Classical Mechanics in Hilbert Space, Part 2

Roberto Beneduci
University of Calabria
rbeneduci@unical.it

James Brooke
University of Saskatchewan
brooke@math.usask.ca

Ray Curran
University of Denver
rcurran4@du.edu

Franklin E. Schroeck, Jr.
University of Denver
fschroec@du.edu

January 4, 2011

Abstract

We continue from Part 1. We will illustrate the general theory of Hamiltonian mechanics in the Lie group formalism. We then obtain the Hamiltonian formalism in the Hilbert spaces of square integrable functions on the symplectic spaces. We illustrate this general theory with several concrete examples, two of which are the representations of the Lorentz group and the Poincaré group with interactions.

1 Introduction

Keeping the notation, we continue from Part 1. Having considered the historical Hamiltonian formalism, defined a general symplectic (phase) space, and made a connection between a symplectic space and the orbit of G/H, where G is the group of symmetries and H is a certain subgroup of G, then in the Section 2 of this part, we will illustrate all of this with several examples. In Section 3, we look at the connection of G/H with that of $L^2(G/H)$ (or the orbit of such) and its irreducible components. In Section 4, we will consider the action of the Hamiltonian dynamics on $\mathcal{H}_{Koopman}$ and the Fock space on $\mathcal{H}_{Koopman}$. In Section 5, we will return to the examples and illustrate the procedure in (the Fock space of) $\mathcal{H}_{Koopman}$ with specific Hamiltonians and some of the groups already considered. One important group is that of Poincaré relativity with interactions(!), in apparent contradiction to a common interpretation of a result of Currie, Jordan, et al. [1],[2],[3],[4],[5]

2 Some Examples

2.1 Example 1: Heisenberg Motions

The motions in the p and q directions constitute the Heisenberg group, G. If G has corresponding self-adjoint generators P_j and Q_j, $j = 1, \cdots, n$, then by

$$[P_j, Q_k] = i\delta_{j,k}\mathbf{1}_\Gamma$$

it also has a generator $\mathbf{1}_\Gamma$. All other commutation relations among the basis elements vanish. Thus the basis of self-adjoint generators is $\{i\mathbf{1}_\Gamma, iP_j, iQ_j \mid j = 1, \cdots, n\}$. (Equivalently, you may write the Lie algebra for the generators which is an algebra over $\mathbb{R}$.) From this (or for this), you obtain the Heisenberg group, G:

$$G = \left\{ \begin{array}{c} \exp[i(\alpha\mathbf{1} + p \cdot P + q \cdot Q)] = (\alpha, p, q) \\ \text{such that } \alpha \in \mathbb{R}; p, q \in \mathbb{R}^n, \\ (\alpha', p', q') \circ (\alpha, p, q) = \\ (\alpha' + \alpha + [q' \cdot p - q \cdot p']/2, p' + p, q' + q) \end{array} \right\}.$$

The inverse is thus

$$(\alpha, p, q)^{-1} = (-\alpha, -p, -q).$$

If we choose $\Gamma = \mathbb{R}^{2n}$ and $x = (\mathbf{0}, \mathbf{0}) \in \Gamma$, then

$$(0, p, q) \cdot (\mathbf{0}, \mathbf{0}) = \exp[i(p \cdot P + q \cdot Q)] \cdot (\mathbf{0}, \mathbf{0}) \tag{1}$$

may be defined as (p, q). Notice that we have a variety of choices in defining (p, q) since we have

$$\begin{aligned} \exp[i(p \cdot P + q \cdot Q)] &= \exp[i(q \cdot p/2)\mathbf{1}_\Gamma] \circ \exp[ip \cdot P] \circ \exp[iq \cdot Q] \\ &= \exp[-i(q \cdot p/2)\mathbf{1}_\Gamma] \circ \exp[iq \cdot Q] \circ \exp[ip \cdot P], \end{aligned}$$

and we may define (p, q) as one of the three $\exp[i(p \cdot P + q \cdot Q)](0, 0)$, $(\exp[ip \cdot P] \circ \exp[iq \cdot Q]) \cdot (0, 0)$, or $(\exp[iq \cdot Q] \circ \exp[ip \cdot P]) \cdot (0, 0)$ as they differ by a constant phase. We will choose the first.

Now, if we use the left operation, then

$$\begin{aligned} (\alpha', p', q') \cdot (p, q) &= (\alpha', p', q') \cdot [(0, p, q) \cdot (\mathbf{0}, \mathbf{0})] \\ &= [(\alpha', p', q') \circ (0, p, q)] \cdot (\mathbf{0}, \mathbf{0}) \\ &= (\alpha' + [q' \cdot p - q \cdot p']/2, p + p', q + q') \cdot (\mathbf{0}, \mathbf{0}) \\ &= \exp\{i(\alpha' + [q' \cdot p - q \cdot p']/2)\mathbf{1}_\Gamma\}(p + p', q + q'). \end{aligned} \tag{2}$$

We have that $H = \{\exp(i\lambda\mathbf{1}_\Gamma) \mid \lambda \in \mathbb{R}\}$ commutes with the action of the group, as H is the center of the group. Thus

$$\begin{aligned} (\alpha', p', q') * (p, q) &\equiv (0, p, q) \circ H \circ (\alpha', p', q')^{-1} \\ &= (\alpha', p', q') * (p, q) = [(0, p, q) \circ (\alpha', p', q')^{-1}] \circ H, \end{aligned}$$

providing us with the fact that we may use either form (a) or (b) as the action from the right based on the Heisenberg group. (This is the problem with using this example of the Heisenberg group and led to the many misstatements in the literature.) Hence, operating from the right, we obtain

$$\begin{aligned}(\alpha', p', q') * (p, q) &= [(0, p, q) \circ (\alpha', p', q')^{-1}] * (\mathbf{0}, \mathbf{0}) \\ &= (-\alpha' - [q' \cdot p - q \cdot p']/2, p - p', q - q') * (\mathbf{0}, \mathbf{0}) \\ &= \exp\{i(-\alpha' - [q' \cdot p - q \cdot p']/2)\mathbf{1}_\Gamma\}(p - p', q - q'). \quad (3)\end{aligned}$$

Either way, we obtain the same vector except for a different phase and a different interpretation of the action of (α', p', q') on the base vectors.

We may place this closer to the setting developed in Part 1 by setting

$$H = \{\exp(-i\lambda\mathbf{1}_\Gamma) \mid \lambda \in \mathbb{R}\}, \quad (4)$$

where we have that H is a closed subgroup of the Heisenberg group. Then we might identify Γ as G/H. In particular, defining the equivalence classes in G/H by

$$[[p, q]] \equiv (0, p, q) \circ H, \quad (5)$$

we have

$$[[\mathbf{0}, \mathbf{0}]] = H$$

and

$$[[p, q]] = (0, p, q) \circ [[\mathbf{0}, \mathbf{0}]]. \quad (6)$$

Consequently, for this group, we have one of two choices:

$$\begin{aligned}(\alpha', p', q') \cdot [[p, q]] &= (\alpha', p', q') \cdot \{(0, p, q) \circ [[\mathbf{0}, \mathbf{0}]]\} \\ &= \{(\alpha', p', q') \circ (0, p, q)\} \circ [[\mathbf{0}, \mathbf{0}]] \\ &= ((\alpha' + [q' \cdot p - q \cdot p']/2, p + p', q + q') \circ [[\mathbf{0}, \mathbf{0}]] \\ &= \{(0, p' + p, q' + q) \circ (\alpha' + [q' \cdot p - q \cdot p']/2, \mathbf{0}, \mathbf{0})\} \circ [[\mathbf{0}, \mathbf{0}]] \\ &= \{(0, p' + p, q' + q) \circ (\alpha' + [q' \cdot p - q \cdot p']/2, \mathbf{0}, \mathbf{0})\} \circ H \\ &= (0, p' + p, q' + q) \circ H \\ &= [[p' + p, q' + q]], \quad (7)\end{aligned}$$

corresponding to the action on the left; or

$$\begin{aligned}(\alpha', p', q') * [[p, q]] &= [(0, p, q) \circ (\alpha', p', q')^{-1}] * [[\mathbf{0}, \mathbf{0}]] \\ &= (-\alpha' - [q' \cdot p - q \cdot p']/2, p - p', q - q') * [[\mathbf{0}, \mathbf{0}]] \\ &= \{(-\alpha' - [q' \cdot p - q \cdot p']/2, \mathbf{0}, \mathbf{0}) \circ (0, p - p', q - q')\} * [[\mathbf{0}, \mathbf{0}]] \\ &= (0, p - p', q - q') * [[\mathbf{0}, \mathbf{0}]] \\ &= [[p - p', q - q']]\end{aligned}$$

corresponding to the action on the right. In the second case, it may be preferable to define

$$[[p, q]] = (0, -p, -q) \circ H = (0, -p, -q) \circ [[\mathbf{0}, \mathbf{0}]] \quad (8)$$

and obtain

$$(\alpha', p', q') * [[p, q]] = [[p' + p, q' + q]] \tag{9}$$

so that both expressions (7) and (9) agree. Henceforth, we will adopt the notation (8) for the right actions.

We may obtain H by the Lie group theoretical method we have described. From [6, pp. 391-392] we obtain, for the Heisenberg group,

$$\begin{aligned} & Z^2(\mathfrak{g}^*) \\ = & \left\{ \sum_{j,k} (\alpha_{j,k} P_j^* \wedge P_k^* + \beta_{j,k} Q_j^* \wedge P_k^* + \gamma_{j,k} Q_j^* \wedge Q_k^* \mid \alpha_{j,k}, \beta_{j,k}, \gamma_{j,k} \in \mathbb{R} \right\}. \end{aligned}$$

If we choose

$$\omega = \sum_j Q_j^* \wedge P_j^*,$$

then we have (22) of Part 1 satisfied, and for $J = -i\mathbf{1}_\Gamma$ in any space Γ,

$$\mathfrak{h}_\omega = \{i\lambda J \mid \lambda \in \mathbb{R}\},$$

from which we obtain the closed subgroup

$$H_\omega = \{\exp[i\lambda J] \mid \lambda \in \mathbb{R}\}.$$

Hence, our phase space is

$$\begin{aligned} G/H_\omega &= \{\exp[-i(p \cdot P + q \cdot Q)] \circ H_\omega \mid p_j, q_j \in \mathbb{R}\} \\ &\simeq \mathbb{R}^{3n}_{mom} \times \mathbb{R}^{3n}_{config}, \ n \text{ the number of particles,} \end{aligned}$$

which is isomorphic with the space with which we have started. We also have that the invariant measure is

$$\mu = \left[\sum_{j=1}^{m/2} Q_j^* \wedge P_j^* \right]^{m/2},$$

which is the correct historical form, and so $(G/H_\omega, \mu) = (\Gamma, \Omega)$. Furthermore, we may choose canonical coordinates to be any set $\{Q_j, P_j\}$ such that $\omega = \sum_j Q_j^* \wedge P_j^*$.

