every $A, B \in \mathcal{A}$. We say that $(H, \mathcal{E}, \psi)$ is *cyclic* if ψ is a cyclic vector for $\mathcal{E}$. We call $\mathcal{E}(A)$ the *event* or *class operator* at A. It is not hard to show that if $(H, \mathcal{E}, \psi)$ is an operator representation for D, then $\mathcal{F}(A) = \mathcal{E}(A)\psi$ gives a vector representation for D. However, the operator representation gives more information because it specifies the class operator at every $A \in \mathcal{A}$. Moreover, we do not know whether every vector representation $(H, \mathcal{F})$ has a corresponding operator representation $(H, \mathcal{E}, \psi)$ such that $\mathcal{F}(A) = \mathcal{E}(A)\psi$ for all $A \in \mathcal{A}$. Two operator representations $(H, \mathcal{E}, \psi)$ and $(K, \mathcal{F}, \phi)$ are *equivalent* if there exists a unitary operator $U\colon H\to K$ such that $U\psi=\phi$ and $U\mathcal{E}(A)U^*=\mathcal{F}(A)$ for all $A\in\mathcal{A}$. For example, if $(H,\mathcal{E},\psi)$ is an operator representation for D and $\alpha\in\mathbb{C}$ with $|\alpha|=1$, then $(H,\mathcal{E},\alpha\psi)$ is an equivalent operator representation for D. In this case, the unitary operator is $U=\alpha I$.

We shall show that a decoherence functional on a finite measurable space possesses an operator representation. It is an open problem whether this result holds for an arbitrary decoherence functional. It should be pointed out that although finiteness is a strong restriction, there are important applications for finite quantum systems. For example, models for quantum computation and information are usually finite. Moreover, measurement based quantum computation has a structure that is similar to that of the history approach to quantum mechanics [10].

Theorem 3.1. *If $(\Omega,\mathcal{A})$ is a finite measurable space and $D\colon\mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is a decoherence functional, then there exists a cyclic operator representation for D.*

Proof. By Lemma 2.1, there exists a spanning set $e_i,\ldots,e_n$ in a Hilbert space H such that $D(\omega_i,\omega_j)=\langle e_i,e_j\rangle$, $i,j=1,\ldots,n$. We show by induction on m that there is a $\phi\in H$ such that $\langle e_i,\phi\rangle\neq 0$ for all $e_i\neq 0$, $i=1,\ldots,m\leq n$. The result clearly holds for $m=1$. Assume that the result holds for m. Then there is a ϕ such that $\langle e_i,\phi\rangle\neq 0$ for all $e_i\neq 0$, $i=1,\ldots,m$. Suppose $e_{m+1}\neq 0$ and $\langle e_{m+1},\phi\rangle=0$. By continuity, we can find a small ball $B\subseteq H$ centered at ϕ such that $\langle e_i,f\rangle\neq 0$ for all $f\in B$ and $e_i\neq 0$, $i=1,\ldots,m$. If $\langle e_{m+1},f\rangle=0$ for all $f\in B$ then $e_{m+1}=0$ which is a contradiction. Hence, there is an $f\in B$ such that $\langle e_i,f\rangle\neq 0$ for all $e_i\neq 0$, $i=1,\ldots,m+1$. This completes the induction proof. Letting $\psi=\phi/\|\phi\|$ we conclude that $\psi\in H$ is a unit vector satisfying $\langle e_i,\psi\rangle\neq 0$ for all $e_i\neq 0$, $i=1,\ldots,n$. Define $P_i\in B(H)$, $i=1,\ldots,n$, as follows. If $e_i=0$, then $P_i=0$ and if $e_i\neq 0$, then

$$P_i=\frac{1}{\langle e_i,\psi\rangle}\,|e_i\rangle\langle e_i|$$

We then have that

$$\langle P_i\psi,P_j\psi\rangle=D(\omega_i,\omega_j)$$

for $i,j=1,\ldots n$. Defining $\mathcal{E}\colon\mathcal{A}\to B(H)$ by

$$\mathcal{E}(\mathcal{A})=\sum\{P_i\colon\omega_i\in\mathcal{A}\}$$

we have that $(H,\mathcal{E},\psi)$ is a cyclic operator representation for D. □

We now give an example which shows that there may exist inequivalent cyclic operator representations for D. Let $\Omega = \{1, 2\}$ and let $D\colon 2^\Omega \times 2^\Omega \to \mathbb{C}$ be the decoherence functional given by $D(\emptyset, A) = D(A, \emptyset) = 0$, $D(\Omega, \Omega) = 1$

$$D(i, j) = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$
$$D(\Omega, 1) = D(1, \Omega) = 2/5$$
$$D(\Omega, 2) = D(2, \Omega) = 3/5$$

Let $H = \mathbb{C}^2$ with the usual inner product and standard basis e_1, e_2. Define the operator-valued measure $\mathcal{F}\colon 2^\Omega \to H$ by $\mathcal{F}(\emptyset) = 0$, $\mathcal{F}(1) = c|e_1\rangle\langle e_1|$, $\mathcal{F}(2) = cI$ and

$$\mathcal{F}(\Omega) = c|e_1\rangle\langle e_1| + cI$$

where $c = \sqrt{2/5}$. Let ϕ be the unit vector $\phi = 2^{-1/2}(1, 1)$. Since $\mathcal{F}(1)\phi = \frac{c}{\sqrt{2}}\, e_1$ and $\mathcal{F}(2)\phi = c\phi$ we see that ϕ is cyclic for $\mathcal{F}$. Moreover,

$$\langle \mathcal{F}(1)\phi, \mathcal{F}(1)\phi\rangle = \frac{c^2}{2} = \frac{1}{5} = D(1, 1)$$
$$\langle \mathcal{F}(2), \phi, \mathcal{F}(2)\phi\rangle = c^2 = \frac{2}{5} = D(2, 2)$$
$$\langle \mathcal{F}(1)\phi, \mathcal{F}(2)\phi\rangle = \frac{c^2}{\sqrt{2}}\,\langle e_1, \phi\rangle = \frac{c^2}{2} = \frac{1}{5} = D(1, 2) = D(2, 1)$$

It follows that $\langle \mathcal{F}(A), \mathcal{F}(B)\rangle = D(A, B)$ for all $A, B \in 2^\Omega$. We conclude that $(H, \mathcal{F}, \phi)$ is a cyclic operator representation for D. Since $\operatorname{rank}(\mathcal{F}(2)) = 2$ and $\operatorname{rank}(\mathcal{E}(2)) = 1$ where $\mathcal{E}(2)$ is the operator defined in Theorem 3.1, $(H, \mathcal{F}, \phi)$ is not equivalent to $(H, \mathcal{E}, \psi)$ of Theorem 3.1.

