Non-Extendible Latin Cuboids

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Abstract

We show that for all integers $m \geqslant 4$ there exists a $2m \times 2m \times m$ latin cuboid that cannot be completed to a $2m \times 2m \times 2m$ latin cube. We also show that for all even $m \notin \{2, 6\}$ there exists a $(2m-1) \times (2m-1) \times (m-1)$ latin cuboid that cannot be extended to any $(2m-1) \times (2m-1) \times m$ latin cuboid.

1 Introduction

There is a celebrated result due to Marshall Hall [6] that every latin rectangle is completable to a latin square. However, the equivalent statement in higher dimensions is not true. The purpose of this paper is to investigate the extent to which it fails.

We think of a 3-dimensional array as having *layers* stacked on top of each other. It also has *lines* of cells in three directions, obtained from fixing two

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coordinates and allowing the third to vary. The lines obtained by varying the first, second and third coordinates will be known respectively as *columns, rows* and *stacks*. The first, second and third coordinates themselves will be referred to as the indices of the rows, columns and layers.

An $n \times n \times k$ *latin cuboid* is a 3-dimensional array containing n different symbols positioned so that every symbol occurs exactly once in each row and column and at most once in each stack. An $n \times n \times n$ latin cuboid is a *latin cube* of order n. Every layer of a latin cuboid is a latin square, and we will present our cuboids by displaying the latin squares corresponding to each layer. Each individual layer is composed of a set of n^2 *entries*, each of which is a triple (r, c, s) where s is the symbol in row r and column c. The layer in which a given entry resides will always be made clear by the context.

We say that an $n \times n \times k$ latin cuboid has order n. It is *extendible* if it is contained in some $n \times n \times (k + 1)$ latin cuboid and it is *completable* if it is contained in some latin cube of order n. Our aim is to investigate how "thin" (that is, how small k can be, relative to n) non-extendible and non-completable latin cuboids can be. We will refer to an $n \times n \times k$ latin cuboid as being less than half-full, half-full or more than half-full if $k < \frac{1}{2}n$, $k = \frac{1}{2}n$ or $k > \frac{1}{2}n$, respectively. Although we find some non-extendible examples that are less than half-full, many questions will remain open.

In the 1980s, several authors [5, 7, 8] considered the problem of constructing non-completable $n \times n \times (n - 2)$ latin cuboids. Subsequently Kochol [9] proved that for any k and n satisfying $\frac{1}{2}n < k \leqslant n - 2$ there is a non-completable $n \times n \times k$ latin cuboid. Although he did not say so, it is simple to use such examples to create non-completable $n \times n \times \cdots \times n \times k$ latin hypercuboids in higher dimensions. Kochol conjectured that all non-completable latin cuboids are more than half-full, but examples of non-completable $5 \times 5 \times 2$, $6 \times 6 \times 2$, $7 \times 7 \times 3$ and $8 \times 8 \times 4$ latin cuboids were subsequently given in [10]. Our results will show that Kochol's conjecture fails for all orders except possibly those that are 1 mod 4.

Cutler and Öhman [2] showed for all m that every $2mk \times 2mk \times m$ latin cuboid is extendible, provided k is sufficiently large. Little else is known about extendibility aside from the elementary observations that all $n \times n \times 1$ and $n \times n \times (n - 1)$ latin cuboids are extendible (in fact completable). Of course, by extending any non-completable latin cuboid as far as possible we will obtain at least one non-extendible latin cuboid, but often only one of dimensions $n \times n \times (n - 2)$.

2 Half full non-completable latin cuboids

In this section we build non-completable latin cuboids of even order that are exactly half full. To do this we show that the sets of symbols that are missing from the stacks of the latin cuboid have a particular configuration. For an $n \times n \times k$ latin cuboid where $k \leqslant n$, the set of *available symbols* in each stack consists of those symbols that do not occur in that stack.

For each integer $m \geqslant 1$, let $U = U(m) = \{1, 2, \ldots, m\}$ and $U^* = U^*(m) = \{1^*, 2^*, \ldots, m^*\}$. Let $R(m)$ be a $2m \times 2m \times m$ latin cuboid with rows, columns and symbols indexed by $U \cup U^*$ and layers indexed by U.

Consider the $m \times m \times m$ sub-array of $R(m)$ with rows, columns and layers indexed by U. Suppose that the sets of available symbols in this sub-array's stacks are as follows:

$$\begin{array}{cccc} X & X & \cdots & X \\ Y & U^* & U^* & U^* \\ \vdots & U^* & \ddots & \vdots \\ Y & U^* & \cdots & U^* \end{array} \tag{1}$$

where $X = (U^* \setminus \{1^*\}) \cup \{1\}$ and $Y = (U^* \setminus \{2^*\}) \cup \{1\}$. Then we say that $R(m)$ is *awkward*. Note that an awkward latin cuboid has even order.

Lemma 1. *For $m \geqslant 2$, no $2m \times 2m \times m$ awkward latin cuboid is completable.*

Proof. To complete $R(m)$, we restrict our attention to the problem of choosing entries for the sub-array T with rows and columns indexed by U and layers indexed by U^*. Thus the stacks of T must use the available symbols described in (1). To prove our lemma, it suffices to show that a completion of T is impossible.

Now, some layer of T must include symbol 1 in row and column 1. This will be the only occurrence of 1 in this layer of T. Symbols 1^* and 2^* can each occur at most $(m-1)$ times in this layer of T while symbols 3^* to m^* can each occur m times. But this allows at most $2(m-1)+(m-2)m+1 = m^2-1$ cells to be filled in this layer of T, a contradiction. □

By computer search, we found that awkward latin cuboids of orders less than 8 do not exist. We did find examples of orders $8, 10, 12$ and 14 (that is, for $m = 4, 5, 6$ and 7) and these are given in the Appendix. Hence we have the following lemma.