2.2 Example 2: Rotations

The rotation group in $\mathbb{R}^3$ about one point, say $\mathbf{0}$, is a non-commutative Lie group, G. We may take it as $SO(3)$ and derive the phase space on which it works by using the "G/H_ω" form of the phase space and Lie algebra, $\mathfrak{g}$, with a basis of generators $\{J_1, J_2, J_3\}$ satisfying

$$[J_1, J_2] = J_3, \tag{10}$$

or by dealing with the Lie algebra, $\mathfrak{g}$, with a basis of generators in self-adjoint form J_1, J_2, J_3 satisfying

$$[J_1, J_2] = iJ_3 \tag{11}$$

and cyclically in $1, 2, 3$. The first has the advantage of being "more mathematical" and latter has the advantage of being "more physical." We shall treat both cases, as they both have something to say.

2.2.1 Case 1

Taking the first form, and after some work we obtain in [6]

$$\begin{aligned} Z^2(\mathfrak{g}^*) &= B^2(\mathfrak{g}^*) \\ &= \{\alpha J_1^* \wedge J_2^* + \beta J_2^* \wedge J_3^* + \gamma J_3^* \wedge J_1^* \mid \alpha, \beta, \gamma \in \mathbb{R}\}. \end{aligned}$$

Now, if we choose

$$\omega = J_1^* \wedge J_2^*,$$

then we have (22) of Part 1 satisfied and

$$\mathfrak{h}_\omega = \{\alpha J_3 \mid \alpha \in \mathbb{R}\}.$$

Thus, with $R_3(\gamma) = \exp(\gamma J_3)$, $\gamma \in \mathbb{R}$, we have

$$H_\omega = \{R_3(\gamma) \mid \gamma \in \mathbb{R}\}$$

which is a closed subgroup of G. Hence G/H_ω is a phase space with the left invariant measure $J_1^* \wedge J_2^*$ or in other notation $d\Omega(J_1, J_2)$ where $d\Omega$ is the invariant measure on the sphere. Now $h \in H_\omega$ has the form

$$h = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the standard basis. Thus, for an element $g \circ H_\omega \in G/H_\omega$, we may take g to be in the form

$$\begin{aligned} g &= \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \circ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & y_{13} \\ v_2 & y_{13} \\ v_3 & y_{13} \end{bmatrix} \circ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} v_1 \cdot (\cos\theta, \sin\theta) & v_1 \cdot (-\sin\theta, \cos\theta) & y_{13} \\ v_2 \cdot (\cos\theta, \sin\theta) & v_2 \cdot (-\sin\theta, \cos\theta) & y_{23} \\ v_3 \cdot (\cos\theta, \sin\theta) & v_3 \cdot (-\sin\theta, \cos\theta) & y_{33} \end{bmatrix}, \end{aligned}$$

where $v_j = (y_{j1}, y_{j2})$. We may choose θ such that $v_1 \cdot (-\sin\theta, \cos\theta) = 0$ and $v_1 \cdot (\cos\theta, \sin\theta) = \| v_1 \| \geq 0$. We now treat the case ">0." Then the form is

$$g = \begin{bmatrix} \alpha & 0 & y_{13} \\ x_{21} & x_{22} & y_{13} \\ x_{31} & x_{32} & y_{13} \end{bmatrix}.$$

Using the properties of $SO(3)$, we may reduce this, after four pages of work orthogonalizing the rows and columns and setting the determinant equal to zero, to

$$g = \begin{bmatrix} \sqrt{1-(y_{13})^2} & 0 & y_{13} \\ \frac{-y_{13}y_{23}}{\sqrt{1-(y_{13})^2}} & \frac{\sqrt{1-(y_{13})^2-(y_{23})^2}}{\sqrt{1-(y_{13})^2}} & y_{23} \\ \frac{-y_{13}\sqrt{1-(y_{13})^2-(y_{23})^2}}{\sqrt{1-(y_{13})^2}} & \frac{-y_{23}}{\sqrt{1-(y_{13})^2}} & \sqrt{1-(y_{13})^2-(y_{23})^2} \end{bmatrix}.$$

From here, we may take the partial derivatives at $(y_{13}, y_{23}) = (0,0)$ to obtain

$$J_1 = \frac{\partial}{\partial y_{13}} g|_{(y_{13},y_{23})=(0,0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

and

$$J_2 = \frac{\partial}{\partial y_{23}} g|_{(y_{13},y_{23})=(0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then using the commutator, we obtain

$$J_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in $\mathfrak{h}_\omega$. Consequently, we may label the points of G/H_ω by

$$[[(y_{13}, y_{23})]] = g \circ H_\omega$$

with g as above and then we have $[J_1, J_2] = J_3 \in [[(0,0)]] = \{\alpha J_3 | \alpha \in \mathbb{R}\} = \mathfrak{h}_\omega$ in G/H_ω. Thus we have (40) of Part 1 satisfied by "$[J_1, J_2] = 0$" at the origin in G.

Finally, we have to treat the case $\| v_1 \| = 0$, but then we must have $y_{13} = \pm 1$; so by the normalization of the third column, $y_{23} = y_{33} = 0$. The problem then reduces to a triviality.

Next consider the right action of $k \in G$ on $g \circ H_\omega$:

$$\begin{aligned} k * (g \circ H_\omega) &= g \circ H_\omega \circ k^{-1} \\ &= (g \circ k^{-1}) \circ (k \circ H_\omega \circ k^{-1}). \end{aligned}$$

Now $k \circ H_\omega \circ k^{-1}$ is another closed subgroup of G generated from

$$\omega' = r_k^* \omega;$$

i.e., from $\omega'(X,Y) = r_k^*\omega(X,Y) = \omega((r_k)_*X, (r_k)_*Y)$. This subgroup is isomorphic to H_ω. On $k \circ H_\omega \circ k^{-1}$, we may perform a similar operation to that which we have just performed on H_ω, and obtain forms for the matrices based on the vectors in $\mathbb{R}^3$ that are $r_k e_j$ rather than the e_j, with $< r_k e_j, r_k e_l > = \delta_{jl}$. We obtain

$$[(r_k)_*J_1, (r_k)_*J_2] = (r_k)_*J_3 \in [[0]]_k = \{\alpha (r_k)_*J_3 \mid \alpha \in \mathbb{R}\} = \mathfrak{h}_{(r_k)_*\omega} \text{ in } G/H_{r_k^*\omega'}.$$

In summary, we have obtained the condition "$[(r_k)_*J_1, (r_k)_*J_2] = 0$" on each $G/H_{r_k^*\omega}$ for $k \in G$. We may paste these together to get $\mathfrak{g}, T(\mathfrak{g})$, and $T(\mathfrak{g}^*)$. We also have canonical (i.e., commuting) coordinates in the $(r_k)_*J_1$ and $(r_k)_*J_2$ defined on each $k \circ H_\omega \circ k^{-1}$ in the above way.

2.2.2 Case 2

Alternatively, treating the Lie algebra, $\mathfrak{g}$, with a basis of generators in self-adjoint form J_1, J_2, J_3 and $[J_1, J_2] = iJ_3$, then a rotation of θ degrees about the kth axis is given by

$$R_k(\theta) = \exp(-i\theta J_k), \ \theta \in \mathbb{R}, \tag{12}$$

and a general element g of G is of the form

$$g = \exp(-ir \cdot J), \ r \in \mathbb{R}^3 \tag{13}$$

where

$$r \cdot J = r_1 J_1 + r_2 J_2 + r_3 J_3. \tag{14}$$

We list several crucial facts about the rotations:

Fact 1: The operator

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

commutes with all the J_k individually. Therefore, J^2 is a constant in any irreducible representation, π, of the rotation group. We will drop the notation π. Consequently, in any irreducible representation space, there are only two of the J_ks which are independent. We will choose to focus on J_2 and J_3. Furthermore, we may take J_3 to have (at least) one eigenvector in this representation:

$$J_3\psi = m\psi, \text{ some } \psi \neq 0. \tag{15}$$

Thus (40') of Part 1 is satisfied.

Fact 2: The group is compact, which has as a consequence that every irreducible representation is finite dimensional and any representation is a union of irreducible representations.

Fact 3: By considering

$$J_\pm = J_1 \pm iJ_2,$$

one finds algebraically that the values of J^2 equal $\{j(j+1) \mid j \in \{0\} \cup \mathbb{N}/2\}$, and $m \in \{-j, \cdots, j\}$. Explicitely:

$$[J_3, J_+] = J_+,\ [J_3, J_-] = J_-,\ [J_+, J_-] = 2J_3, \tag{16}$$

$$J^2 = J_3^2 - J_3 + J_+J_- = J_3^2 + J_3 + J_-J_+, \tag{17}$$

and

$$J_\pm^\dagger = J_\mp.$$

Now

$$\begin{aligned} J_3J_+\psi &= J_+J_3\psi + [J_3, J_+]\psi \\ &= mJ_+\psi + J_+\psi \\ &= (m+1)J_+\psi. \end{aligned}$$

Hence, if $J_+\psi \neq 0$, then $m+1$ is also an eigenvalue of J_3. If so, then $J_3J_+^2\psi = (m+2)J_+^2\psi$, etc. Because the representation is finite dimensional, this must stop somewhere, say when $J_3J_+^s\psi = (m+s)J_+^s\psi$, $J_+^s\psi \neq 0$, and $J_+^{s+1}\psi = 0$. Let $m+s \equiv j$, $J_+^s\psi \equiv \varphi \neq 0$. Then $J_3\varphi = j\varphi$. Consider the sequence $\varphi, J_-\varphi, \cdots$. By similar reasoning, you obtain $J_3J_-^s\varphi = (j-s)J_-^s\varphi$, which must also stop somewhere, say when $s = N$, $J_-^N\varphi \neq 0$, $J_-^{N+1}\varphi = 0$. Now

$$J^2\varphi = (J_3^2 + J_3 + J_-J_+)\varphi = (j^2 + j)\varphi.$$

Consequently, $J^2 = (j^2 + j)\mathbf{1}$ in this representation. Hence,

$$\begin{aligned} 0 &= J_+J_-J_-^N\varphi = (J^2 - J_3^2 + J_3)J_-^N\varphi \\ &= [(j^2 + j) - \{(j-N)^2 - (j-N)\}]J_-^N\varphi \\ &= N[2j - N + 1]J_-^N\varphi, \end{aligned}$$

or $2j = N - 1$, $N \geq 1$.