4 History Hilbert Space

Let $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ be a decoherence functional and K_0 the set of complex-valued functions on $\mathcal{A}$ that vanish except for a finite number of sets in $\mathcal{A}$. For $f, g \in K_0$ define

$$\langle f, g\rangle = \sum_{A,B\in\mathcal{A}} D(A, B) f(A)\overline{g(B)}$$

As before, we define the subspace

$$N = \{f \in K_0 \colon \|f\| = 0\}$$

The quotient space $K_1 = K_0/N$ consists of equivalence classes $[f] = f + N$, $f \in K_0$. Again $\langle [f], [g] \rangle = \langle f, g \rangle$ becomes an inner product on K_1. We denote the completion of K_1 by K and call K the *history Hilbert space* for D [3]. The space K corresponds to the history approach to quantum mechanics [8, 9, 12].

Let $(H, \mathcal{E})$ be a vector representation for D. We think of H as the standard Hilbert space of the usual quantum formulation. A natural connection between K and H was introduced in [2]. We define the *natural map* $U \colon K_0 \to H$ by

$$Uf = \sum_{A \in \mathcal{A}} f(A)\mathcal{E}(A)$$

It is clear that U is linear and moreover,

$$\begin{aligned}\langle Uf, Ug \rangle &= \left\langle \sum_{A \in \mathcal{A}} F(A)\mathcal{E}(A), \sum_{B \in \mathcal{A}} g(B)\mathcal{E}(B) \right\rangle \\ &= \sum_{A,B \in \mathcal{A}} f(A)\overline{g(B)}\langle \mathcal{E}(A), \mathcal{E}(B) \rangle \\ &= \sum_{A,B \in \mathcal{A}} f(A)\overline{g(B)}D(A,B) = \langle f, g \rangle\end{aligned}$$

Hence, $U \colon K_1 \to H$ given by $U[f] = Uf$ is well-defined and is an isometry. It follows that U has a unique extension to an isometry, that we also denote by U, from K into H. The next result shows that K is isomorphic to a closed subspace of H and characterizes when K is isomorphic to all of H. This proves a conjecture posed in [3].

Theorem 4.1. *The operator $P = UU^*$ is an orthogonal projection on H and $U \colon K \to PH$ is unitary. The natural map $U \colon K \to H$ is unitary if and only if $(H, \mathcal{E})$ is spanning.*

Proof. The operator P is clearly self-adjoint and since $U^*U = I_K$ we have that

$$P^2 = UU^*UU^* = UU^* = P$$

Clearly $PH \subseteq \text{Range}(U)$. Conversely, if $\phi \in \text{Range}(U)$ then $\phi = U\phi'$ for some $\phi' \in K$. Again, $U^*U = I_K$ gives

$$P\phi = PU\phi' = UU^*U\phi = U\phi' = \phi$$

Hence, $PH = \text{Range}(U)$. Thus, $U\colon K \to PH$ is unitary from K to the closed subspace PH of H. Now it is clear that

$$\overline{\text{span}}\,\{\mathcal{E}(A)\colon (A \in \mathcal{A}\} = \text{Range}(U)$$

Hence, $\text{Range}(U) = H$ if and only if $\mathcal{E}$ is spanning. It follows that $U\colon K \to H$ is unitary if and only if $(H, \mathcal{E})$ is spanning. □

We can proceed in a similar way for an operator representation $(H, \mathcal{E}, \psi)$ for D. Then the corresponding vector representation $(H, \mathcal{F})$ given by $\mathcal{F}(A) = \mathcal{E}(A)\psi$ is spanning if and only if $(H, \mathcal{E}, \psi)$ is cyclic. By Theorem 4.1 the natural map $U\colon K \to H$ given by

$$Uf = \sum_{A \in \mathcal{A}} f(A)\mathcal{E}(A)\psi \tag{4.1}$$

is unitary if and only if $(H, \mathcal{E})$ is cyclic.

We now introduce an example presented in [3]. Consider a system consisting of a single particle that has n possible positions $\{1, 2, \ldots, n\}$ at any time. We assume that the particle evolves in $N - 1$ discrete time steps at times $0 = t_1 < t_2 < \cdots < t_N = T$. Each history ω of the system is represented by an N-tuple of integers $\omega = (\omega_1, \ldots, \omega_N)$ with $1 \le \omega_i \le n$, $i = 1, \ldots, N$, where ω_i is the location of the particle at time t_i. The corresponding sample space Ω is the collection of n^N possible histories and $\mathcal{A} = 2^\Omega = \{A\colon A \subseteq \Omega\}$. For this example, the standard Hilbert space is $H = \mathbb{C}^n$ with the usual inner product

$$\langle \phi, \phi' \rangle = \sum_{i=1}^{n} \phi_i \overline{\phi'_i}$$

where $\phi = (\phi_1, \ldots, \phi_n)$. The initial state is given by a fixed unit vector $\psi \in H$.

To describe the decoherence functional, we assume that states propagate from time t to time t' according to a unitary evolution operator $U(t', t)$ that satisfies

$$U(t'', t')U(t', t) = U(t'', t)$$

Let $P_1, \dots, P_n$ be the projection operators given by

$$P_i(\phi_1, \dots, \phi_n) = (0 \dots, 0, \phi_i, 0, \dots, 0)$$

$i = 1, \dots, n$. These projections form the spectral measure for the position operator. For a path $\omega = (\omega_1, \dots, \omega_N)$ we define the path operator

$$\mathcal{E}(\omega) = P_{\omega_N} U(t_N, t_{N-1}) P_{\omega_{N-1}} \cdots P_{\omega_3} U(t_3, t_2) P_{\omega_2} U(t_2, t_1) P_{\omega_1} \tag{4.2}$$

We next define the event operator (or class operator) $\mathcal{E}(A)$, $A \in \mathcal{A}$, by

$$\mathcal{E}(A) = \sum_{\omega \in A} \mathcal{E}(\omega)$$

Then $\mathcal{E}\colon \mathcal{A} \to L(H)$ becomes an operator-valued measure and $(H, \mathcal{E}, \psi)$ is an operator representation for the decoherence functional $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ given by

$$D(A, B) = \langle \mathcal{E}(A)\psi, \mathcal{E}(B)\psi \rangle$$

So far we have presented the standard quantum formulation for the system. We now construct the history Hilbert space K for the decoherence functional D just defined. We have seen that the natural map $U\colon K \to H$ given by (4.1) is an isometry from K into H. Theorem 4.1 tells us that U is unitary if and only if $(H, \mathcal{E}, \psi)$ is cyclic. Another sufficient condition for U to be unitary is given in [3]. We now show that this condition is also necessary.