Lemma 2. *For each $m \in \{4, 5, 6, 7\}$, there exists a $2m \times 2m \times m$ awkward latin cuboid.*

To construct awkward latin cuboids of orders greater than 14 we apply an embedding construction. We first construct awkward latin cuboids of orders 0 mod 4.

Theorem 3. *If there exists a* $2m \times 2m \times m$ *awkward latin cuboid* $R(m)$*, then there exists a* $4m \times 4m \times 2m$ *awkward latin cuboid* $R'(2m)$*.*

Proof. Let $m \geqslant 4$ and let $R(m)$ be a $2m \times 2m \times m$ awkward latin cuboid. Define $U_1 = U(m)$, $U_2 = U(2m) \setminus U(m)$, $U_1^* = U^*(m)$ and $U_2^* = U^*(2m) \setminus U^*(m)$.

We build $R'(2m)$ from $R(m)$ and some latin cubes of order m. Let the rows and columns of $R'(2m)$ be indexed by $U_1 \cup U_2 \cup U_1^* \cup U_2^*$ and the layers be indexed by $U_1 \cup U_2$. In the sub-array indexed by rows and columns $U_1 \cup U_1^*$ and layers U_1, we embed a copy of $R(m)$.

Let A_1, A_2, A_1^* and A_2^* be latin cubes of order m on symbol sets U_1, U_2, U_1^* and U_2^*, respectively. Furthermore, let B, C^*, D and E^* be latin cubes of order m on symbol sets U_1, U_1^*, U_1 and U_1^*, respectively, with the following properties. Latin cubes B and C^* contain symbols 1 and 1^*, respectively, in the first row and ith column of layer i, for each i, while latin cubes D and E^* contain symbols 1 and 2^*, respectively, in the ith row and first column of layer i. It is trivial to construct such cubes by starting with any latin cube of the correct size and symbol set, and then permuting the layers appropriately.

We construct a $4m \times 4m \times 2m$ latin cuboid from these smaller latin cubes as indicated in the following diagrams, with empty cells taken up by the embedding of $R(m)$. (It is understood that the "ith" element of sets U_2, U_1^* or U_2^* refers to $m + i$, i^* or $(m + i)^*$, respectively.)

	U_1	U_2	U_1^*	U_2^*
U_1		A_2		A_2^*
U_2	A_2	A_1	A_2^*	A_1^*
U_1^*		A_2^*		A_2
U_2^*	A_2^*	A_1^*	A_2	A_1

Layers U_1

	U_1	U_2	U_1^*	U_2^*
U_1	A_2	B	A_2^*	C^*
U_2	D	A_2	E^*	A_2^*
U_1^*	A_2^*	C^*	A_2	B
U_2^*	E^*	A_2^*	D	A_2

Layers U_2

Next, for each $u \in U_2$, we exchange the symbols within two latin subsquares of order 2 from layer u of the $4m \times 4m \times 2m$ latin cuboid described above. To be precise, we replace the following entries of layer u

$$\begin{array}{llll} (1, u, 1), & (1, u^*, 1^*), & (u, 1, 1), & (u, 1^*, 2^*), \\ (1^*, u, 1^*), & (1^*, u^*, 1), & (u^*, 1, 2^*), & (u^*, 1^*, 1), \end{array} \qquad (2)$$

with the entries

$$\begin{array}{llll} (1,u,1^*), & (1,u^*,1), & (u,1,2^*), & (u,1^*,1), \\ (1^*,u,1), & (1^*,u^*,1^*), & (u^*,1,1), & (u^*,1^*,2^*). \end{array} \quad (3)$$

It is routine to check that the resulting latin cuboid $R'(2m)$ is awkward. □

Next, using a similar approach of embedding the awkward cuboid $R(m)$ and swapping symbols in order 2 subsquares, we construct awkward latin cuboids of orders 2 mod 4.

Theorem 4. *If there exists a $2m \times 2m \times m$ awkward latin cuboid $R(m)$, then there exists a $(4m+2) \times (4m+2) \times (2m+1)$ awkward latin cuboid $R''(2m+1)$.*

Proof. Let $m \geqslant 4$ and let $R(m)$ be a $2m \times 2m \times m$ awkward latin cuboid. Define $U_1 = U(m)$, $U_2 = U(2m+1) \setminus U(m)$, $U_1^* = U^*(m)$ and $U_2^* = U^*(2m+1) \setminus U^*(m)$. Note that when performing modular arithmetic in this proof, we always take the answer to be the least *positive* residue in the congruence class.

We introduce a new operation, $\oplus$, defined as follows. For integers x, y and m, let

$$x \oplus y \;=\; \begin{cases} x+y & \text{if } x+y \leqslant 2m+1, \\ x+y-m-1 & \text{otherwise.} \end{cases}$$

So, for instance, $(2m+1) \oplus 1 = m+1$. We now define a quasigroup $(Q, \circ)$, where $Q = U_1 \cup U_2$, as follows.

$$\begin{array}{lcll} i \circ j & = & i+j \mod m & \text{if } i,j \in U_1; \\ i \circ j & = & i \oplus j & \text{if } i = U_1, j \in U_2 \text{ or } i \in U_2, j \in U_1; \\ i \circ i & = & i & \text{if } i \in U_2; \\ i \circ j & = & i-j \mod (m+1) & \text{if } i,j \in U_2, i \neq j. \end{array}$$

We use Q, in turn, to define a latin cube S of order $2m+1$ with rows, columns and layers indexed by $U_1 \cup U_2$, where the cell in row i and column j of layer l contains symbol $(i \circ j) \circ l$ for each $i, j, l \in U_1 \cup U_2$. Next, we extend S to a $(4m+2) \times (4m+2) \times (2m+1)$ latin cuboid $R^\dagger(2m+1)$ as follows. Let S^* be a copy of S in which every symbol has been starred. Form $R^\dagger(2m+1)$ by placing copies of S and S^* in the following arrangement

$$\begin{array}{cc} S & S^* \\ S^* & S \end{array}$$

where rows and columns are indexed in the obvious way by $U_1 \cup U_2 \cup U_1^* \cup U_2^*$. Note that the layer indices remain unstarred.