The other alternative is $J_+\psi = 0$, $J_-\psi = 0$, and $J_3\psi = 0$. Hence, the representation is $2j+1$ dimensional, $j \in \{0, 1/2, 1, 3/2, \cdots\}$ and there is a basis $\{\phi_{j,m}\}$ such that

$$\begin{aligned} J^2\phi_{j,m} &= j(j+1)\phi_{j,m}, \\ J_3\phi_{j,m} &= m\phi_{j,m}, \\ J_\pm\phi_{j,m} &= [j(j+1) - m(m+1)]^{1/2}\phi_{j,m\pm1}. \end{aligned}$$

We also have

$$< \phi_{j,m'}, \phi_{j,m} > \ = 0,\ m' \neq m,$$

as $\phi_{j,m'}, \phi_{j,m}$ are eigenvectors of J_3 belonging to different eigenvalues.

Fact 4: Every rotation may be written in terms of the Euler angles as

$$\exp(-ir \cdot J) = R_3(\alpha)R_2(\beta)R_3(\gamma)$$

by rotating $\exp(-ir \cdot J)$ through the "line of nodes". [7, pp. 107-108]

Fact 5: In the jth irreducible space, the group of rotations acts as

$$R_3(\alpha)R_2(\beta)R_3(\gamma)\phi_{j,m} = \sum_{m'} \left[D^j(\alpha,\beta,\gamma)\right]_m^{m'} \phi_{j,m'}$$

where

$$\left[D^j(\alpha,\beta,\gamma)\right]_m^{m'} = \exp(-i\alpha m') \left[d^j(\beta)\right]_m^{m'} \exp(-i\gamma m),$$

and

$$\left[d^j(\beta)\right]_m^{m'} = \ <\phi_{j,m'}, \exp(-i\beta J_2)\phi_{j,m}> .$$

Thus, the space G/H_ω decomposes into subspaces of various dimensions.

Example 0: The spherically symmetric case: $j = 0$. $J_1 = J_2 = J_3 = 0$.

Example 1/2: The "spin 1/2" case: For $j = 1/2$, we obtain $J_k = \frac{1}{2}\sigma_k$, σ_k the Pauli spin matrices:

$$\sigma_k^2 = 1,\ \sigma_1\sigma_2 = i\sigma_3 \text{ and cyclically, } \sigma_k^\dagger = \sigma_k. \tag{18}$$

Thus

$$\exp(-ir \cdot J) = \exp(-i(r/2)\cdot\sigma) = \cos(|r|/2)1 - i\sin(|r|/2)\widehat{r}\cdot\sigma.$$

In particular, for the standard basis

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{19}$$

we have

$$\exp(-[r/2]\sigma_2) = \begin{pmatrix} \cos[r/2] & -\sin[r/2] \\ \sin[r/2] & \cos[r/2] \end{pmatrix}.$$

We next use these five facts to obtain the various symplectic manifolds for the rotation group. Now

$$G/H_\omega = \{R_3(\alpha) \circ R_2(\beta) \circ H_\omega \equiv [[\alpha,\beta]] \mid \alpha,\beta \in \mathbb{R}\}.$$

Hence, there are (again) just two canonical variables for the rotations.

As is well known, we have that the representation on the left will give the commutation relations on $\mathfrak{g}$. However, the representation on the right with

$$[[\alpha,\beta]] \equiv R_3(-\alpha) \circ R_2(-\beta) \circ H_\omega, \tag{20}$$

will give

$$\begin{aligned}
& [R_3(\alpha') \circ R_2(\beta') \circ R_3(\gamma')] * [[\alpha,\beta]] \\
= \ & [R_3(\alpha') \circ R_2(\beta') \circ R_3(\gamma')] * \{R_3(-\alpha) \circ R_2(-\beta) \circ H_\omega\} \\
= \ & [R_3(\alpha') \circ R_2(\beta') \circ R_3(\gamma')] * \{R_3(-\alpha) \circ R_2(-\beta) \circ R_3(\gamma') \circ H_\omega\} \\
= \ & R_3(-\alpha) \circ R_2(-\beta) \circ R_3(\gamma') \circ R_3(-\gamma') \circ R_2(-\beta') \circ R_3(-\alpha') \circ H_{\omega'} \\
= \ & R_3(-\alpha) \circ R_2(-[\beta+\beta']) \circ H_{\omega'} \\
= \ & R_{3'}(f) \circ R_{2'}(g) \circ R_{3'}(h) \circ H_{\omega'} \\
= \ & R_{3'}(f) \circ R_{2'}(g) \circ H_{\omega'},
\end{aligned}$$

where f, g, and h are complicated functions of α, $\beta+\beta'$ and $k=$ a function of α', β' and γ', and $\omega' = (r_{(\alpha',\beta',\gamma')})^*\omega$. It is not easy to see, but valid, that the commutation relations all vanish for the canonical variables α and β as you have for the $\frac{\partial}{\partial x_k}$s.

Before we leave this example as well as the one before, we should make the following general remark:

The representations from the left obey the commutation relations of $\mathfrak{g}$, while the commutation relations from the right for the canonical variables all effectively vanish on $G/H_{r_g^*\omega}$. This is the difference between having a "quantum representation" versus a "classical representation."

2.3 Example 3: Galilei Motions

From the group

$$G = \{(t, x, v, R) \mid t \in \mathbb{R}, x \text{ and } v \in \mathbb{R}^3, R \in SO(3))\}$$

where t represents time shift, x represents translations in position, v represents boosts in velocity, and R represents rotation, with multiplication

$$\begin{aligned} & (t', x', v', R')(t, x, v, R) \\ = \; & (t'+t, x'+R'x+tv', v'+R'v, R'R) \end{aligned}$$

one derives the Lie algebra $\mathfrak{g}$ with basis $\{\tau, k_j, u_j, M_j \mid j \in \{1,2,3\}\}$ and

$$\begin{aligned} [M_i, M_j] &= \sum_k e_{ijk} M_k, \; [M_i, u_j] = \sum_k e_{ijk} u_k, \\ [M_i, k_j] &= \sum_k e_{ijk} k_k, \; [u_i, \tau] = k_i, \end{aligned}$$

and the other commutators are all zero, where τ is the generator of time shifts, $k=(k_1,k_2,k_3)$ are the generators of translations, $u=(u_1,u_2,u_3)$ are the generators of boosts, and $M=(M_1,M_2,M_3)$ are the generators of rotations. We shall use the alternate notation $k=(k^x,k^y,k^z)$, etc. in Section 5.1. From this, the computation of $Z^2(\mathfrak{g})$ follows easily. [6] One element of $Z^2(\mathfrak{g})$ is

$$\omega = m \sum_i u_i^* \wedge k_i^* + M_1^* \wedge M_2^*$$

from which we obtain

$$\mathfrak{h}_\omega = \{aM_3 + b\tau \mid a, b \in \mathbb{R}\}.$$

From this,

$$H_\omega = \{(t, 0, 0, R_3) \mid t \in \mathbb{R} \text{ and } R_3 \text{ is any rotation about the 3-axis}\}.$$

We obtain that G/H_ω is topologically isomorphic to $(\mathbb{R}^3 \oplus \mathbb{R}^3) \times S^2$. We may take any representation π of G on G/H_ω in which $\pi(u_3) = \lambda(u_3)1$ and $\pi(k_3) = \lambda(k_3)1$, where $\lambda(u_3)$ and $\lambda(k_3) \in \mathbb{R}$. Then $\pi\{M_1, M_2, u_1, u_2, (u_3), k_1, k_2, (k_3)\}$ are canonical variables. We have placed u_3 and k_3 in parenthesis as they are, strictly speaking, not canonical variables since they are not variable at all.

2.4 Example 4: Lorentz Motions

We will treat just the case of the Lorentz group for spinning particles. Other cases are treated similarly. We will also treat only the case of time running forward. We will also use the Einstein convention that when a letter is used twice to denote a variable from {0,1,2,3) we mean that we sum over this variable.

The (homogeneous) Lorentz group is the group $SO(1,3)$. (See [8, pp.38-44].) Here, we take $x = \{x_\mu\} \in \mathbb{R}^4$ in which $\mathbb{R}^4$ is equipped with the metric $\{g^\mu_\nu\} = diag(1,-1,-1,-1)$; i.e., with $x^\mu x_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Then $A \in SO(1,3)$ has the matrix form $\{A^\mu_\nu \mid (x')^\mu = A^\mu_\nu x^\nu\}$ with the A^μ_ν real and

$$A^\mu_\nu g_{\mu\rho} A^\rho_\sigma = g_{\nu\sigma}. \tag{21}$$

(We take $A^0_0 \geq 1$ for the time running forward.) Hence, every A has the form

$$A = R_2 L_1 R_1,$$

where R_2 and R_1 are rotations of the form $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ where R is a rotation matrix, and L_1 is a pure Lorentz transformation (boost) in the $0-1$ plane:

$$L_1(u) = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Call the generators of the Lorentz group in matrix form $\{M^\mu_\nu\}$. Accordingly, we have

$$A = \exp\{\alpha^\nu_\mu M^\mu_\nu\},\ \alpha^{\mu\nu} = -\alpha^{\nu\mu}$$

where

$$M^\mu_\nu = \frac{d}{d\alpha^\nu_\mu} A|_{A=id},\ M_{\mu\nu} = -M_{\nu\mu}.$$

(The condition $M_{\mu\nu} = -M_{\nu\mu}$ is to obtain equation (21).) Then we have

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} - g_{\mu\sigma} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma}.$$

We may extend these relations to a general representation.

For example, in the representation $\mathbb{R}^4$ we have

$$M^0_1 = M_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We may take

$$\boldsymbol{M} = (M_{32}, M_{13}, M_{21}) \equiv (M_1, M_2, M_3),$$

and

$$\boldsymbol{N} = (M_{01}, M_{02}, M_{03}) = (N_1, N_2, N_3)$$

to obtain

$$\begin{aligned} [M_j, M_k] &= \varepsilon_{jkl} M_l, \\ [N_j, N_k] &= -\varepsilon_{jkl} M_l, \\ [M_j, N_k] &= \varepsilon_{jkl} N_l. \end{aligned}$$

The $\varepsilon_{j,k,l} = 1$ if (j, k, l) is a cyclic permutation of (1,2,3); = -1 if it is a cyclic permutation of (1,3,2); and = 0 otherwise. Likewise for ε_{jklm}.