Theorem 4.2. *For this example, U is unitary if and only if for every $i = 1, \dots, n$ there exists an $\omega \in \Omega$ such that*

$$[\mathcal{E}(\omega)\psi](i) \neq 0 \tag{4.3}$$

Proof. Let $\{\psi^1, \dots, \psi^n\}$ be the standard basis for $\mathbb{C}^n$. By (4.2) we have that $\mathcal{E}(\omega)\psi = c(\omega)\psi^{\omega_N}$ for some $c(\omega) \in \mathbb{C}$. If (4.3) holds, then $\mathcal{E}(\omega)\psi = c(\omega)\psi^i$ for $c(\omega) \neq 0$. It follows that ψ is cyclic so by Theorem 4.1, U is unitary. Conversely, suppose $[\mathcal{E}(\omega)\psi](i_0) = 0$ for every $\omega \in \Omega$. It follows that if

$$\phi \in \operatorname{span}\{\mathcal{E}(A)\psi\colon A \in \mathcal{A}\}$$

then $\phi(i_0) = 0$. Hence, ψ is not cyclic so by Theorem 4.1, U is not unitary. $\square$

5 Classical Decoherence Functionals

A decoherence functional $D\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is *weakly classical* if $\mu(A) = D(A,A)$ is a probability measure on $\mathcal{A}$. A decoherence functional $D\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is *classical* if $D(A,B)=\mu(A\cap B)$ for some probability measure on $\mathcal{A}$. Of course, D is weakly classical if D is classical.

Theorem 5.1. (a) *A decoherence functional $D\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is weakly classical if and only if there exists a probability measure $\mu\colon\mathcal{A}\to\mathbb{R}^+$ such that* $\operatorname{Re}D(A,B)=\mu(A\cap B)$. (b) *If $D\colon\mathcal{A}\times\mathcal{A}\to\mathbb{C}$ has the form $D(A,B)=\mu(A\cap B)$ for some probability measure $\mu\colon\mathcal{A}\to\mathbb{R}^+$, then D is a classical decoherence functional.*

Proof. (a) If $\operatorname{Re}D(A,B)=\mu(A\cap B)$ for some probability measure μ, it is clear that D is weakly classical. Conversely, suppose D is weakly classical so that $\mu(A)=D(A,A)$ is a probability measure. By Theorem 2.3, there is spanning vector representation $(H,\mathcal{E})$ so that $D(A,B)=\langle\mathcal{E}(A),\mathcal{E}(B)\rangle$. If $A,B\in\mathcal{A}$ are disjoint, then

$$\begin{aligned}\langle\mathcal{E}(A),\mathcal{E}(A)\rangle+\langle\mathcal{E}(B),\mathcal{E}(B)\rangle &= \mu(A)+\mu(B)=\mu(A\cup B)\\ &=\langle\mathcal{E}(A\cup B),\mathcal{E}(A\cup B\rangle=\langle\mathcal{E}(A)+\mathcal{E}(B),\mathcal{E}(A)+\mathcal{E}(B)\rangle\\ &=\langle\mathcal{E}(A),\mathcal{E}(A)\rangle+\langle\mathcal{E}(B),\mathcal{E}(B)\rangle+2\operatorname{Re}\langle\mathcal{E}(A),\mathcal{E}(B)\rangle\end{aligned}$$

Hence,

$$\operatorname{Re}D(A,B)=\operatorname{Re}\langle\mathcal{E}(A),\mathcal{E}(B)\rangle=0$$

For arbitrary $A,B\in\mathcal{A}$ we have

$$\begin{aligned}\operatorname{Re}D(A,B)&=\operatorname{Re}D\left[(A\cap B)\cup(A\cap B'),(A\cap B)\cup(A'\cap B)\right]\\ &=\operatorname{Re}\left[D(A\cap B,A\cap B)+D(A\cap B,A'\cap B)\right.\\ &\qquad\left.+D(A\cap B',A\cap B)+D(A\cap B',A'\cap B)\right]\\ &=\operatorname{Re}D(A\cap B,A\cap B)=\mu(A\cap B)\end{aligned}$$

(b) Suppose $D(A,B)=\mu(A\cap B)$ for a probability measure $\mu\colon\mathcal{A}\to\mathbb{R}^+$. We only need to show that D is a decoherence functional. It is clear that $D(\Omega,\Omega)=1$ and that $A\mapsto D(A,B)$ is a complex measure for every $B\in\mathcal{A}$. Let $A_1,\ldots,A_k\in\mathcal{A}$ and let $\mathcal{A}_0$ be the Boolean algebra generated by $\{A_1,\ldots,A_k\}$. Since $|\mathcal{A}_0|<\infty$, by Stone's theorem there is a finite set

$\Omega = \{\omega_1, \ldots, \omega_n\}$ and an isomorphism $h\colon 2^\Omega \to \mathcal{A}_0$. Define $D'\colon 2^\Omega \times 2^\Omega \to \mathbb{C}$ by $D'(A,B) = D\left(h(A), h(B)\right)$. In particular,

$$D'_{ij} = D'(\omega_i, \omega_j) = D\left(h(\omega_i), h(\omega_j)\right)$$

Now, $\sum\limits_{i,j} D'_{ij} = 1$ and for $i \neq j$ we have

$$D'_{ij} = D\left(h(\omega_i), h(\omega_j)\right) = \mu\left(h(\omega_i) \cap h(\omega_i)\right) = 0$$

Hence, $D'(\omega_i, \omega_j) = \mu\left(h(\omega_i)\right)\delta_{ij}$, $i,j = 1, \ldots, n$ so $\left[D'_{ij}\right]$ is a positive semi-definite matrix. It follows from the proof of Theorem 2.2 that there exists a vector representation $(H, \mathcal{E})$ such that

$$D'(\omega_i, \omega_j) = \langle \mathcal{E}(\omega_i), \mathcal{E}(\omega_j)\rangle$$

for $i,j = 1, \ldots, n$. Hence, $D'(A,B) = \langle \mathcal{E}(A), \mathcal{E}(B)\rangle$ so D' is a decoherence functional. Hence,

$$D(A_i, A_j) = D'\left(h^{-1}(A_i), h^{-1}(A_j)\right)$$

is a positive semi-definite matrix. We conclude that D is a decoherence functional. □