A number of adjustments are needed to turn $R^\dagger(2m+1)$ into an awkward latin cuboid $R''(2m+1)$. Firstly, observe that $R^\dagger(2m+1)$ contains a $2m \times 2m \times m$ subcuboid with row and column indices from $U_1 \cup U_1^*$ and layer indices from U_1, based on the symbols $U_1 \cup U_1^*$. We replace this subcuboid with the awkward latin cuboid $R(m)$.

Next, for each row $i \in U_2$ and layer $l \in U_1 \cup U_2$, we swap the symbols in columns 1^* and 2^*. Similarly, for each row $i \in U_2^*$ and layer $l \in U_1 \cup U_2$, we swap the symbols in columns 1 and 2. Observe that overall each row, column and stack remains latin.

Finally, for each $u \in U_2$, we exchange the symbols within two latin subsquares of order 2 from layer u of the $(4m+2) \times (4m+2) \times (2m+1)$ latin cuboid described above. To be precise, we replace the entries of layer u listed in (2) with the entries listed in (3). It is routine to check that the resulting latin cuboid $R''(2m+1)$ is awkward. □

Combining the results in this section, we have the following:

Theorem 5. *For all $m \geqslant 4$, there exists a non-completable $2m \times 2m \times m$ latin cuboid.*

We next show that the latin cuboids constructed in this section, while not completable, are extendible by at least one layer.

Corollary 6. *For all $m \geqslant 8$, there exists a non-completable $2m \times 2m \times m$ latin cuboid which is extendible to a $2m \times 2m \times (m+1)$ latin cuboid.*

Proof. For each latin cuboid constructed in Theorems 3 and 4, we describe a latin square that can be added as an extra layer without causing repeated entries in a stack. The rows and columns of our latin square will be indexed by $U_1 \cup U_2 \cup U_1^* \cup U_2^*$ as in the proofs above. For the sake of economy we give our construction below in terms of U_1, U_2, U_1^*, U_2^*; note however that these have distinct definitions in Theorems 3 and 4.

Fix an element $u \in U_2$. In the intersection of rows $U_1 \cup U_2$ with columns $U_1 \cup U_2$ (respectively, rows $U_1^* \cup U_2^*$ with columns $U_1^* \cup U_2^*$), we place a latin square on the symbol set $U_1^* \cup U_2^*$ containing the entries $(1, u, 1^*)$ and $(u, 1, 2^*)$ (respectively, $(1^*, u^*, 1^*)$ and $(u^*, 1^*, 2^*)$). We also specify that cells from rows U_1 and columns U_1 (respectively, rows U_1^* and columns U_1^*) contain only symbols from U_2^*.

Next, in the intersection of rows $U_1 \cup U_2$ with columns $U_1^* \cup U_2^*$ (respectively, rows $U_1^* \cup U_2^*$ with columns $U_1 \cup U_2$), we place a latin square on the symbol set $U_1 \cup U_2$ containing the entries $(1, u^*, 1)$ and $(u, 1^*, 1)$ (respectively, $(1^*, u, 1)$ and $(u^*, 1, 1)$). We also specify that cells from rows U_1 and columns U_1^* (respectively, rows U_1^* and columns U_1) contain only symbols from U_2.

Finally, we exchange the entries

$$\begin{array}{llll} (1,u,1^*), & (1,u^*,1), & (u,1,2^*), & (u,1^*,1), \\ (1^*,u,1), & (1^*,u^*,1^*), & (u^*,1,1), & (u^*,1^*,2^*), \end{array}$$

with the entries

$$\begin{array}{llll} (1,u,1), & (1,u^*,1^*), & (u,1,1), & (u,1^*,2^*), \\ (1^*,u,1^*), & (1^*,u^*,1), & (u^*,1,2^*), & (u^*,1^*,1). \end{array}$$

Given a latin cuboid constructed in Theorems 3 or 4, the above construction yields a latin square that can serve as an additional layer. □

The obstruction used to prove Theorem 5 is similar to that used by Kochol [9]. In the next section we will use a very different argument to construct non-extendible latin cuboids; an argument which is reminiscent of the Δ-lemma arguments in [1], [3], [4] and [11].

3 Thin non-extendible cuboids

We now turn our attention to the problem of finding thin non-extendible latin cuboids. A *species* (also known as a *main class*) is an equivalence class of latin squares or rectangles. We shall use the term "species" for the natural generalisation of this well-known notion to latin cubes and cuboids (also see [10]).

For $k < n \leqslant 4$, all $n \times n \times k$ latin cuboids are completable, and hence all are extendible. Of the 31 species of $5 \times 5 \times 2$ latin cuboids, there is only one that is non-extendible; it is the non-completable example given in [10]. To find the thinnest example of a non-extendible latin cuboid of order 6, we compiled a catalogue of all 601 115 species of $6 \times 6 \times 2$ latin cuboids. We then counted the number of extensions that each had to a $6 \times 6 \times 3$ latin cuboid. The fewest number of extensions was 3932, which was achieved by the following example.

1	2	3	4	5	6		3	5	1	6	2	4
2	1	4	3	6	5		6	2	5	1	4	3
3	5	1	6	4	2		1	4	3	2	5	6
4	6	5	1	2	3		5	1	4	3	6	2
5	3	6	2	1	4		4	6	2	5	3	1
6	4	2	5	3	1		2	3	6	4	1	5

The most number of extensions was 41984, which was achieved by two species; the one shown below, and the one that can be obtained from this one by interchanging the shaded symbols.