We next define the self-adjoint form of the rotation operators by

$$M_k = -iJ_k$$

to obtain

$$[J_j, J_k] = i\varepsilon_{jkl} J_l$$

in accordance with the commutation relations for the rotation group. With these conventions, we define

$$S_k = \frac{1}{2}(J_k + N_k) = \frac{1}{2}(iM_k + N_k),\ T_k = \frac{1}{2}(J_k - N_k) = \frac{1}{2}(iM_k - N_k) \tag{22}$$

to obtain

$$\begin{aligned} [S_j, S_k] &= i\varepsilon_{jkl} S_l,\ [T_j, T_k] = i\varepsilon_{jkl} T_l, \\ [S_j, T_k] &= 0 \end{aligned} \tag{23}$$

from which all the irreducible representations of the Lorentz group may be found just as we found the irreducible representations of the rotation group, but now there are two sets of parameters like that of the rotations. The operators $\frac{1}{2} M_{\mu\nu} M^{\mu\nu} = \boldsymbol{M}^2 - \boldsymbol{N}^2$ and $\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} = -\boldsymbol{M} \cdot \boldsymbol{N}$ commute with all the M_j and N_k, and hence are invariants of the Lorentz group. They are multiples of the identity in any irreducible representation.

We obtain that $Z^2(\mathfrak{g}^*) = B^2(\mathfrak{g}^*)$; so that every $\omega \in Z^2(\mathfrak{g}^*)$ is of the form $\omega = \delta\Theta$, $\Theta \in \wedge^1(\mathfrak{g}^*)$; i.e.,

$$\omega(X, Y) = -\Theta([X, Y]),\ X, Y \in \mathfrak{g}.$$

We may take, for example,

$$\omega_1 = -\delta N_3^* - \delta J_3^* = -J_1^* \wedge N_2^* + J_2^* \wedge N_1^* - J_1^* \wedge J_2^*$$

and note that (22) of Part 1 is again satisfied. We obtain

$$\begin{aligned} \mathfrak{h}_{\omega_1} &= \{\alpha M_{03} + \beta M_{12} \mid \alpha, \beta \in \mathbb{R}\} \\ &= \{\alpha N_3 + i\beta J_3 \mid \alpha, \beta \in \mathbb{R}\}, \end{aligned}$$

and

$$H_{\omega_1} = \{\exp\{\alpha N_3 + i\beta J_3\} \mid \alpha, \beta \in \mathbb{R}\},$$

which is a closed subgroup of G, and so then G/H_{ω_1} is what it is. We obtain from ω_1 that

$$\begin{aligned} d\mu &= dM_{01}dM_{02}dM_{23}dM_{13} \\ &= dN_1dN_2dM_1dM_2 \\ &= d\mathcal{S}_1d\mathcal{S}_2d\mathcal{T}_1d\mathcal{T}_2, \end{aligned}$$

(up to a sign) in whatever coordinates you wish to work. This choice of G/H_{ω_1} makes G/H_{ω_1} behave like the vectors in the representation $\mathcal{S} \times \mathcal{T}$ where each of the vectors in $\mathcal{S}$ resp. $\mathcal{T}$ are as in the case of the symplectic space for the rotation group.

Physically, the choice of ω_1 corresponds to what we have in the Stern-Gerlach experiment in which the classical magnetic moment when averaged over the precession, or the spin, is only in the direction of the three axis and is independent of any boost along the three axis.

Alternatively we may take

$$\omega_2 = -\delta J_3 = -J_1^* \wedge J_2^*$$

from which we obtain equation (22) of Part 1 satisfied,

$$\begin{aligned} \mathfrak{h}_{\omega_2} &= \{\alpha M_{12} \mid \alpha \in \mathbb{R}\} \\ &= \{i\alpha J_3 \mid \alpha \in \mathbb{R}\}, \end{aligned}$$

and

$$H_{\omega_2} = \{\exp\{i\alpha J_3\} \mid \alpha \in \mathbb{R}\},$$

which is a closed subgroup of G, and from which we obtain G/H_ω with $d\mu = d\omega_2$.

Warning: In analogy with fact 4 of the rotation group,

$$G/H_\omega = \{g \circ H_\omega \mid g \in G\},$$

and we have an arbitrary $g \in G$ of the form

$$g = R_1A_3R_3,$$

where the R_j are rotations with axis in the jth direction of some angles and A_3 is a pure Lorentz transformation in the x_3 direction. (This holds for any $\omega \in Z^2(\mathfrak{g}^*)$.) This seems to say that

$$G/H_\omega = \{R_1H_\omega \mid R_1 \text{ a rotation in the 1 direction}\},$$

which is one dimensional. But then when we act on this with group elements from the right, once again we will obtain a parameterization of more dimensions, in this case a four dimensional space.

2.5 Example 5: Poincaré Motions

The Poincaré group (the inhomogeneous Lorentz group), $\mathcal{P}$, is taken as

$$\mathcal{P} = \mathbb{R}^4 \ltimes SO(1,3),$$

in which $\mathbb{R}^4$ is equipped with the metric $diag(1,-1,-1,-1)$. It is viewed as the group of translations on the four dimensional affine space $\mathbb{R}^4$ in semidirect product with $SO(1,3)$. Elements of the space $\mathbb{R}^4$ will be denoted $\{q_\lambda\}$. $\mathcal{P}$ has the basis for its Lie algebra, $\mathfrak{p}$, the elements $\{P_0, P_j, N_j = M_{0j}, M_j = \varepsilon_{jkl}M_{kl} \mid j = 1,2,3\}$, where for all $\mu,\nu,\rho,\sigma \in \{0,1,2,3\}$

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, P_\rho] &= g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\mu} - g_{\nu\rho}M_{\mu\sigma};
\end{aligned}$$

i.e., for all $j,k,l \in \{1,2,3\}$,

$$\begin{aligned}
[P_0, P_0] &= 0;\ [P_k, P_0] = 0;\ [N_k, P_0] = -P_k;\ [M_k, P_0] = 0; \\
[P_j, P_k] &= 0;\ [N_j, P_k] = \delta_{j,k}P_0;\ [M_j, P_k] = -\varepsilon_{j,k,l}P_l; \\
[N_j, N_k] &= -\varepsilon_{j,k,l}M_l;\ [M_j, N_k] = \varepsilon_{j,k,l}N_l;\ [M_j, M_k] = \varepsilon_{j,k,l}M_l. \quad (24)
\end{aligned}$$

(The $\{M_{\mu\nu}\}$ are the generators of the Lorentz group.) We interpret the operator P_0 as the generator of time translations, the operations P_k as the generators of space translations (in q_k), the operations N_k as the generator of boosts in the q_k direction, and the operations M_k as the generator of rotations about the direction q_k. We will treat only the case of the time going forward, the mass being positive, and spin also positive.

Just as in the case of the pure Lorentz group, we may write

$$S_k = \frac{1}{2}(iM_k + N_k) \text{ and } T_k = \frac{1}{2}(iM_k - N_k). \quad (25)$$

Then we obtain the relations

$$\begin{aligned}
[P_0, P_0] &= 0;\ [P_k, P_0] = 0;\ [P_j, P_k] = 0; \\
[S_k, P_0] &= -\frac{1}{2}P_k = -[T_k, P_0]; \\
[S_j, P_k] &= -\frac{i}{2}\varepsilon_{j,k,l}P_l = [T_j, P_k] \text{ for } j \neq k; \\
[S_k, P_k] &= \frac{1}{2}P_0 = -[T_k, P_k]; \\
[S_j, T_k] &= 0; \\
[S_j, S_k] &= i\varepsilon_{j,k,l}S_l;\ [T_j, T_k] = i\varepsilon_{j,k,l}T_l. \quad (26)
\end{aligned}$$

Now from [6, pp. 445-447], we have $Z^2(\mathfrak{p}^*) = B^2(\mathfrak{p}^*)$; so that every $\omega \in Z^2(\mathfrak{p}^*)$ is of the form $\omega = \delta\Theta$, $\Theta \in \wedge^1(\mathfrak{p}^*)$; i.e.,

$$\omega(X,Y) = -\Theta([X,Y]),\ X,Y \in \mathfrak{p}. \quad (27)$$

Then from the relations (24), we derive

$$
\begin{aligned}
\delta P_0^* &= N_1^* \wedge P_1^* + N_2^* \wedge P_2^* + N_3^* \wedge P_3^*, \\
\delta P_1^* &= -N_1^* \wedge P_0^* - M_2^* \wedge P_3^* + M_3^* \wedge P_2^* - N_2^* \wedge N_3^*, \\
\delta P_2^* &= -N_2^* \wedge P_0^* - M_3^* \wedge P_1^* + M_1^* \wedge P_3^* - N_3^* \wedge N_1^*, \\
\delta P_3^* &= -N_3^* \wedge P_0^* - M_1^* \wedge P_2^* + M_2^* \wedge P_1^* - N_1^* \wedge N_2^*, \\
\delta N_1^* &= M_2^* \wedge N_3^* - M_3^* \wedge N_2^*, \\
\delta N_2^* &= M_3^* \wedge N_1^* - M_1^* \wedge N_3^*, \\
\delta N_3^* &= M_1^* \wedge N_2^* - M_2^* \wedge N_1^*, \\
\delta M_1^* &= M_2^* \wedge M_3^*, \\
\delta M_2^* &= M_3^* \wedge M_1^*, \\
\delta M_3^* &= M_1^* \wedge M_2^*.
\end{aligned}
$$

From this we choose the infinity of elements of $Z^2(\mathfrak{p}^*)$.

2.5.1 Example 1:

We choose

$$\omega_1 = -\delta M_3^* = -M_1^* \wedge M_2^*,$$

obtaining that (22) of Part 1 is satisfied, $\mathfrak{h}_{\omega_1} = < P_0, P_1, P_2, P_3, N_1, N_2, N_3, M_3 >$ and obtaining G/H_{ω_1} parameterized by M_1, M_2 with $d\mu = d\omega_1$. Here $< X, \cdots >$ denotes the sub-Lie algebra with a basis of generators given by the Xs. Here M_1, M_2 are the canonical coordinates, just as in the rotation or Lorentz groups.