Theorem 5.2. *If $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a decoherence functional, the following statements are equivalent.* (a) *D is classical.* (b) *$D(A \cap B, C) = D(B, A \cap C)$ for all $A, B, C \in \mathcal{A}$.* (c) *If $A \cap B = \emptyset$, then $D(A,B) = 0$.* (d) *D has a spanning vector representation $(H, \mathcal{E})$ where $\mathcal{E}(A) \perp \mathcal{E}(B)$ whenever $A \cap B = \emptyset$.*

Proof. For (a)⇒(b), if D is classical, then $D(A,B) = \mu(A \cap B)$ for a probability measure $\mu\colon \mathcal{A} \to \mathbb{R}^+$. Hence

$$D(A \cap B, C) = \mu\left((A \cap B) \cap C\right) = \mu\left(B \cap (A \cap C)\right) = D(B, A \cap C)$$

For (b)⇒(c), suppose (b) holds and $A \cap B = \emptyset$. We have that

$$D(A,B) = D(A \cap B, B) = D(\emptyset, B) = 0$$

For (c)⇒(d), suppose (c) holds. By Theorem 2.3, D has a spanning vector representation $(H, \mathcal{E})$ such that $D(A,B) = \langle \mathcal{E}(A), \mathcal{E}(B)\rangle$ for all $A, B \in \mathcal{A}$. If $A \cap B = \emptyset$, then $D(A,B) = 0$ so that $\mathcal{E}(A) \perp \mathcal{E}(B)$.

For (d)⇒(a), suppose (d) holds. We conclude that

$$\begin{aligned} D(A,B) &= \langle \mathcal{E}(A), \mathcal{E}(B)\rangle = \langle \mathcal{E}(A\cap B)+\mathcal{E}(A\cap B'), \mathcal{E}(A\cap B)+\mathcal{E}(B\cap A')\rangle \\ &= \langle \mathcal{E}(A\cap B), \mathcal{E}(A\cap B)\rangle = \|\mathcal{E}(A\cap B)\|^2 \end{aligned}$$

Defining $\mu\colon \mathcal{A}\to\mathbb{R}^+$ by $\mu(A)=\|\mathcal{E}(A)\|^2$ we have that $D(A,B)=\mu(A\cap B)$. To show that μ is a probability measure, we have

$$\mu(\Omega)=\mu(\Omega\cap\Omega)=D(\Omega,\Omega)=1$$

Moreover, if $A_i\in\mathcal{A}$ are mutually disjoint, then

$$\begin{aligned} \mu\left(\cup A_i\right) &= \left\|\mathcal{E}\left(\cup A_i\right)\right\|^2 = \left\|\lim_{n\to\infty}\sum_{i=1}^n \mathcal{E}(A_i)\right\|^2 = \lim_{n\to\infty}\left\|\sum_{i=1}^n \mathcal{E}(A_i)\right\|^2 \\ &= \lim_{n\to\infty}\sum_{i=1}^n \|\mathcal{E}(A_i)\|^2 = \sum_{i=1}^\infty \mu(A_i) \end{aligned}$$ □

The importance of Theorem 5.2 is that it characterizes classical decoherence functionals in terms of their vector representations. In fact, we have the following corollary.

Corollary 5.3. *A decoherence functional $D\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ is classical if and only if for any vector representation $(H,\mathcal{E})$ for D we have $\mathcal{E}(A)\perp\mathcal{E}(B)$ whenever $A\cap B=\emptyset$.*

The following is another characterization of classicality.

Corollary 5.4. *A decoherence functional D is classical if and only if $D(A,B)=D(A\cap B,A\cap B)$ for all $A,B\in\mathcal{A}$.*

Proof. If $D(A,B)=D(A\cap B,A\cap B)$, then $A\cap B=\emptyset$ implies that

$$D(A,B)=D(\emptyset,\emptyset)=0$$

By Theorem 5.2, D is classical. Conversely, if D is classical by Theorem 5.2 we have

$$D(A\cap B,A\cap B)=D(A,A\cap B)=D(A,B)$$ □

6 Quantum Measures

This section applies our previous work on decoherence functionals to the study of quantum measures. For $(\Omega, \mathcal{A})$ a measurable space, a map $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is *grade-2 additive* if

$$\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C) \quad (6.1)$$

for all mutually disjoint $A, B, C \in \mathcal{A}$. A *q-measure* is a grade-2 additive set function $\mu\colon \mathcal{A} \to \mathbb{R}^*$ that satisfies the following conditions.

(C1) If $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence in $\mathcal{A}$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

(C2) If $A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing sequence in $\mathcal{A}$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Using the notation $A \triangle B = (A \cap B') \cup (A' \cap B)$, it is shown in [4] that $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is grade-2 additive if and only if

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) + \mu(A \triangle B) - \mu(A \cap B') - \mu(A' \cap B) \quad (6.2)$$

for all $A, B \in \mathcal{A}$.

Due to quantum interference, a q-measure need not satisfy the usual additivity condition of an ordinary measure but satisfies the more general grade-2 additivity condition (6.1) instead [5, 7, 8, 11, 12]. We have already mentioned that (2.1) and (2.3) are examples of q-measures. If μ is a q-measure on $\mathcal{A}$, we call $(\Omega, \mathcal{A}, \mu)$ a *q-measure space*. We shall not assume that a q-measure μ satisfies $\mu(\Omega) = 1$. For this reason we relax Condition (D1) for a decoherence functional and our previous results still hold.

Let $(\Omega, \mathcal{A}, \mu)$ be a q-measure space in which $\Omega = \{\omega_1, \ldots, \omega_n\}$ is finite and $\mathcal{A}$ is the power set 2^Ω. The *two-point interference term* for μ is defined by

$$I_{ij}^\mu = \mu\left(\{\omega_i, \omega_j\}\right) - \mu(\omega_i) - \mu(\omega_j)$$

for $i \neq j = 1, \ldots, n$, where $\mu(\omega_i) = \mu(\{\omega_i\})$. The *decoherence matrix* D is given by

$$\begin{aligned} D_{ii} &= D(\omega_i, \omega_j) = \mu(\omega_i), \qquad i = 1, \ldots, n \\ D_{ij} &= D(\omega_i, \omega_j) = \tfrac{1}{2} I_{ij}^{\mu}, \qquad i \neq j = 1, \ldots, n \end{aligned}$$

The q-measure μ is *strongly positive* if D is positive semi-definite. Of course, if μ is a measure, then $I_{ij}^{\mu} = 0$ for $i \neq j$ so μ is strongly positive. However, there are many examples of q-measures that are not strongly positive. For instance, let $\Omega = \{\omega_1, \omega_2\}$ and define the q-measure $\mu \colon \mathcal{A} \to \mathbb{R}^+$ by $\mu(\Omega) = 1$ and

$$\mu(\emptyset) = \mu(\omega_1) = \mu(\omega_2) = 0$$

Then μ is not strongly positive because

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is not positive semi-definite. For another example, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and define the q-measure $\mu \colon \mathcal{A} \to \mathbb{R}^+$ by $\mu(\emptyset) = \mu(\Omega) = 0$ and $\mu(A) = 1$ for $A \neq \emptyset, \Omega$. Then μ is not strongly positive because

$$D = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

is not positive semi-definite.