1	2	3	4	5	6		2	1	4	3	6	5
2	1	4	3	6	5		1	2	3	4	5	6
3	4	5	6	1	2		4	3	6	5	2	1
4	3	6	5	2	1		3	4	5	6	1	2
5	6	1	2	3	4		6	5	2	1	4	3
6	5	2	1	4	3		5	6	1	2	3	4

Thus, the thinnest non-extendible latin cuboid of order 6 is a $6 \times 6 \times 3$ latin cuboid. An example of such a latin cuboid is given below; it is an extension of the non-completable $6 \times 6 \times 2$ latin cuboid given in [10].

1	2	3	4	5	6		2	1	4	3	6	5		3	4	5	6	1	2
2	1	4	3	6	5		1	2	5	6	4	3		4	3	2	1	5	6
3	4	5	6	1	2		4	5	6	2	3	1		6	1	3	5	2	4
4	3	6	5	2	1		3	6	2	1	5	4		5	2	1	4	6	3
5	6	1	2	3	4		6	4	3	5	1	2		2	5	6	3	4	1
6	5	2	1	4	3		5	3	1	4	2	6		1	6	4	2	3	5

Hence, order 6 is the smallest order for which there exists a non-completable latin cuboid that has strictly fewer layers than any non-extendible latin cuboid. For order 7, the construction below gives a non-extendible $7 \times 7 \times 3$ latin cuboid, but we do not know if there is one with fewer layers.

We now give a construction that produces a family of non-extendible latin cuboids that are slightly less than half-full. This represents significant progress given that all previous general constructions produced examples that were at least half-full and that only satisfied the weaker condition of being non-completable.

Theorem 7. *For all even $m \notin \{2, 6\}$, there exists a $(2m-1) \times (2m-1) \times (m-1)$ non-extendible latin cuboid.*

Proof. The $2m-1$ symbols used in our construction will consist of unstarred symbols $U = \{1, 2, \ldots, m\}$ and starred symbols $S = \{1^*, 2^*, \ldots, (m-1)^*\}$. We may add and subtract symbols within and between symbol sets so long as we indicate whether the result is a starred or unstarred symbol and compute modulo $m-1$ or m, respectively. We will place starred or unstarred brackets around calculations to indicate the intended meaning. For example, if $m = 8$, then $[5 + 6^*]^* = 4^*$ while $[5 + 6^*] = 3$. As in the previous section, when performing modular arithmetic we always take the answer to be the least *positive* residue in the congruence class.

Since $m \notin \{2, 6\}$, there exists a pair of $m \times m$ orthogonal latin squares L and M whose rows, columns and symbols are indexed by U. By permuting rows and symbols if necessary, we can insist that m appears in every cell on the main diagonal of M while the main diagonal of L lists the unstarred symbols in order. For each $u \in U$ there is a transversal T_u of L that corresponds to the positions of the symbol u in M. For example, T_m is the main diagonal of L.

We will construct a $(2m-1) \times (2m-1) \times (m-1)$ latin cuboid in four blocks A, B, C, and D arranged as follows.

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

The rows of A and B and the columns of A and C will be indexed by S while the rows of C and D and the columns of B and D will be indexed by U. The layers of the cuboid will be indexed by $U \setminus \{m\}$. Block A is any latin cube of order $m-1$ on the starred symbols S. The structure of layer u of blocks B, C and D depends entirely upon the transversal T_u of L. Suppose this transversal contains the entries $\{(i, c_i, s_i) : i \in U\}$. Then in layer u, for all $i \in U$ and $k^* \in S$:

- Block B contains the entry $(k^*, c_i, [s_i + k^*])$;
- Block C contains the entry $(i, k^*, [s_i + k^*])$;
- Block D contains the entries (i, c_i, s_i) and $(i, [c_i + k^*], k^*)$.

In other words, layer u of B and C consists of unstarred symbols developed cyclically in each column and row, respectively. Meanwhile, layer u of D

consists of a copy of T_u with the subsequent entries in each row listing the starred symbols in order.

First we show that this construction yields a latin cuboid, that is, no line contains a repeated symbol. It is straightforward to see that each layer is a latin square on the symbols $U \cup S$, so no row or column contains a repeated symbol.

We now show that no stack contains a repeated symbol. Since A is a latin cube, none of its stacks contain a repeated symbol. Suppose the entry (x^*, y, s) occurs in both layers u and v of block B. By construction, $[s - x^*]$ is the symbol contained in column y of both T_u and T_v. Since disjoint transversals of L must contain different symbols in any particular column, we have $u = v$. Up to row-column symmetry, the same reasoning applies to block C. For block D, suppose entry (x, y, s) occurs in layers u and v where s is an unstarred symbol. It follows that $(x, y, s) \in T_u$ and $(x, y, s) \in T_v$ and thus $u = v$. Now suppose entry (x, y, s^*) occurs in layers u and v of block D. It then follows that there is an unstarred symbol in row x and column $[y - s^*]$. Thus, by the same reasoning as above, we have that T_u and T_v must intersect, and so $u = v$. Hence our construction does give a $(2m-1) \times (2m-1) \times (m-1)$ latin cuboid.

Now suppose the constructed latin cuboid can be extended by a single layer. Let R_X, C_X and S_X be the sum, modulo m, of the row indices, column indices, and symbols, respectively, for the occurrences of unstarred symbols in this new layer in block X, for $X = B, C$. Since the new layer in block A must contain only unstarred symbols, exactly one additional unstarred symbol will be needed in each row of B and in each column of C (the unique unstarred symbol not used in the corresponding row or column of A). Thus $R_B = C_C = [1^* + 2^* + \cdots + (m-1)^*] \equiv \binom{m}{2} \mod m$. Since each unstarred symbol occurs the same number of times in B as it does in C (namely $m - 1$ minus the number of times it occurs in A), we have $S_B = S_C$.

The main diagonal of L is not only a transversal but lists the unstarred symbols in order so that they align with the column indexing. Therefore, the only unstarred symbol available in row x^* and column y of B is $[x^* + y]$, and hence $S_B = R_B + C_B$. Similarly $S_C = R_C + C_C$. Given the previous identities, we have $R_C = C_B$.