2.5.2 Example 2:

We choose

$$\omega_2 = -\delta N_3^* - \delta M_3^* = -M_1^* \wedge N_2^* + M_2^* \wedge N_1^* - M_1^* \wedge M_2^*,$$

obtaining that (22) of Part 1 is satisfied, $\mathfrak{h}_{\omega_2} = < P_0, P_1, P_2, P_3, N_3, M_3 >$ and obtaining G/H_{ω_2} parameterized by N_1, N_2, M_1, M_2 or alternatively by S_1, T_1, S_2, T_2 with $d\mu = d\omega_2 \wedge d\omega_2 = \pm N_1^* \wedge N_2^* \wedge M_1^* \wedge M_2^* = \pm S_1^* \wedge S_2^* \wedge T_1^* \wedge T_2^*$. Here the canonical coordinates are the S_1, T_1, S_2, T_2 as in the treatment of the Lorentz group.

2.5.3 Example 3:

We may choose

$$
\begin{aligned}
\omega_3 &= \delta(mP_0^* + s_3(-i)[S_3^* + T_3^*]) = \delta(mP_0^* + s_3 M_3^*) \\
&= m(N_1^* \wedge P_1^* + N_2^* \wedge P_2^* + N_3^* \wedge P_3^*) + s_3(M_1^* \wedge M_2^*).
\end{aligned}
$$

This corresponds to a particle with non-zero mass m and a non-zero spin S with $S^2 = -s_3^2$. We have for $p = (m, 0, 0, 0)$ and $s = (0, 0, 0, s_3)$, then $p_\lambda p^\lambda = m^2$, $s_\lambda s^\lambda = -s_3^2 = S^2$, and $p_\lambda s^\lambda = 0$.

A physical justification of ω_3 should be made here: The equation $p_\lambda s^\lambda = 0$ is precisely the condition of the momentum being perpendicular to the orbit of the classical magnetic moment when averaged over the precession as in the classical interpretation of the Stern-Gerlach experiment. [It also is consistent with (the mass zero case with arbitrary helicity of the particles and in particular) the photon which has the momentum perpendicular to the direction of the $E-$ and $B-$fields.]

We again have (22) of Part 1 satisfied, and we compute

$$\mathfrak{h}_{\omega_3} = \{\alpha P_0 + \beta M_3 \mid \alpha, \beta \in \mathbb{R}\}$$

to obtain

$$H_{\omega_3} = \{\exp(\alpha P_0 + \beta M_3) \mid \alpha, \beta \in \mathbb{R}\}.$$

H_{ω_3} is again a closed subspace of $\mathcal{P}$. Also

$$\mu = -N_1^* \wedge P_1^* \wedge N_2^* \wedge P_2^* \wedge N_3^* \wedge P_3^* \wedge M_1^* \wedge M_2^*$$

is an invariant measure of the space $\mathcal{P}/H_{\omega_3}$. Hence, $(\mathcal{P}/H_{\omega_3}, \mu)$ is a phase space of dimension 8. In other notation,

$$\mathcal{P}/H_{\omega_3} = \{(p,q,s) \mid p_\lambda p^\lambda = m^2,\ s_\lambda s^\lambda = -s_3^2 = S^2,\ \text{and } p_\lambda s^\lambda = 0\} \tag{28}$$

and

$$\mu = d^3p d^3q d\Omega(s), \tag{29}$$

where $d\Omega(s)$ is the invariant measure on the sphere.

We add here a commentary that we may also take $\omega = m(\delta P_0^* + \delta P_3^*) + s_3 \delta M_3^* = m\delta(P_0^* + P_3^*) + s_3 \delta M_3^*$ and find that $\mathfrak{h}_\omega = \{\alpha(P_0+P_3)+\beta M_3 \mid \alpha, \beta \in \mathbb{R}\}$ and $\mu = \pm N_1^* \wedge P_1^* \wedge N_2^* \wedge P_2^* \wedge N_3^* \wedge (P_0^* - P_3^*) \wedge M_1^* \wedge M_2^*$. The difficulty or ease is the same as for ω_3. But $\omega = m\delta(P_0^* + P_j^*) + s_3 \delta M_3$ is much different for $j = 1, 2$.

One may check for canonical variables corresponding to linear combinations of the $\{P_0, P_k, N_k, M_k\}$ which have some of the P_μs in them, some of the N_js and M_js in them, and that have Lie brackets "equal to zero" in the sense that they are in $\pi(\mathfrak{h}_{\omega_3}) + \mathbb{R}1$. That is, we search for canonical variables on $\mathcal{P}/H_{\omega_3}$. For example, we may start with M_1 and M_2, which have Lie bracket in $\mathfrak{h}_{\omega_3}$, and then expand that to a basis of canonical variables on $\mathcal{P}/H_{\omega_3}$. Then we may write any Hamiltonian in terms of the canonical variables with an interaction in terms of the M_1 and M_2.

Alternatively, we may start with ω_3 and derive $\mathcal{P}/H_{\omega_3}$. The "Lie algebra" for $\mathcal{P}/H_{\omega_3}$, $\mathfrak{p}/\mathfrak{h}_{\omega_3}$, has a (local) basis that is orthogonal at the identity of $\mathcal{P}$. We will take this basis as being orthogonal in the sense of having commutation relations satisfying

$$[\pi(\mathfrak{p} - \mathfrak{h}_{\omega_3}), \pi(\mathfrak{p} - \mathfrak{h}_{\omega_3})] \subseteq \pi(\mathfrak{h}_{\omega_3}) + \mathbb{R}1.$$

(This guarantees that we will have canonical variables on $\mathcal{P}/H_{\omega_3}$.) We may take this basis and translate it to the point $g \in \mathcal{P}$, and then it will be a basis for the "Lie algebra" for $\mathcal{P}/H_{r_g^*\omega_3} = g \circ (\mathcal{P}/H_{\omega_3}) \circ g^{-1}$ with two-form $r_g^*\omega_3$. Next, consider $x \in \mathcal{P}/H_{\omega_3}$. We may write this as $x = k \circ H_{\omega_3}$ for some $k \in \mathcal{P}$. Then we have, for $g \in P$,

$$\begin{aligned} x &= k \circ H_{\omega_3} \\ &\mapsto (k \circ H_{\omega_3}) \circ g^{-1} \\ &= (k \circ g^{-1}) \circ (g \circ H_{\omega_3} \circ g^{-1}) \\ &= (k \circ g^{-1}) \circ H_{g^*\omega_3} \\ &= (k \circ g^{-1}) \circ H_{r_g^*\omega_3} \end{aligned}$$

which is in $g \circ (\mathcal{P}/H_{\omega_3}) \circ g^{-1} = \mathcal{P}/H_{r_g^*\omega_3}$. We may also check that this is independent of which k we choose in kH_{ω_3} to obtain x. In this way, we again illustrate the results in Section 4 of Part 1.

2.5.4 Example 4:

We may choose, for $m, s_3 = s$, and $b \in \mathbb{R}$, where m and $s_3 \neq 0$,

$$\begin{aligned} \omega_4 &= m\delta P_0^* + m\delta P_3^* + s\delta M_3^* + b\delta N_3^* \\ &= m(N_1^* \wedge P_1^* + N_2^* \wedge P_2^* + N_3^* \wedge P_3^* - N_3^* \wedge P_0^* \\ &\quad + M_1^* \wedge P_2^* + M_2^* \wedge P_1^* - N_1^* \wedge N_2^*) \\ &\quad + sM_1^* \wedge M_2^* + b(M_1^* \wedge N_2^* - M_2^* \wedge N_1^*). \end{aligned}$$

This is not in the form of equation (22) of Part 1 but may be placed in that form by the induction procedure in the proof. We obtain

$$\begin{aligned} \omega_4 &= \frac{1}{s}(sM_1^* + bN_1^* - mP_1^*) \wedge (sM_2^* + bN_2^* + mP_2^*) \\ &\quad + \frac{s}{ms+b^2}(\frac{ms+b^2}{s}N_1^* - \frac{mb}{s}P_1^* + mP_2^*) \wedge \\ &\quad (-\frac{ms+b^2}{s}N_2^* + mP_1^* - \frac{mb}{s}P_2^*) \\ &\quad + \frac{m^2(m-s)}{ms+b^2}P_1^* \wedge P_2^* + mN_3^* \wedge (P_3^* - P_0^*). \end{aligned}$$

From this, it is easy to count the dimesion of the space $\mathfrak{h}_{\omega_4} = 2$. We obtain

$$\mathfrak{h}_{\omega_4} = \{\alpha M_3 + \beta(P_3 + P_0) \mid \alpha, \beta \in \mathbb{R}\}.$$

We will leave the problem of obtaining the canonical coordinates to the reader.

Note: We may make a transition from $\mathbb{R}^4$ to the 2×2 complex matrices via the Cayley-Klein representation (or spinor representation). Then $SO(1,3)$ becomes $SL(2,\mathbb{C})$, referred to as the double cover of $SO(1,3)$. Therefore, we may write

$$\mathcal{P} = \mathbb{R}^4 \ltimes SL(2,\mathbb{C}).$$

We shall not delve further into this in this paper, as it gives us just another example for this general procedure.

2.6 Summary from the Examples:

We take the Lie group G and derive the Lie algebra $\mathfrak{g}$ from it. Then we form $Z^2(\mathfrak{g}^*)$. In many, but not all cases, we have $Z^2(\mathfrak{g}^*) = B^2(\mathfrak{g}^*)$; so, we have an easier time choosing an arbitrary $\omega \in Z^2(\mathfrak{g}^*)$. Putting ω in the canonical form allows one to easily count the dimension of $\mathfrak{h}_\omega$, simplifying the derivation of it, and thus obtaining H_ω. Next we check whether or not the H_ω is a closed subgroup of G. If it is, then G/H_ω is a symplectic vector (phase) space. We search for canonical variables on it (if they exist). The commutation relations of the canonical variables on it of relevance to us are of the form $[\pi(\mathfrak{g} - \mathfrak{h}_\omega), \pi(\mathfrak{g} - \mathfrak{h}_\omega)] \subseteq \pi(\mathfrak{h}_\omega) + \mathbb{R}1$, insuring that the canonical variables have vanishing commutation relations on $\pi(G/H_\omega)$. (This last step is not always easy!) If we must view the phase space as a symplectic manifold $\cup'_{g\in G}(g(G/H_\omega)g^{-1}, g^*\omega)$, then we may take an orthogonal basis $\{X_j\}$ for $T(G/H_\omega)$ (presumably corresponding to the canonical variables) and transform it to $\{(r_g)_* X_j\}$ which will be a basis for $T(g(G/H_\omega)g^{-1})$.