Theorem 6.1. *Let $(\Omega, \mathcal{A})$ be a finite measurable space. A map $\mu \colon \mathcal{A} \to \mathbb{R}^+$ is a strongly positive q-measure if and only if there exists a finite-dimensional complex Hilbert space H and a spanning vector-valued measure $\mathcal{E} \colon \mathcal{A} \to H$ such that*

$$\mu(A) = \|\mathcal{E}(A)\|^2 \tag{6.3}$$

for all $A \in \mathcal{A}$.

Proof. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$. It is straightforward to check that if μ has the form (6.3), then μ is a strongly positive q-measure. Conversely, suppose that $\mu \colon \mathcal{A} \to \mathbb{R}^+$ is a strongly positive q-measure and let D_{ij} be the corresponding positive semi-definite decoherence matrix. By Lemma 2.1 and the proof of Theorem 2.2, there exists a decoherence functional $D\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ given by

$$D(A,B)=\sum\left\{D_{ij}\colon \omega_1\in A,\omega_j\in B\right\}$$

a finite-dimensional complex Hilbert space H and a spanning vector-valued measure $\mathcal{E}\colon \mathcal{A}\to H$ such that

$$D(A,B)=\langle\mathcal{E}(A),\mathcal{E}(B)\rangle$$

for all $A,B\in\mathcal{A}$. Notice that (6.3) holds if $A=\{\omega_i\}$, $i=1,\dots,n$. To show that (6.3) holds for a general $A\in\mathcal{A}$, we can assume without loss of generality that $A=\{\omega_1,\dots,\omega_m\}$, $2\le m\le n$. It follows from Theorem 2.2 of reference [3] that

$$\mu(A)=\sum_{i<j=1}^m\mu\left(\{\omega_i,\omega_j\}\right)-(m-2)\sum_{i=1}^m\mu(\omega_i)$$

We then have that

$$\begin{aligned}\|\mathcal{E}(A)\|^2&=D(A,A)=\sum_{i,j=1}^m D_{ij}=\sum_{i=1}^m D_{ii}+2\sum_{i<j=1}^m D_{ij}\\&=\sum_{i=1}^m\mu(\omega_i)+\sum_{i<j=1}^m I_{ij}^\mu\\&=\sum_{i=1}^m\mu(\omega_i)+\sum_{i<j=1}^m\mu\left(\{\omega_i,\omega_j\}\right)-\sum_{i<j=1}^m\left[\mu(\omega_i)-\mu(\omega_j)\right]\\&=\sum_{i=1}^m\mu(\omega_i)+\sum_{i<j=1}^m\mu\left(\{\omega_i,\omega_j\}\right)-(m-1)\sum_{i=1}^m\mu(\omega_i)\\&=\sum_{i<j=1}^m\mu\left(\{\omega_i,\omega_j\}\right)-(m-2)\sum_{i=1}^m\mu(\omega_i)=\mu(A)\end{aligned}$$

□

If the sample space Ω is infinite, then we must proceed differently than we did for the finite case. For example, when Ω is infinite the singleton and doubleton subsets may not be measurable (i.e., may not be in $\mathcal{A}$) and even if they are measurable, they frequently all have measure zero.

Let $(\Omega, \mathcal{A}, \mu)$ be a q-measure space. For $A, B \in \mathcal{A}$ define

$$\triangle(A, B) = \tfrac{1}{2}\left[\mu(A \cup B) + \mu(A \cap B) - \mu(A \cap B') - \mu(A' \cap B)\right] \tag{6.4}$$

Notice that if $\{\omega_i\}$ and $\{\omega_j\}$ are measurable, then

$$\triangle\left(\{\omega_i\}, \{\omega_j\}\right) = D_{ij}$$

so $\triangle(A, B)$ is a generalization of the decoherence matrix. We say that μ is *strongly positive* if for any $A_1, \ldots, A_k \in \mathcal{A}$, the matrix $\triangle(A_i, A_j)$, $i, j = 1, \ldots, k$ is positive semi-definite. It follows that this definition reduces to the definition of strongly positive in the finite case. Also, observe that if μ is a measure, then (6.4) gives $\triangle(A, B) = \mu(A \cap B)$ so $\triangle$ is a classical decoherence functional. Applying Theorem 5.2, there exists a vector-valued measure $\mathcal{E}\colon \mathcal{A} \to H$ satisfying $\mathcal{E}(A) \perp \mathcal{E}(B)$ whenever $A \cap B = \emptyset$ such that $\mu(A) = \|\mathcal{E}(A)\|^2$ for all $A \in \mathcal{A}$. Although the next result generalizes Theorem 6.1, we gave an independent proof of Theorem 6.1 because the decoherence matrix D_{ij} is physically more intuitive than $\triangle$.

Theorem 6.2. *Let $(\Omega, \mathcal{A})$ be a measurable space. A map $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is a strongly positive q-measure if and only if there exists a complex Hilbert space H and a spanning vector-valued measure $\mathcal{E}\colon \mathcal{A} \to H$ such that* (6.3) *holds.*

Proof. Suppose μ has the form (6.3). It is straightforward to check that μ is a q-measure. To show that μ is strongly positive, let $A_1, \ldots, A_k \in \mathcal{A}$. Applying (6.4) we have that

$$\begin{aligned}
\triangle(A, B) &= \tfrac{1}{2}\left[\langle \mathcal{E}(A \cup B), \mathcal{E}(A \cup B)\rangle + \langle \mathcal{E}(A \cap B), \mathcal{E}(A \cap B)\rangle\right.\\
&\qquad \left. - \langle \mathcal{E}(A \cap B'), \mathcal{E}(A \cap B')\rangle - \langle \mathcal{E}(A' \cap B), \mathcal{E}(A' \cap B)\rangle\right]\\
&= \tfrac{1}{2}\Big[\|\mathcal{E}(A \cap B) + \mathcal{E}(A \cap B') + \mathcal{E}(A' \cap B)\|^2 + \|\mathcal{E}(A \cap B)\|^2\\
&\qquad - \|\mathcal{E}(A \cap B')\|^2 - \|\mathcal{E}(A' \cap B)\|^2\Big]\\
&= \operatorname{Re}\left[\|\mathcal{E}(A \cap B)\|^2 + \langle \mathcal{E}(A \cap B), \mathcal{E}(A \cap B')\rangle\right.\\
&\qquad \left. + \langle \mathcal{E}(A \cap B), \mathcal{E}(A' \cap B)\rangle + \langle \mathcal{E}(A \cap B'), \mathcal{E}(A' \cap B)\rangle\right]\\
&= \operatorname{Re}\left[\langle \mathcal{E}(A \cap B) + \mathcal{E}(A \cap B'), \mathcal{E}(A \cap B) + \mathcal{E}(A' \cap B)\rangle\right]\\
&= \operatorname{Re}\langle \mathcal{E}(A), \mathcal{E}(B)\rangle
\end{aligned}$$