Let R_D, C_D, and S_D be the sum, modulo m, of the row indices, column indices, and symbols, respectively, for the occurrences of starred symbols in the new layer in D. As already noted, exactly one unstarred symbol must occur in each row of B and in each row of C in the new layer. Thus when the remaining $(m-1)^2$ entries of B (or C) are filled with starred symbols, each starred symbol is missing from precisely one column of B (and from precisely one row of C). Therefore, each starred symbol must be used in the new layer in D precisely once and these occurrences must fall in the same columns used by unstarred symbols in B and the same rows used by unstarred symbols in C. Hence $R_D = R_C$, $C_D = C_B$ and $S_D = [1^* + \cdots + (m-1)^*] \equiv \binom{m}{2} \mod m$.

Finally, consider which starred symbols are available for the new layer in block D. Suppose s^* is available in row x and column y in the new layer. Thus, no earlier layer has an unstarred symbol in row x and column $[y - s^*]$. Note that since the transversal T_m of L is the main diagonal, we have that $x \neq y$. Since, for every position off the main diagonal, there is a layer that contains an unstarred symbol in that position, we have that $x = [y - s^*]$. Thus $x \equiv y - s^* \mod m$ and $s^* = (y - x \mod m)^*$ where $(y-x) \mod m \in \{1, 2, \ldots, m-1\}$ as required. Therefore $S_D = C_D - R_D$.

Putting it all together, we have

$$\tfrac{1}{2}m(m-1) \equiv S_D \equiv C_D - R_D = C_B - R_C \equiv 0 \mod m.$$

This is a contradiction since m is assumed to be even. □

We illustrate Theorem 7 by giving an example of the construction for $m = 4$. Suppose that we choose the orthogonal latin squares

$$L = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 4 & 2 & 3 & 1 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

Then layers 1, 2 and 3 of our latin cuboid could be as shown below.

$$\begin{array}{ccc|cccc} 1^* & 2^* & 3^* & 4 & 1 & 2 & 3 \\ 2^* & 3^* & 1^* & 1 & 2 & 3 & 4 \\ 3^* & 1^* & 2^* & 2 & 3 & 4 & 1 \\ \hline 3 & 4 & 1 & 1^* & 2^* & 3^* & 2 \\ 2 & 3 & 4 & 2^* & 3^* & 1 & 1^* \\ 1 & 2 & 3 & 3^* & 4 & 1^* & 2^* \\ 4 & 1 & 2 & 3 & 1^* & 2^* & 3^* \end{array} \qquad \begin{array}{ccc|cccc} 2^* & 3^* & 1^* & 1 & 4 & 3 & 2 \\ 3^* & 1^* & 2^* & 2 & 1 & 4 & 3 \\ 1^* & 2^* & 3^* & 3 & 2 & 1 & 4 \\ \hline 4 & 1 & 2 & 3^* & 3 & 1^* & 2^* \\ 1 & 2 & 3 & 4 & 1^* & 2^* & 3^* \\ 2 & 3 & 4 & 1^* & 2^* & 3^* & 1 \\ 3 & 4 & 1 & 2^* & 3^* & 2 & 1^* \end{array} \qquad \begin{array}{ccc|cccc} 3^* & 1^* & 2^* & 3 & 2 & 1 & 4 \\ 1^* & 2^* & 3^* & 4 & 3 & 2 & 1 \\ 2^* & 3^* & 1^* & 1 & 4 & 3 & 2 \\ \hline 1 & 2 & 3 & 2^* & 3^* & 4 & 1^* \\ 4 & 1 & 2 & 1^* & 2^* & 3^* & 3 \\ 3 & 4 & 1 & 2 & 1^* & 2^* & 3^* \\ 2 & 3 & 4 & 3^* & 1 & 1^* & 2^* \end{array}$$

Together these layers form a non-extendible $7 \times 7 \times 3$ latin cuboid.

4 Concluding remarks

The case $m = 2$ is a genuine exception in Theorem 7, in the sense that there are certainly no $3 \times 3 \times 1$ non-extendible latin cuboids. We do not know whether $m = 6$ is a genuine exception, or just an artifact of our construction. It would also be interesting to know whether an analogue of Theorem 7 holds for odd m. The same construction cannot be used since, at least when $m \in \{3, 5\}$, the cuboid is not just extendible, it is completable.

The existence of an at most half-full non-extendible latin cuboid of an order $n > 6$ with $n \not\equiv 3 \mod 4$ remains open. It does not seem feasible to use a random search to answer such questions even for quite small orders. We searched for random examples of non-extendible latin cuboids of orders $8, 9, 10, 11$ but did not find any that are at most half-full.

If one is interested in non-extendible latin cuboids that are more than half-full, such things can be obtained by varying the construction in Theorem 7 slightly. Suppose we had a latin cuboid that agreed with the one from Theorem 7 in all but c cells. Then it could be extended by at most c layers since each additional layer must make use of one of the changes, otherwise it could have been added to the original. By this method it will often be possible to create thicker non-extendible examples given any thin non-extendible specimen.