3 Connection of $\cup'_{g\in G} G/H_{r_g^*\omega}$ with $\cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu)$ and Its Irreducible Components

We wish to show that there is a map from states on one space to states on the other, and another map from classical functions on the first to operators on the second such that the expectation values of both agree. We will show that by a series of steps.

Now that we have the phase space,

$$\Gamma \equiv \cup'_{g\in G} G/H_{r_g^*\omega}, \tag{30}$$

and have given many examples of it, we may next form the space of complex valued functions that are square integrable on $G/H_{r_g^*\omega}$ with respect to the invariant measure μ. We shall call this $L^2(G/H_{r_g^*\omega}, d\mu)$, and it is equal to the Hilbert space $\mathcal{H}_{Koopman}$ in the case that Koopman considered, namely the Heisenberg case. It is necessarily separable. Then form the disjoint union

$$\begin{aligned} L^2(\Gamma, \nu) &\equiv \cup'_{g\in G} L^2(G/H_{r_g^*\omega}, \mu), \\ < \Phi, \Psi > &= \int d\nu(g) < \Phi_g, \Psi_g >, \end{aligned} \tag{31}$$

where the inner product in the integral is taken in $L^2(G/H_{r_g^*\omega}, \mu)$ and Φ_g, Ψ_g are the representatives of Φ, Ψ in $L^2(G/H_{r_g^*\omega}, \mu)$. $d\nu(g)$ is a Borel measure over the sets $G/H_{r_g^*\omega}$. This is an example of a G-Hilbert bundle, the basic properties of which have been treated in the literature. [9] (There, the disjoint union is symbolized, alternatively, as the direct integral of the Hilbert spaces, and we may take V_g^{-1} as the "basic unitary," V defined by (33) below.) It is also a Hilbert space with all the continuity properties necessary for this discussion.

Because we have the disjoint union, it suffices to specify the action of G on $L^2(G/H_{r_g^*\omega}, d\mu)$ for each $g \in G$. Let V be the representation of G on $L^2(\Gamma, \nu)$, and W the action on $G/H_{r_g^*\omega}$ from the right for classical mechanics and from the left for quantum mechanics. We will ignore the fact that W is properly W_g, defined as a function of the g in $r_g^*\omega$ for the part of x in $G/H_{r_g^*\omega}$. Then the action is built up from $[V^1(g)\Psi](x) = \Psi(W(g^{-1})x)$:

Let h be a cocycle of the group; i.e., for $x \in G/H_{r_g^*\omega}$, we have

$$\sigma : G/H_{r_g^*\omega} \to G$$

as a Borel cross-section of the group. We again ignore in the notation that σ is defined separately for each $g \in G$. Let [6, pp. 302-303] $h : G \times G/H_{r_g^*\omega} \to H_{r_g^*\omega}$, $h(g_1, x) \in H_{r_g^*\omega}$ for all $g \in G$, satisfy

$$g_1 \circ \sigma(x) \circ h(g_1, x) = \sigma(g_1 x);$$

i.e.,

$$h(g_1, x) = \sigma(x)^{-1} \circ g_1^{-1} \circ \sigma(g_1 x).$$

Consequently, $h(g_1, x)$ satisfies

$$h(g_1 \circ g_2, x) = h(g_2, x) \circ h(g_1, g_2 x). \tag{32}$$

Now let

$$\alpha : H_{r_g^*\omega} \to \{z \in \mathbb{C} \mid |z| \ = 1\}$$

be a unitary representation of $H_{r_g^*\omega}$. Then V^α is a unitary representation of G on $L^2(\Gamma, \nu)$ defined by

$$[V^\alpha(g)\Psi](x) = \alpha(h(g^{-1}, x))\Psi(W(g)^{-1} \cdot x) \tag{33}$$

for all $g \in G$, where the action $\cdot$ may be either from the left or from the right as exemplified in (37) and (38) of Part 1. We may prove that V^α is a bona fide representation by using (32). The representation from the right takes you from one of the spaces to another in the phase space $\Gamma = \cup'_{g\in G} G/H_{r_g^*\omega}$. [The representation from the left is well known to give the usual Lie algebra relations, and is the form from which the quantum realizations of G all derive. See [6] for this.]

The factors $\alpha(h(g^{-1}, x))$ give projective representations of G. For $\alpha \equiv 1$ we have the ordinary left-regular or right-regular representations. These projective representations are related to the Casimir invariants in irreducible representations.

Let f be a classical observable on the phase space $\Gamma = \cup'_{g\in G} G/H_{r_g^*\omega}$; i.e., a real, measurable function. This may be done in the following manner: First define the value of a classical function $f(x)$ for $x = g_0 H_\omega \in G/H_\omega$. Then obtain $f(x)$ for $x = g_0 \circ g H_{r_g^*\omega} \in G/H_{r_g^*\omega}$ by defining it as the same value. Thus, f is defined as a class function on the right G-orbits. Next, define an operator, $A(f)$, on each of the spaces $L^2(G/H_{r_g^*\omega}, d\mu)$ by

$$[A(f)\psi](x) = f(x)\psi(x), \ \psi \in L^2(G/H_{r_g^*\omega}, d\mu). \tag{34}$$

In this way, we obtain the "classical observables on the space $\cup'_{g\in G}L^2(G/H_{r_g^*\omega}, d\mu)$" from the classical observables on $\cup'_{g\in G}G/H_{r_g^*\omega}$, (or on G/H_ω). We have that these $A(f)$'s commute pairwise.

In general, we define the states on any separable Hilbert space $\mathcal{H}$, including $\cup'_{g\in G}L^2(G/H_{r_g^*\omega}, d\mu)$, as a trace class positive operator, ρ, of trace one. This is in keeping with the definition of state in quantum mechanics. We next define the expectation value of an operator A on the Hilbert space $\mathcal{H}$ in state ρ as

$$Exp(A, \rho) = Tr(\rho A), \tag{35}$$

presuming that the trace exists. We say that the set of self-adjoint operators $\{A_\alpha \mid \alpha \in I\}$ is "informationally complete" if for ρ and ρ' any states, $Tr(\rho A_\alpha) = Tr(\rho' A_\alpha)$ for all $\alpha \in I$ implies $\rho = \rho'$. A theorem states that every set of commuting operators in a Hilbert space is necessarily informationally incomplete. [10]

In the setting of $\cup'_{g\in G}L^2(G/H_{r_g^*\omega}, d\mu)$, we have that the set $\{A(f)\}$ is informationally incomplete. We define the "classically physically equivalent classes of states" by defining classical physical equivalence of two states, ρ and ρ', by ρ and ρ' are equivalent if $Tr(\rho A(f)) = Tr(\rho' A(f))$ for all (bounded, real-valued) measurable functions f. We write this as $\rho \approx \rho'$. We may be more explicit by first noting a theorem of J. von Neumann [11, p. 296]: We have for any state ρ,

$$\rho = \sum \rho_j P_{\Psi_j}, \tag{36}$$

for some orthonormal basis $\{\Psi_j\}$ of $\mathcal{H}$ where P_{Ψ_j} is the projection onto the vector Ψ_j: $P_{\Psi_j}\Phi = <\Psi_j, \Phi> \Psi_j \ \forall \Phi \in \mathcal{H}$, and the ρ_j are real numbers satisfying $\rho_j \geq 0$, $\Sigma_j \rho_j = 1$. Then we have

$$\begin{aligned} Tr(\rho A(f)) &= \sum \rho_j Tr(P_{\Psi_j} A(f)) \\ &= \sum \rho_j <\Psi_j, A(f)\Psi_j>, \end{aligned}$$

presuming that this exists. In particular,

$$<\Psi_j, A(f)\Psi_j> \ = \int d\nu(g) \int_{G/H_{r_g^*\omega}} f(x) \mid \Psi_j(x) \mid^2 d\mu(x).$$

From this, we may see what is the classical physical equivalence in terms of the $\sum_j \rho_j \mid \Psi_j(x) \mid^2$ for almost all $x \in G/H_{r_g^*\omega}$, all $g \in G$. Alternatively, it says that if we write ρ in terms of its kernel, $\rho(x, y)$,

$$[\rho\Phi](x) = \int d\nu(g) \int_{G/H_{r_g^*\omega}} \rho(x, y)\Phi(y) d\mu(y)$$

then $\rho \approx \rho'$ iff $\rho(x, x) = \rho'(x, x)$ for almost all $x \in G/H_{r_g^*\omega}$, all $g \in G$. It says nothing about the off-diagonal elements of the form $\rho(x, y)$, $x \neq y$.

For the action of the group on the states, we have

$$\begin{aligned}\rho &= \sum \rho_j P_{\Psi_j} \to^g \sum \rho_j P_{V^\alpha(g)\Psi_j} \\ &= \sum \rho_j V^\alpha(g) P_{\Psi_j} V^\alpha(g)^{-1} \\ &= V^\alpha(g)\rho V^\alpha(g)^{-1}. \end{aligned} \tag{37}$$

Thus we have the action of the generators $X \in \mathfrak{g}$ being

$$\rho \to^X X\rho - \rho X = [X, \rho]. \tag{38}$$

Therefore, we obtain the action of the generators given by the commutator, and all subsequent calculations will be in terms of commutators. We also have that we may use the $V^\alpha(g)$ to shift the x's around obtaining $\rho(g.x, g.x) \approx [V^\alpha(g)\rho V^\alpha(g)^{-1}](x,x) = [V^1(g)\rho V^1(g)^{-1}](x,x)$. Thus, the existance of the projective representations plays no role in the theory of classical mechanics in Hilbert space.