Hence, for $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ we have

$$\begin{aligned}\sum_{i,j} \triangle(A_i, A_j)\alpha_i\overline{\alpha_j} &= \sum_{i,j} \operatorname{Re}\langle \mathcal{E}(A_i), \mathcal{E}(A_j)\rangle \alpha_i\overline{\alpha_j} \\ &= \operatorname{Re}\left\langle \sum \alpha_i \mathcal{E}(A_i), \sum \alpha_j \mathcal{E}(A_j)\right\rangle \geq 0\end{aligned}$$

We conclude that $\triangle(A_i, A_j)$ is a positive semi-definite matrix so μ is strongly positive.

Conversely, suppose that μ is a strongly positive q-measure. We show that $A \mapsto \triangle(A, B)$ is a complex-valued measure for every $B \in \mathcal{A}$. If $A_1, A_2 \in \mathcal{A}$ are disjoint we have

$$\begin{aligned}\triangle(A_1 \cup A_2, B) = \tfrac{1}{2}\{&\mu\left[(A_1 \cup B) \cup (A_2 \cup B)\right] + \mu\left[(A_1 \cap B) \cup (A_2 \cap B)\right] \\ &-\mu\left[(A_1 \cap B') \cup (A_2 \cap B')\right] - \mu(A'_1 \cap A'_2 \cap B)\}\end{aligned} \tag{6.5}$$

By (6.2) we have

$$\begin{aligned}&\mu\left[(A_1 \cup B) \cup (A_2 \cup B)\right] \\ &\quad= \mu\left[(A_1 \cup B)\triangle(A_2 \cup B)\right] - \mu\left[(A_1 \cup B) \cap (A_2 \cup B)'\right] \\ &\qquad- \mu\left[(A_1 \cup B)' \cap (A_2 \cap B)\right] + \mu(A_1 \cup B) + \mu(A_2 \cup B) \\ &\qquad- \mu\left[(A_1 \cup B) \cap (A_2 \cup B)\right] \\ &\quad= \mu\left[(A_1 \cap B') \cup (A_2 \cap B')\right] - \mu(A_1 \cap B') - \mu(A_2 \cap B') \\ &\qquad+ (A_1 \cup B) + \mu(A_2 \cup B) - \mu(B)\end{aligned} \tag{6.6}$$

Since μ is grade-2 additive we have

$$\begin{aligned}\mu(B) &= \mu\left[(B \cap A_1) \cup (B \cap A_2) \cup (B \cap A'_1 \cap A'_2)\right] \\ &= \mu\left[(B \cap A_1) \cup (B \cap A_2)\right] + \mu\left[(B \cap A_1) \cup (B \cap A'_1 \cap A'_2)\right] \\ &\quad+ \mu\left[(B \cap A_2) \cup (B \cap A'_1 \cap A'_2)\right] - \mu(B \cap A_1) - \mu(B \cap A_2) \\ &\quad- \mu(B \cap A'_1 \cap A'_2) \\ &= \mu\left[(B \cap A_1) \cup (B \cap A_2)\right] + \mu(B \cap A'_2) + \mu(B \cap A'_1) \\ &\quad- \mu(B \cap A_1) - \mu(B \cap A_2) - \mu(B \cap A'_1 \cap A'_2)\end{aligned} \tag{6.7}$$

Substituting (6.7) into (6.6) gives

$$\begin{aligned}&\mu\left[(A_1 \cup B) \cup (A_2 \cup B)\right] \\ &\quad= \mu\left[(A_1 \cap B') \cup (A_2 \cap B')\right] - \mu(A_1 \cap B') - \mu(A_2 \cap B') \\ &\qquad+ \mu(A_1 \cup B) + \mu(A_2 \cup B) - \mu\left[(B \cap A_1) \cup (B \cap A_2)\right] - \mu(B \cap A'_2) \\ &\qquad- \mu(B \cap A'_1) + \mu(B \cap A_1) + \mu(B \cap A_2) + \mu(B \cap A'_1 \cap A'_2)\end{aligned} \tag{6.8}$$

Substituting (6.8) into (6.5) gives

$$\begin{aligned}\triangle(A_1\cup A_2,B)&=\tfrac{1}{2}\left[\mu(A_1\cup B)+\mu(A_2\cup B)+\mu(A_1\cap B)+\mu(A_2\cap B)\right.\\&\quad\left.-\mu(A_1\cap B')-\mu(A_2\cap B')-\mu(A_2'\cap B)-\mu(A_1'\cap B)\right]\\&=\triangle(A_1,B)+\triangle(A_2,B)\end{aligned}$$

We conclude by induction that

$$\triangle\left(\bigcup_{i=1}^{n}A_i,B\right)=\sum_{i=1}^{n}\triangle(A_i,B)$$

whenever $A_1,\ldots,A_n\in\mathcal{A}$ are mutually disjoint. Let $A_i\in\mathcal{A}$ with $A_1\subseteq A_2\subseteq\cdots$. Since μ is continuous, we have

$$\begin{aligned}&\lim\triangle(A_i,B)\\&\quad=\tfrac{1}{2}\left[\lim\mu(A_i\cup B)+\lim\mu(A_i\cap B)-\lim\mu(A_i\cap B')-\lim(A_i'\cap B)\right]\\&\quad=\tfrac{1}{2}\left\{\mu\left[(\cup A_i)\cup B\right]+\mu\left[(\cup A_i)\cap B\right]-\mu\left[(\cup A_i)\cap B'\right]-\mu\left[(\cap A_i)'\cap B\right]\right\}\\&\quad=\triangle(\cup A_i,B)\end{aligned}$$

It follows that $A\mapsto D(A,B)$ is a complex-valued measure for all $B\in\mathcal{B}$. Hence, D is a decoherence functional (except for Condition (D1)) and the result follows from Theorem 2.3 $\square$

7 Operator Quantum Measures

This section briefly considers a generalization of q-measures to operator q-measures. Let $(\Omega,\mathcal{A})$ be a measurable space and $\mathcal{E}\colon\mathcal{A}\to B(H)$ be an operator-valued measure. We define the *decoherence operator* $\mathcal{D}\colon\mathcal{A}\times\mathcal{A}\to B(H)$ by

$$\mathcal{D}(A,B)=\mathcal{E}(B)^*\mathcal{E}(A)$$

Notice that if $\|\mathcal{E}(\Omega)\psi\|=1$, then $D(A,B)=\langle\mathcal{D}(A,B)\psi,\psi\rangle$ is a decoherence functional. We call $\mathcal{Q}\colon\mathcal{A}\to B(H)$ given by

$$\mathcal{Q}(A)=\mathcal{D}(A,A)=\mathcal{E}(A)^*\mathcal{E}(A)\tag{7.1}$$

an *operator q-measure*. The next result summarizes some of the interesting properties of $\mathcal{Q}$.