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Appendix

In this appendix we give the examples that prove Lemma 2. To save space, we write $\bar{x}$ instead of x^* for $x = 1, 2, \ldots, 7$. We begin with an awkward $8 \times 8 \times 4$ latin cuboid:

$$
\begin{array}{c|c}
2\bar{1}34 & 1\bar{2}\bar{3}\bar{4} \\
\bar{2}243 & \bar{1}1\bar{4}\bar{3} \\
3412 & \bar{3}\bar{4}\bar{1}\bar{2} \\
4321 & \bar{4}\bar{3}\bar{2}\bar{1} \\
\hline
1\bar{2}\bar{3}\bar{4} & 2\bar{1}34 \\
\bar{1}\bar{3}\bar{4}\bar{2} & 3241 \\
\bar{3}\bar{4}\bar{2}\bar{1} & 4312 \\
\bar{4}1\bar{1}\bar{3} & \bar{2}423
\end{array}
\qquad
\begin{array}{c|c}
\bar{1}243 & \bar{3}\bar{4}1\bar{2} \\
2134 & \bar{4}\bar{3}\bar{2}\bar{1} \\
4321 & \bar{1}\bar{2}\bar{3}\bar{4} \\
3412 & \bar{2}\bar{1}\bar{4}\bar{3} \\
\hline
\bar{2}\bar{1}\bar{4}\bar{3} & 1243 \\
1\bar{2}\bar{3}\bar{4} & 23\bar{1}4 \\
\bar{4}\bar{3}\bar{1}\bar{2} & 3421 \\
\bar{3}\bar{4}\bar{2}\bar{1} & 4132
\end{array}
\qquad
\begin{array}{c|c}
342\bar{1} & \bar{4}1\bar{2}\bar{3} \\
4312 & \bar{2}\bar{1}\bar{3}\bar{4} \\
\bar{2}234 & 1\bar{3}\bar{4}\bar{1} \\
2143 & \bar{3}\bar{4}\bar{1}\bar{2} \\
\hline
\bar{4}\bar{3}\bar{1}\bar{2} & 4321 \\
\bar{3}\bar{4}\bar{2}1 & \bar{1}432 \\
1\bar{1}\bar{4}\bar{3} & 2\bar{2}43 \\
\bar{1}\bar{2}\bar{3}\bar{4} & 3214
\end{array}
\qquad
\begin{array}{c|c}
43\bar{1}2 & \bar{2}\bar{3}\bar{4}1 \\
3421 & \bar{3}\bar{4}\bar{1}\bar{2} \\
2143 & \bar{4}\bar{1}\bar{2}\bar{3} \\
\bar{2}234 & \bar{1}1\bar{3}\bar{4} \\
\hline
\bar{3}\bar{4}\bar{2}\bar{1} & 3412 \\
\bar{4}\bar{1}1\bar{3} & 4\bar{2}23 \\
\bar{1}\bar{2}\bar{3}\bar{4} & 1234 \\
1\bar{3}\bar{4}\bar{2} & 234\bar{1}
\end{array}
$$

Here is an awkward $10 \times 10 \times 5$ latin cuboid:

$\bar{1}2345$	$1\bar{2}\bar{3}\bar{4}\bar{5}$	$5\bar{1}234$	$\bar{2}\bar{3}\bar{4}\bar{5}1$	$45\bar{1}23$	$\bar{5}\bar{4}1\bar{2}\bar{3}$	$345\bar{1}2$	$\bar{3}\bar{5}\bar{2}1\bar{4}$	$2345\bar{1}$	$\bar{4}1\bar{5}\bar{3}\bar{2}$
23451	$\bar{5}\bar{1}\bar{2}\bar{3}\bar{4}$	$\bar{2}2345$	$\bar{1}1\bar{3}\bar{4}\bar{5}$	51234	$\bar{3}\bar{5}\bar{4}\bar{1}\bar{2}$	45123	$\bar{4}\bar{3}\bar{5}\bar{2}\bar{1}$	34512	$\bar{2}\bar{4}\bar{1}\bar{5}\bar{3}$
34512	$\bar{4}\bar{5}\bar{1}\bar{2}\bar{3}$	23451	$\bar{5}\bar{1}\bar{2}\bar{3}\bar{4}$	$\bar{2}2345$	$1\bar{3}\bar{5}\bar{4}\bar{1}$	51234	$\bar{1}\bar{4}\bar{3}\bar{5}\bar{2}$	45123	$\bar{3}\bar{2}\bar{4}\bar{1}\bar{5}$
45123	$\bar{3}\bar{4}\bar{5}\bar{1}\bar{2}$	34512	$\bar{4}\bar{5}\bar{1}\bar{2}\bar{3}$	23451	$\bar{1}\bar{2}\bar{3}\bar{5}\bar{4}$	$\bar{2}2345$	$1\bar{1}\bar{4}\bar{3}\bar{5}$	51234	$\bar{5}\bar{3}\bar{2}\bar{4}\bar{1}$
51234	$\bar{2}\bar{3}\bar{4}\bar{5}\bar{1}$	45123	$\bar{3}\bar{4}\bar{5}\bar{1}\bar{2}$	34512	$\bar{4}\bar{1}\bar{2}\bar{3}\bar{5}$	23451	$\bar{5}\bar{2}\bar{1}\bar{4}\bar{3}$	$\bar{2}2345$	$\bar{1}\bar{5}\bar{3}1\bar{4}$
$1\bar{2}\bar{3}\bar{4}\bar{5}$	$\bar{1}2543$	$\bar{1}1\bar{4}\bar{5}\bar{3}$	$5\bar{2}432$	$\bar{3}\bar{4}\bar{5}\bar{2}\bar{1}$	45321	$\bar{5}\bar{3}\bar{2}1\bar{4}$	$342\bar{1}5$	$\bar{4}\bar{5}\bar{1}\bar{3}\bar{2}$	23154
$\bar{5}\bar{1}\bar{2}\bar{3}\bar{4}$	23451	$\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}$	12345	$\bar{1}\bar{3}\bar{4}\bar{5}\bar{2}$	51234	$\bar{4}\bar{5}\bar{3}\bar{2}\bar{1}$	45123	$1\bar{4}\bar{5}\bar{1}\bar{3}$	$345\bar{2}2$
$\bar{4}\bar{5}\bar{1}\bar{2}\bar{3}$	35214	$\bar{5}\bar{3}\bar{2}\bar{1}\bar{4}$	24153	$1\bar{1}\bar{3}\bar{4}\bar{5}$	$\bar{2}3542$	$\bar{1}\bar{4}\bar{5}\bar{3}\bar{2}$	52431	$\bar{3}\bar{2}\bar{4}\bar{5}1$	$4\bar{1}325$
$\bar{3}\bar{4}\bar{5}\bar{1}\bar{2}$	41325	$\bar{4}\bar{5}\bar{3}\bar{2}\bar{1}$	35214	$\bar{5}\bar{2}1\bar{3}\bar{4}$	$24\bar{1}53$	$1\bar{1}\bar{4}\bar{5}\bar{3}$	$\bar{2}3542$	$\bar{1}\bar{3}\bar{2}\bar{4}\bar{5}$	52431
$\bar{2}\bar{3}\bar{4}\bar{5}\bar{1}$	54132	$1\bar{4}\bar{5}\bar{3}\bar{2}$	$4352\bar{1}$	$\bar{4}\bar{5}\bar{2}1\bar{3}$	32415	$\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}$	21354	$\bar{5}\bar{1}\bar{3}\bar{2}\bar{4}$	15243