We may decompose $\cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu)$ into the irreducible components $\mathcal{H}_{irr}$, with irreducible representation given by U; i.e., we take an $\eta \in \mathcal{H}_{irr}$ and define, for $x \in \cup'_{g\in G} G/H_{r_g^*\omega}$, and for all $\psi \in \mathcal{H}_{irr}$,

$$\begin{aligned}\mathcal{W}^\eta \quad &: \quad \mathcal{H}_{irr} \to \cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu), \\ (\mathcal{W}^\eta[\psi])(x) \quad &= \quad < U(\sigma(x))\eta, \psi > . \end{aligned} \tag{39}$$

When operating from the left for quantum mechanics, again we may use just one of the factors, say $L^2(G/H_\omega, d\mu)$. Then there are some technical conditions on η; namely that it is "α-admissible". See [6] for this. In this way we obtain every quantum mechanical representation by reducing $L^2(G/H_\omega, d\mu)$ for the various ωs. We expect that we obtain every irreducible representation from the right with a similar calculation. Now we have

$$\cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu) = \oplus'_{j=irrep} \mathcal{H}_j \tag{40}$$

where the irreducible representations $\mathcal{H}_j$ occur as often as their dimension. Let U be the irreducible representation on one such space, $\mathcal{H}_j$. Let $\sigma : G/H_{r_g^*\omega} \to G$ be the choice function as before. Define

$$[\mathcal{W}^\eta \varphi](x) = \ < U(\sigma(x))\eta, \varphi > \tag{41}$$

for any $\varphi \in \mathcal{H}_j$. Then $\mathcal{W}^\eta$ is a map from $\mathcal{H}_j$ to $\cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu)$ subject to some conditions on η. [6, p. 321] Furthermore, $\mathcal{W}^\eta$ is a closed operator, intertwines U on $\mathcal{H}_j$ with V^α on $\cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu)$:

$$V^\alpha(g)\mathcal{W}^\eta = \mathcal{W}^\eta U(g) \text{ for all } g \in G, \tag{42}$$

and is independent of the section σ chosen if $\alpha = 1$. As above, we may assume that $\alpha = 1$.

Let $P^\eta : \cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu) \to \mathcal{W}^\eta(\mathcal{H}_j)$ be the canonical projection. Let

$$A^\eta(f) = (\mathcal{W}^\eta)^{-1} P^\eta A(f) \mathcal{W}^\eta, \;\; A^\eta(f) : \mathcal{H}_j \to \mathcal{H}_j. \tag{43}$$

Then [6, pg. 380]

$$\begin{aligned} A^\eta(f) &= \int f(x) T^\eta(x) d\mu(x). \\ T^\eta(x) &= U(\sigma(x)) P_\eta U(\sigma(x))^{-1}. \end{aligned} \tag{44}$$

Now define, for any state ρ_j on $\mathcal{H}_j$,

$$r_j(x) = Tr(\rho_j T^\eta(x)). \tag{45}$$

Then we have that r_j defines a Kolmogorov probability density on G/H_ω. This is a classical state! (However, this formulation does not allow any interpretation of r being a delta function at some point in G/H_ω, as there is no way we may generate such an r since ρ is a density operator on $\mathcal{H}_j$.) Now a simple calculation shows that

$$Tr(\rho A^\eta(f)) = \int_{G/H_\omega} f(x) r(x) d\mu(x); \tag{46}$$

i.e., the Hilbert space expectation and the expectation in the phase space agree. (Also we have the interpretation of ρ in terms of its expectation values only; this opens the way to discuss the informational completeness of the set of the $A^\eta(f)$'s.)

Furthermore, we generally obtain a positive operator, $A^\eta(f)$, if f is positive (like the Hamiltonian); etc. There are no negative energy states, no ultraviolate or infrared catastrophies. Moreover, defining the αth part of the current by choosing $f(x) = p_\alpha$, we obtain a conserved current (p_0 is positive implies $A^\eta(p_0)$ is positive), not a quasi-current in which the zeroth element is just real-valued. There are many ramifications, but we will stop here.

We state that these results hold no matter if the representation of G on G/H_ω is from the left (quantum mechanical) [6] or on the right (classical mechanical), as was done here.

4 Hamiltonian Action on $L^2(\Gamma, \nu)$ and $\mathcal{F}(L^2(\Gamma, \nu))$

It is the representation from the right which has the classical mechanics inherent in it; there the Lie algebra relations all give zero or $\lambda 1$; i.e., the X_j corresponding to the canonical variables are all "reduced" and commute when acting on G/H_ω from the right. (We will speak as though we have a symplectic vector space here. The generalization to the symplectic manifold, Γ, is quite straightforward.) Therefore, when acting on a vector in $L^2(G/H_\omega, \mu)$, they also commute. Thus, we may write the X_j in terms of partial derivatives. Consequently, we may write $[p,q]$ for $x \in G/H_\omega$, and then, when G acts on $\Psi \in L^2(G/H_\omega, \mu)$,

$$\begin{aligned} [V^\alpha(g)\Psi](p,q) &= \alpha(h(g^{-1},[p,q])) \exp\left\{ \Sigma_{j=1}^{n/2} i\beta_j \frac{\partial}{\partial p_j} + \Sigma_j^{n/2} i\gamma_j \frac{\partial}{\partial q_j} \right\} \Psi(p,q), \\ \beta_j, \gamma_j &\in \mathbb{R}. \end{aligned}$$

It is this representation with which we shall be concerned here.

To make the connection with the Hamiltonian formalism and classical mechanics, take any state ρ on $L^2(\Gamma,\mu)$. Recall that by "state" we mean a positive, trace class operator of trace 1. By an old theorem of J. von Neumann [11, p. 296], we have

$$\rho = \sum \rho_j P_{\Psi_j},$$

for some orthonormal basis $\{\Psi_j\}$ of $L^2(\Gamma,\mu)$ where P_{Ψ_j} is the projection onto the vector Ψ_j: $P_{\Psi_j}\Phi = <\Psi_j, \Phi> \Psi_j \ \forall \Phi \in L^2(\Gamma,\mu)$, and the ρ_j are real numbers satisfying $\rho_j \geq 0$, $\Sigma_j \rho_j = 1$.

Now, the action of G on ρ is given by

$$\mathcal{V}^\alpha(g)\rho \equiv V^\alpha(g)\rho V^\alpha(g)^\dagger = \sum \rho_j P_{V^\alpha(g)\Psi_j}.$$

Hence, the action of $\mathfrak{g}$ on ρ is given by

$$\mathcal{X}_j \rho \equiv [X_j, \rho], \ X_j \in \mathfrak{g}.$$

When we choose $\Gamma = G/H_\omega$ and we take the action on G/H_ω from the right, the action of $\mathcal{X}_j$ on the states in $L^2(\Gamma,\mu)$ is commutative, thanks to the commutative nature of the X_j on $L^2(G/H_\omega,\mu)$:

$$\begin{aligned} \mathcal{X}_j \mathcal{X}_k \rho &= [X_j, [X_k, \rho]] \\ &= X_j X_k \rho - X_j \rho X_k - X_k \rho X_j + \rho X_k X_j \\ &= X_k X_j \rho - X_k \rho X_j - X_j \rho X_k + \rho X_j X_k \\ &= \mathcal{X}_k \mathcal{X}_j \rho; \end{aligned}$$

i.e., they also commute on ρ. (Therefore, once again, they may be represented by partial derivatives.)

Next, we may introduce a Hamiltonian $H_0 + V$ as a C^∞-function of the canonical variables in G/H_ω, and then use the Hamiltonian flow, X_{H_0+V}, to obtain

$$U(t) = \exp\{-itX_{H_0+V}\}, \ t \in \mathbb{R}.$$

Then with $\alpha \equiv 1$, we have

$$\begin{aligned} [U(t)\Psi]([p,q]) &= [V^1(-p,-q)U(t)\Psi](H_\omega) \\ &= [\exp\left[i\left\{-p\cdot\nabla_p - q\cdot\nabla_q - t\beta\cdot\nabla_p - t\gamma\cdot\nabla_q\right\}\right]\Psi](H_\omega) \\ &= \Psi([p+t\beta, q+t\gamma]). \end{aligned}$$

(From this, we may compute the *expectation values* of the ps to be displaced by $t\beta$ and the qs by $t\gamma$.) In this way we may introduce the Hamiltonian formalism into $L^2(G/H_\omega, \mu)$. Furthermore, it doesn't matter what group G we take; this is particularly important when we consider the Poincaré group for which there is a folk theorem that there are no interactions in the context of Poincaré relativity. There *are* interactions when considering the action from the right on G/H_ω.

Next, place all of this in $\Gamma = \cup'_{g\in G} G/H_{r_g^*\omega}$ and $\mathcal{H} = \cup'_{g\in G} L^2(G/H_{r_g^*\omega}, d\mu)$ in the usual fashion.

To make a Fock space interaction, we take any one of the $\mathcal{H} = L^2(\Gamma, \nu)$ and then compute $\mathcal{H}^{\otimes n}$, for $n = 0, 1, 2, 3, \cdots$ and where $\otimes n$ denotes the n-th tensor product. Accordingly, we have the Fock space $\mathcal{F}(\mathcal{H}) \equiv \sum_{n=0}^{\infty} \mathcal{H}^{\otimes n}$. Then we may simply take the canonical variables to have the form of partial derivatives on the various components of the $\mathcal{H}^{\otimes n}$. This is necessary when we treat interactions of a particle in $\mathcal{H}$ with another particle or particles of the same symmetry. Now given any n-body interaction, H, we may form the operator $\Omega(H, n)$ acting on $\mathcal{F}(\mathcal{H})$. [12][13] The details of $\Omega(H, n)$ are left to the reader. We may also consider the Fock space of symmetric or anti-symmetric (or more general) particles which may be handled in a very general manner; but this goes too far away from the purposes of this article.

5 Examples of Classical Mechanics in Hilbert Space

We will just illustrate examples with Galilei and Poincaré invariance. We shall use the notation from previous sections.