Theorem 7.1. *If $\mathcal{E}\colon \mathcal{A} \to B(H)$ is an operator-valued measure, then the operator q-measure* (7.1) *is a positive operator-valued function that satisfies the following conditions.* (a) *(Grade-2 additivity) For any mutually disjoint sets $A, B, C \in \mathcal{A}$ we have*

$$\mathcal{Q}(A \cup B \cup C) = \mathcal{Q}(A \cup B) + \mathcal{Q}(A \cup C) + \mathcal{Q}(B \cup C) - \mathcal{Q}(A) - \mathcal{Q}(B) - \mathcal{Q}(C)$$

(b) *(Regularity) If $\mathcal{Q}(A) = 0$, then $\mathcal{Q}(A \cup B) = \mathcal{Q}(B)$ whenever $A \cap B = \emptyset$. If $A \cap B = \emptyset$ and $\mathcal{Q}(A \cup B) = 0$, then $\mathcal{Q}(A) = \mathcal{Q}(B)$.*
(c) *(Continuity) If $A_1 \subseteq A_2 \subseteq \cdots$ and $\phi, \phi' \in H$, then*

$$\langle \mathcal{Q}(\cup A_i)\phi, \phi' \rangle = \lim \langle \mathcal{Q}(A_i)\phi, \phi' \rangle$$

and if $A_1 \supseteq A_2 \supseteq \cdots$ and $\phi, \phi' \in H$, then

$$\langle \mathcal{Q}(\cap A_i)\phi, \phi' \rangle = \lim \langle \mathcal{Q}(A_i)\phi, \phi' \rangle$$

Proof. It is clear that $\mathcal{Q}(A)$ is a positive operator for all $A \in \mathcal{A}$. (a) Since $\mathcal{Q}(A) = \mathcal{E}(A)^*\mathcal{E}(A)$, $A \in \mathcal{A}$, we have

$$\begin{aligned}
&\mathcal{Q}(A \cup B) + \mathcal{Q}(A \cup C) + \mathcal{Q}(B \cup C) - \mathcal{Q}(A) - \mathcal{Q}(B) - \mathcal{Q}(C)\\
&\quad = 2\mathcal{E}(A)^*\mathcal{E}(A) + 2\mathcal{E}(B)^*\mathcal{E}(B) + 2\mathcal{E}(C)^*\mathcal{E}(C) + \mathcal{E}(A)^*\mathcal{E}(B)\\
&\qquad + \mathcal{E}(B)^*\mathcal{E}(A) + \mathcal{E}(A)^*\mathcal{E}(C) + \mathcal{E}(C)^*\mathcal{E}(A) + \mathcal{E}(B)^*\mathcal{E}(C) + \mathcal{E}(C)^*\mathcal{E}(B)\\
&\qquad - \mathcal{E}(A)^*\mathcal{E}(A) - \mathcal{E}(B)^*\mathcal{E}(B) - \mathcal{E}(C)^*\mathcal{E}(C)\\
&\quad = \mathcal{E}(A \cup B \cup C)^*\mathcal{E}(A \cup B \cup C) = \mathcal{Q}(A \cup B \cup C)
\end{aligned}$$

(b) If $\mathcal{Q}(A) = 0$, then $\mathcal{E}(A)^*\mathcal{E}(A) = 0$. Hence, for every $\phi \in H$ we have

$$\|\mathcal{E}(A)\phi\|^2 = \langle \mathcal{E}(A)\phi, \mathcal{E}(A)\phi \rangle = \langle \mathcal{E}(A)^*\mathcal{E}(A)\phi, \phi \rangle = 0$$

Hence, $\mathcal{E}(A)\phi = 0$ so $\mathcal{E}(A) = 0$. For $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ we have

$$\begin{aligned}
\mathcal{Q}(A \cup B) &= \mathcal{E}(A \cup B)^*\mathcal{E}(A \cup B) = [\mathcal{E}(A) + \mathcal{E}(B)]^* [\mathcal{E}(A) + \mathcal{E}(B)]\\
&= \mathcal{E}(B)^*\mathcal{E}(B) = \mathcal{Q}(B)
\end{aligned}$$

If $A \cap B = \emptyset$ and $\mathcal{Q}(A \cup B) = 0$, then

$$\begin{aligned}
0 = \mathcal{Q}(A \cup B) &= \mathcal{E}(A)^*\mathcal{E}(A) + \mathcal{E}(B)^*\mathcal{E}(B) + \mathcal{E}(A)^*\mathcal{E}(B) + \mathcal{E}(B)^*\mathcal{E}(A)\\
&= [\mathcal{E}(A) + \mathcal{E}(B)]^* [\mathcal{E}(A) + \mathcal{E}(B)]
\end{aligned}$$

As before, $\mathcal{E}(A)+\mathcal{E}(B)=0$. It follows that

$$\mathcal{E}(B)^*\mathcal{E}(A)=-\mathcal{E}(B)^*\mathcal{E}(B)$$

and

$$\mathcal{E}(A)^*\mathcal{E}(B)=-\mathcal{E}(B)^*\mathcal{E}(B)$$

Hence

$$\mathcal{E}(A)^*\mathcal{E}(A)-\mathcal{E}(B)^*\mathcal{E}(B)=0$$

so that $\mathcal{Q}(A)=\mathcal{Q}(B)$.
(c) Let $A_1\subseteq A_2\subseteq\cdots$ be increasing in $\mathcal{A}$ and let $\phi,\phi'\in H$. Define $B_1=A_1$, $B_i=A_i\smallsetminus A_{i-1}$ $i=2,3,\ldots$. Then $B_i\in\mathcal{A}$ are mutually disjoint so we have

$$\begin{aligned}\langle\mathcal{Q}(\cup A_i)\phi,\phi'\rangle=\langle\mathcal{E}(\cup B_i)\phi,\mathcal{E}(\cup B_j)\phi'\rangle&=\sum_{i,j}\langle\mathcal{E}(B_i)\phi,\mathcal{E}(B_j)\phi'\rangle\\&=\lim_{n,m\to\infty}\left\langle\mathcal{E}\left(\bigcup_{i=1}^n B_i\right)\phi,\mathcal{E}\left(\bigcup_{j=1}^m B_j\right)\phi'\right\rangle\\&=\lim_{n,m\to\infty}\langle\mathcal{E}(A_n)\phi,\mathcal{E}(A_m)\phi'\rangle\\&=\lim_{n\to\infty}\langle\mathcal{E}(A_n)^*\mathcal{E}(A_n)\phi,\phi'\rangle=\lim\langle\mathcal{Q}(A_n)\phi,\phi'\rangle\end{aligned}$$