Next we give an awkward $12 \times 12 \times 6$ latin cuboid:

$2\bar{1}3456$	$1\bar{4}\bar{3}\bar{6}\bar{5}\bar{2}$	$\bar{1}62345$	$\bar{5}\bar{2}1\bar{4}\bar{3}\bar{6}$	$65\bar{1}234$	$\bar{4}1\bar{6}\bar{3}\bar{2}\bar{5}$
$\bar{2}34562$	$\bar{1}1\bar{4}\bar{3}\bar{6}\bar{5}$	623451	$\bar{6}\bar{5}\bar{2}\bar{1}\bar{4}\bar{3}$	512346	$\bar{2}\bar{1}\bar{3}\bar{4}\bar{5}\bar{6}$
345621	$\bar{3}\bar{2}\bar{5}\bar{4}\bar{1}\bar{6}$	234516	$\bar{1}\bar{6}\bar{3}\bar{2}\bar{5}\bar{4}$	$\bar{2}23465$	$1\bar{3}\bar{4}\bar{5}\bar{6}\bar{1}$
456213	$\bar{4}\bar{3}\bar{6}\bar{5}\bar{2}\bar{1}$	345162	$\bar{2}\bar{1}\bar{4}\bar{3}\bar{6}\bar{5}$	234651	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{1}\bar{2}$
562134	$\bar{5}\bar{6}\bar{1}\bar{2}\bar{3}\bar{4}$	451623	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{1}\bar{2}$	346512	$\bar{6}\bar{5}\bar{2}\bar{1}\bar{4}\bar{3}$
621345	$\bar{6}\bar{5}\bar{2}\bar{1}\bar{4}\bar{3}$	516234	$\bar{4}\bar{3}\bar{6}\bar{5}\bar{2}\bar{1}$	465123	$\bar{5}\bar{6}\bar{1}\bar{2}\bar{3}\bar{4}$
$1\bar{2}\bar{4}\bar{3}\bar{5}\bar{6}$	$2\bar{1}4356$	$\bar{2}\bar{1}\bar{5}\bar{4}\bar{6}\bar{3}$	163245	$\bar{5}\bar{6}\bar{2}\bar{1}\bar{3}\bar{4}$	652134
$\bar{1}\bar{3}\bar{5}\bar{2}\bar{6}\bar{4}$	342561	$1\bar{5}\bar{6}\bar{3}\bar{4}\bar{2}$	$23\bar{1}456$	$\bar{6}\bar{4}1\bar{5}\bar{2}\bar{3}$	$\bar{1}26345$
$\bar{3}\bar{4}\bar{6}\bar{1}\bar{2}\bar{5}$	435612	$\bar{4}\bar{3}\bar{2}\bar{6}\bar{5}\bar{1}$	324561	$1\bar{5}\bar{3}\bar{4}\bar{1}\bar{6}$	$2\bar{2}3456$
$\bar{5}1\bar{1}\bar{6}\bar{4}\bar{3}$	$\bar{2}56423$	$\bar{3}\bar{2}\bar{4}\bar{5}\bar{1}\bar{6}$	645312	$\bar{4}\bar{3}\bar{6}\bar{2}\bar{5}\bar{1}$	534261
$\bar{4}\bar{6}\bar{2}\bar{5}\bar{3}\bar{1}$	561234	$\bar{6}\bar{4}\bar{3}\bar{1}\bar{2}\bar{5}$	456123	$\bar{3}\bar{1}\bar{5}\bar{6}\bar{4}\bar{2}$	345612
$\bar{6}\bar{5}\bar{3}\bar{4}\bar{1}\bar{2}$	623145	$\bar{5}\bar{6}\bar{1}\bar{2}\bar{3}\bar{4}$	512634	$\bar{1}\bar{2}\bar{4}\bar{3}\bar{6}\bar{5}$	461523