5.1 Example of Interacting, Massive, Spinning, Galilean Particles

Let us consider two point particles of mass m interacting by means of an elastic force. From Section 2.3, we have $\omega = m\sum_i u_i^* \wedge k_i^* + M_1^* \wedge M_2^*$ so that the phase space is 8 dimensional for each particle. For each particle, we have three position coordinates q_i^x, q_i^y, q_i^z, $i = 1, 2$, three momentum coordinates p_i^x, p_i^y, p_i^z, $i = 1, 2$ and two spin coordinates s_i^1, s_i^2, $i = 1, 2$. Moreover, q_1^x corresponds to the generator $k^x \otimes 1$, q_2^x to the generator $1 \otimes k^x$, p_1^x to the generator $u^x \otimes 1$, p_2^x to the generator $1 \otimes u^x$ and so on. Finally, s_1^j corresponds to the generator $M_j \otimes 1$ while s_2^j corresponds to the generator $1 \otimes M_j$.
Now, suppose that the two particles are constrained to move on the axis x without friction. The Hamiltonian of the system will be taken to be $H = H_0 + V$ where,

$$H_0 = \frac{(p_1^x)^2}{2m} + \frac{(p_2^x)^2}{2m}$$

$$V = \frac{1}{2}k(q_2^x - q_1^x)^2.$$

The corresponding vector field is,

$$X_H = k(q_2^x - q_1^x)[-X_{p_1^x} + X_{p_2^x}] - \sum_{j=1}^{2} \frac{p_j^x}{m} X_{q_j^x}.$$

Then, by using the notation $[p,q,s] = [p_i^x, p_i^y, p_i^z, q_i^x, q_i^y, q_i^z, s_i^1, s_i^2]$, $i = 1,2$, we obtain:

$$\begin{aligned}
&[U(t)\Psi]([p,q,s]) \\
&= [exp(-itX_H)\Psi]([p,q,s]) \\
&= [V^1(-p_1,-q_1,-s_1) \otimes V^1(-p_2,-q_2,-s_2) exp(-itX_H)\Psi](H_\omega \otimes H_\omega) \\
&= \left[e^{\left(-i\left\{ \sum_{j=1}^2 (p_j^x X_{p_j^x} + q_j^x X_{q_j^x}) \right\}\right)} e^{\left(-it\left\{ k(q_2^x - q_1^x)[-X_{p_1^x} + X_{p_2^x}] - \sum_{j=1}^2 \frac{p_j^x}{m} X_{q_j^x} \right\}\right)} e^{rest}\Psi \right](H_\omega \otimes H_\omega) \\
&= \left[e^{\left(-i\left\{ (p_1^x - tk(q_2^x - q_1^x))X_{p_1^x} + (p_2^x + tk(q_2^x - q_1^x))X_{p_2^x} + \sum_{j=1}^2 (q_j^x - t\frac{p_j^x}{m}) X_{q_j^x} \right\}\right)} e^{rest}\Psi \right](H_\omega \otimes H_\omega) \\
&= \Psi([p_1^x - tk(q_2^x - q_1^x),\ q_1^x - t\frac{p_1^x}{m},\ p_2^x + tk(q_2^x - q_1^x),\ q_2^x - t\frac{p_2^x}{m},\ p_i^y,\ p_i^z,\ q_i^y,\ q_i^z,\ s_i^1,\ s_i^2]).
\end{aligned}$$

We have used the notation "$rest$" equals whatever is needed to obtain the correct p_i^y, p_i^z, q_i^y, q_i^z, s_i^1, s_i^2.

We should add a remark here. Indeed, it is worth noticing that it is not possible to find an example of a classical system involving an angular momentum which is of the form $r \wedge v$, i.e., which is a classical angular momentum. Then, the presence of M_1 and M_2 as canonical variables for the Galilean particles seems to suggest that the formalism of classical mechanics in Hilbert space which we introduce in this paper, gives us the possibility of describing intrinsic spin. This suggestion will be analyzed in a future work.

5.2 Example of Interacting Poincaré Particles

Here we will treat the case of the Poincaré group with the choice ω_1 of 2.5.1, $\mathcal{F}(\mathcal{H}) \equiv \mathcal{F}(\cup'_{g \in G} L^2(G/H_{r_g^* \omega_1}, \mu))$, and with the fictitious two-body interaction corresponding to the action on $\mathcal{H}^{\otimes 2}$ being given on $[L^2(G/H_\omega, \mu)]^{\otimes 2}$ by

$$\begin{aligned}
[H\Psi](p^1, q^1, p^2, q^2) &= [H_0 + \gamma(p^1 p^2)]\Psi(p^1, q^1, p^2, q^2), \\
\gamma &\in \mathbb{R},
\end{aligned}$$

where p^1 = the value of $M_1 \otimes 1$, q^1 = the value of $M_2 \otimes 1$, p^2 = the value of $1 \otimes M_1$, and q^2 = the value of $1 \otimes M_2$ on $\Psi(p^1, q^1, p^2, q^2)$ a.e.μ. $H_0 = \frac{\beta}{2}[(p^1)^2 + (q^1)^2 + (p^2)^2 + (q^2)^2]$ is taken to be the "free Hamiltonian." This results in

$$\begin{aligned} X_H &= \sum_{j=1}^{2}\left(\frac{\partial H}{\partial q^j}\frac{\partial}{\partial p^j} - \frac{\partial H}{\partial p^j}\frac{\partial}{\partial q^j}\right) \\ &= \sum_{j=1}^{2}\left(\beta q^j \frac{\partial}{\partial p^j} - [\beta p^j + \gamma p^{j\pm 1}]\frac{\partial}{\partial q^j}\right) \\ &= \sum_{j=1}^{2}\left(\beta q^j X_{q^j} + [\beta p^j + \gamma p^{j\pm 1}]X_{p^j}\right). \end{aligned}$$

Then transform it to $L^2(G/H_{r_g^*\omega_1}, \mu)$, etc. We obtain, for $\Psi \in \mathcal{H}^{\otimes 2}$, in simplified notation,

$$\begin{aligned} & [U(t)\Psi]([p^1, q^1, p^2, q^2]) \\ = \; & [\exp(-itX_{H_0+V})\Psi]([p^1, q^1, p^2, q^2]) \\ = \; & [\{V^1(-p^1, -q^1) \otimes V^1(-p^1, -q^1)\}U(t)\Psi](H_\omega \otimes H_\omega) \\ = \; & \left[\begin{array}{c} \exp\left(-i\left\{\sum_{j=1}^{2}(p^j X_{p^j} + q^j X_{q^j})\right\}\right) \times \\ \exp\left(-it\left\{\sum_{j=1}^{2}\left(\beta q^j X_{q^j} + [\beta p^j + \gamma p^{j\pm 1}]X_{p^j}\right)\right\}\Psi\right) \end{array}\right](H_\omega \otimes H_\omega) \\ = \; & \left[\exp\left(-i\left\{\sum_{j=1}^{2}([p^j + t\beta p^j + t\gamma p^{j\pm 1}]X_{p^j} + [q^j + t\beta q^j]X_{q^j}\right\}\right)\Psi\right](H_\omega \otimes H_\omega) \\ = \; & \Psi([p^1 + t\beta p^1 + t\gamma p^2, q^1 + t\beta q^1, p^2 + t\beta p^2 + t\gamma p^1, q^2 + t\beta q^2]). \end{aligned}$$

From here we simply take $\Omega(H, n)$ acting on $\Phi \in \mathcal{F}(\mathcal{H})$, as defined in [12].

We may also check directly if (25) of Part 1 holds for this choice of H, as it must. You obtain

$$\begin{aligned} [\iota_{X_H}\omega_1](Y) &= \omega_1(X_H, Y) \\ &= \sum_{j=1}^{2}[\beta p_j + \gamma p_{(j\pm 1)}]M_2^*(Y) - \sum_{j=1}^{2}\beta q_j M_1^*(Y) \\ &= \sum_{j=1}^{2}[\beta p_j + \gamma p_{(j\pm 1)}]X_{p_j}^*(Y) + \sum_{j=1}^{2}\beta q_j X_{q_j}^*(Y) \\ &= (\delta H)(Y). \end{aligned}$$

Similarly for r_g^* operating on everything. Thus H is a Hamiltonian on $[\cup'_{g\in G} G/H_{r_g^*\omega}]^{\otimes 2}$ and then on $[\cup'_g L^2(G/H_{r_g^*\omega})]^{\otimes 2}$. Hence, we have demonstrated a phase space with a non-trivial Hamiltonian in the Poincaré group setting.

We should comment on what allows us to obtain a non-zero interaction in view of the results of Currie, Jordan, et al. Here we only require (40') of Part 1. Furthermore, we only require (40') of Part 1 for the canonical variables occuring in the interaction; for the remaining variables we do not care whether they are canonical or not. Moreover, in our simple example, we chose our canonical variables to be the first two components of the spin. These are subjects which have not been considered by Currie, Jordan, et al.

We have done all this in the Poincaré group; but notice that we might have done it in the Lorentz group as well. So, we also have an example of an interacting system in the Lorentz group.

6 Summary

We have shown that any Lie group (including the Poincaré and Lorentz groups) has classical interactions, H, on the classical variables defined on the appropriate G/H_ω by $g_0 H_\omega \mapsto g_0 H_\omega e^{-itH} = g_0 e^{-itH} H_\omega$. Then, on $g(G/H_\omega)g^{-1} = G/H_{r_g^* \omega}$ we have the corresponding form of the Hamiltonian given by $g_0 H_{r_g^* \omega} \mapsto g_0 H_{r_g^* \omega} e^{-it(r_g)_* H} = g_0 e^{-it(r_g)_* H} H_{r_g^* \omega}$. Note the dependence on $g \in G$. With this, we then represent the Hamiltonian as an operator on a Hilbert space with the relevant group as a symmetry group.

Acknowledgement 1 *One of us (R.B.) would like to acknowledge the INFN (Istituto Nazionale di Fisica Nucleare) gruppo collegato Cosenza and the GNFM (Gruppo Nazionale per la Fisica Matematica) for supporting him for a stay in Denver for an academic quarter to work on this project. Another of us (F. S.) would like to acknowledge the INFN and COMSON (Coupled Multiscale Simulation and Optimization, a Marie Curie European Project) for supporting him for a stay in Cosenza to work on the early stages of this work. He is furthermore an Emeritus Professor of Mathematics at Florida Atlantic University.*

References

[1] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35 (1963), 350-375.

[2] D. G. Currie, J. Math. Phys. 4 (1963), 1470-1488.

[3] J. T. Cannon and T. F. Jordan, J. Math. Phys. 5 (1964), 299-307.

[4] H. Leutwyler, Nuovo Cimento 37 (1965), 556-567.

[5] D. G. Currie and T. F. Jordan, "Interactions in Relativistic Classical Particle Mechanics," in Quantum Theory and Statistical Physics, A. O. Barut and W. E. Brittin, eds., Gordon and Breach Sci. Pubs., N. Y., 1968, pp. 91-139.

[6] F. E. Schroeck, Jr., Quantum Mechanics on Phase Space, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1996.

[7] H. Goldstein, Classical Mechanics, Addison-Wesley Pub. Co. Inc., Reading, Massachusetts, 1950.

[8] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Harper & Row, Pubs., Inc., New York, 1961 or 1962.

[9] M. C. Abbati, R. Cirelli and F. Gallone, J. Math. Phys. 16 (1975), 2233-2240.

[10] P. Busch and P. J. Lahti, Found. Phys. 19 (1989), 633-678.

[11] J. von Neumann, Mathematical Foundations of Quantum Mechanics, E. P. Wigner and R. Hofstadter eds., Princeton Univ. Press, Princeton, 1955.

[12] F. E. Schroeck, Jr., J. Math. Phys. 12 (1971), 1849-1857.

[13] F. E. Schroeck, Jr., Reports on Mathematical Physics 26 (1988), 197-210.