The result is similar for $A_1\supseteq A_2\supseteq\cdots$. □

Motivated by Section 5 we make the following definitions. A decoherence operator $\mathcal{D}$ is *classical* if $\mathcal{D}(A,B)=0$ whenever $A\cap B=\emptyset$. For an operator $T\in B(H)$ we define $\mathrm{Re}\,T=\frac{1}{2}(T+T^*)$. A decoherence operator $\mathcal{D}$ is *weakly classical* if $\mathrm{Re}\,\mathcal{D}(A,B)=0$ whenever $A\cap B=\emptyset$.

Theorem 7.2. *Let $\mathcal{E}\colon\mathcal{A}\to B(H)$ be an operator-valued measure and let $\mathcal{D}(A,B)=\mathcal{E}(B)^*\mathcal{E}(A)$ and $\mathcal{Q}(A)=\mathcal{D}(A,A)$ be the corresponding decoherence operator and operator q-measure.* (a) *$\mathcal{D}$ is classical if and only if $\mathcal{D}(A,B)=\mathcal{Q}(A\cap B)$ for every $A,B\in\mathcal{A}$.* (b) *$\mathcal{D}$ is weakly classical if and only if $\mathcal{Q}$ is an operator-valued measure.*

Proof. (a) If $\mathcal{D}(A,B)=\mathcal{Q}(A\cap B)$ and $A\cap B=\emptyset$, then

$$\mathcal{D}(A,B)=\mathcal{Q}(\emptyset)=\mathcal{D}(\emptyset,\emptyset)=0$$

so $\mathcal{D}$ is classical. Conversely, if $\mathcal{D}$ is classical, then

$$\begin{aligned}\mathcal{D}(A,B)&=\mathcal{E}(B)^*\mathcal{E}(A)=[\mathcal{E}(A\cap B)^*+\mathcal{E}(B\cap A')^*]\,[\mathcal{E}(A\cap B)+\mathcal{E}(A\cap B')]\\&=\mathcal{D}(A\cap B,A\cap B)+\mathcal{D}(A\cap B,A\cap B')+\mathcal{D}(B\cap A',A\cap B)\\&\quad+\mathcal{D}(B\cap A',A\cap B')\\&=\mathcal{D}(A\cap B,A\cap B)=\mathcal{Q}(A\cap B)\end{aligned}$$

(b) If $\mathcal{Q}$ is an operator-valued measure and $A\cap B=\emptyset$, then

$$\mathcal{E}(A\cup B)^*\mathcal{E}(A\cup B)=\mathcal{Q}(A\cup B)=\mathcal{Q}(A)+\mathcal{Q}(B)=\mathcal{E}(A)^*\mathcal{E}(A)+\mathcal{E}(B)^*\mathcal{E}(B)$$

Hence,

$$\operatorname{Re}\mathcal{D}(A,B)=\tfrac{1}{2}\left[\mathcal{E}(B)^*\mathcal{E}(A)+\mathcal{E}(A)^*\mathcal{E}(B)\right]=0$$

so $\mathcal{D}$ is weakly classical. Conversely, suppose $\mathcal{D}$ is weakly classical. To show that $\mathcal{Q}$ is an operator-valued measure, let A_i be a sequence of mutually disjoint sets in $\mathcal{A}$. For any $\phi,\phi'\in H$ we have that

$$\begin{aligned}\langle\mathcal{Q}(\cup A_i)\phi,\phi'\rangle&=\langle\mathcal{E}(\cup A_i)\phi,\mathcal{E}(\cup A_j)\phi'\rangle\\&=\lim_{m,n\to\infty}\left\langle\sum_{i=1}^m\mathcal{E}(A_i)\phi,\sum_{j=1}^n\mathcal{E}(A_j)\phi'\right\rangle\\&=\lim_{n\to\infty}\sum_{i=1}^n\langle\mathcal{E}(A_i)\phi,\mathcal{E}(A_i)\phi'\rangle=\sum_{i=1}^\infty\langle\mathcal{Q}(A_i)\phi,\phi'\rangle\end{aligned}$$

Hence, $\mathcal{Q}$ is an operator-valued measure. □

References

[1] F. Dowker, S. Johnston and S. Surya, On extending the quantum measure, arXiv: quant-ph (1007.2725), 2010.

[2] F. Dowker and Y. Ghazi-Tabatabai, Dynamical wave function collapse models in quantum measure theory, *J. Phys. A* **41**, 205306 (2008).

[3] F. Dowker, S. Johnston and R. Sorkin, Hilbert spaces from path integrals, arXiv: quant-ph (1002.0589), 2010.

[4] D. Dutkay, Positive definite maps, representations and frames, *Rev. Math. Phys.* **16**, (2004).

[5] S. Gudder, Quantum measure theory, *Math. Slovaca* **60** (2010), 681–700.

[6] S. Gudder, Quantum measure and integration theory, *J. Math. Phys.* **50** (2009), 123509.

[7] S. Gudder, Examples of quantum integrals, *Rep. Math. Phys.* (to appear).

[8] J. J. Halliwell, Partial decoherence of histories and the Diosi test, arXiv: quant-ph (0904.4388), 2009 and *Quantum Information Processing* (to appear).

[9] C. Isham, Quantum logic and the histories approach to quantum theory, *J. Math. Phys.* **35** (1994), 2157–2185.

[10] R. Jozsa, An introduction to measurement based quantum computation, arXiv: quant-ph (0508124), 2009.

[11] R. Sorkin, Quantum mechanics as quantum measure theory, *Mod. Phys. Letts. A* **9** (1994), 3119–3127.

[12] R. Sorkin, Quantum mechanics without the wave function, *J. Phys. A* **40** (2007), 3207–3231.