$546\bar{1}23$	$\bar{2}\bar{5}\bar{4}1\bar{6}\bar{3}$	$4356\bar{1}2$	$\bar{3}\bar{6}\bar{5}\bar{2}1\bar{4}$	$32456\bar{1}$	$\bar{6}\bar{3}\bar{2}\bar{5}41$
461235	$\bar{5}\bar{6}\bar{1}\bar{2}\bar{3}\bar{4}$	356124	$\bar{4}\bar{3}\bar{6}\bar{5}\bar{2}\bar{1}$	245613	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{1}\bar{2}$
612354	$\bar{6}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}$	561243	$\bar{5}\bar{4}\bar{1}\bar{6}\bar{3}\bar{2}$	456132	$\bar{4}\bar{5}\bar{6}\bar{1}\bar{2}\bar{3}$
$\bar{2}23546$	$\bar{1}1\bar{3}\bar{4}\bar{5}\bar{6}$	612435	$\bar{6}\bar{5}\bar{2}\bar{1}\bar{4}\bar{3}$	561324	$\bar{5}\bar{6}\bar{1}\bar{2}\bar{3}\bar{4}$
235461	$\bar{4}\bar{3}\bar{6}\bar{5}\bar{2}\bar{1}$	$\bar{2}24356$	$1\bar{1}\bar{3}\bar{4}\bar{5}\bar{6}$	613245	$\bar{1}\bar{2}\bar{4}\bar{3}\bar{6}\bar{5}$
354612	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{1}\bar{2}$	243561	$\bar{1}\bar{2}\bar{4}\bar{3}\bar{6}\bar{5}$	$\bar{2}32456$	$1\bar{1}\bar{3}\bar{4}\bar{5}\bar{6}$
$\bar{6}\bar{3}\bar{1}\bar{2}\bar{4}\bar{5}$	541623	$\bar{3}\bar{4}\bar{6}\bar{5}1\bar{2}$	$4365\bar{1}2$	$\bar{4}\bar{5}\bar{3}\bar{6}\bar{2}1$	$32546\bar{1}$
$\bar{4}\bar{1}\bar{3}1\bar{5}\bar{6}$	$6\bar{2}5234$	$\bar{5}\bar{2}\bar{4}\bar{6}\bar{3}\bar{1}$	564123	$\bar{3}\bar{6}\bar{2}\bar{4}\bar{1}\bar{5}$	453612
$\bar{1}\bar{6}\bar{4}\bar{5}\bar{3}\bar{2}$	162345	$\bar{6}\bar{1}\bar{5}\bar{2}\bar{4}\bar{3}$	651234	$\bar{5}\bar{2}\bar{1}\bar{3}\bar{6}\bar{4}$	546123
$1\bar{5}\bar{2}\bar{3}\bar{6}\bar{4}$	$423\bar{1}56$	$\bar{1}\bar{6}\bar{3}\bar{4}\bar{2}\bar{5}$	312645	$\bar{6}\bar{4}\bar{5}\bar{1}\bar{3}\bar{2}$	261534
$\bar{5}\bar{2}\bar{6}\bar{4}\bar{1}\bar{3}$	234561	$1\bar{5}\bar{1}\bar{3}\bar{6}\bar{4}$	$\bar{2}23456$	$\bar{1}\bar{3}\bar{4}\bar{2}\bar{5}\bar{6}$	612345
$\bar{3}\bar{4}\bar{5}\bar{6}\bar{2}\bar{1}$	356412	$\bar{4}\bar{3}\bar{2}\bar{1}\bar{5}\bar{6}$	245361	$1\bar{1}\bar{6}\bar{5}\bar{4}\bar{3}$	$\bar{2}34256$

Finally, we present an awkward $14 \times 14 \times 7$ latin cuboid:

$\bar{1}234567$	$1\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}$	$7\bar{1}23456$	$\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}1$	$67\bar{1}2345$	$\bar{3}\bar{4}1\bar{6}\bar{7}\bar{2}\bar{5}$	$567\bar{1}234$	$\bar{6}\bar{7}\bar{5}1\bar{2}\bar{3}\bar{4}$
2345671	$\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}$	$\bar{2}234567$	$\bar{1}1\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}$	7123456	$\bar{5}\bar{3}\bar{4}\bar{1}\bar{6}\bar{7}\bar{2}$	6712345	$\bar{4}\bar{6}\bar{7}\bar{5}\bar{1}\bar{2}\bar{3}$
3456712	$\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}$	2345671	$\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}$	$\bar{2}234567$	$1\bar{5}\bar{3}\bar{4}\bar{1}\bar{6}\bar{7}$	7123456	$\bar{3}\bar{4}\bar{6}\bar{7}\bar{5}\bar{1}\bar{2}$
4567123	$\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}$	3456712	$\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}$	2345671	$\bar{7}\bar{2}\bar{5}\bar{3}\bar{4}\bar{1}\bar{6}$	$\bar{2}234567$	$1\bar{3}\bar{4}\bar{6}\bar{7}\bar{5}\bar{1}$
5671234	$\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}$	4567123	$\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}$	3456712	$\bar{6}\bar{7}\bar{2}\bar{5}\bar{3}\bar{4}\bar{1}$	2345671	$\bar{1}\bar{2}\bar{3}\bar{4}\bar{6}\bar{7}\bar{5}$
6712345	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}$	5671234	$\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}$	4567123	$\bar{1}\bar{6}\bar{7}\bar{2}\bar{5}\bar{3}\bar{4}$	3456712	$\bar{5}\bar{1}\bar{2}\bar{3}\bar{4}\bar{6}\bar{7}$
7123456	$\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}$	6712345	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}$	5671234	$\bar{4}\bar{1}\bar{6}\bar{7}\bar{2}\bar{5}\bar{3}$	4567123	$\bar{7}\bar{5}\bar{1}\bar{2}\bar{3}\bar{4}\bar{6}$
$1\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}$	$\bar{1}257346$	$\bar{1}1\bar{4}\bar{5}\bar{6}\bar{7}\bar{3}$	$7\bar{2}46235$	$\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}$	6735124	$\bar{4}\bar{5}\bar{6}\bar{7}\bar{3}\bar{2}\bar{1}$	5624713
$\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}$	3126457	$\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}\bar{6}\bar{7}$	2715346	$1\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}$	$\bar{2}674235$	$\bar{1}\bar{4}\bar{5}\bar{6}\bar{7}\bar{3}\bar{2}$	7563124
$\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}$	4712563	$\bar{7}\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}\bar{6}$	3671452	$\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}$	2567341	$1\bar{1}\bar{4}\bar{5}\bar{6}\bar{7}\bar{3}$	$\bar{2}456237$
$\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}$	6471235	$\bar{6}\bar{7}\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}$	5367124	$\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}$	4256713	$\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}\bar{6}\bar{7}$	3145672
$\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}$	5634721	$\bar{5}\bar{6}\bar{7}\bar{3}\bar{2}\bar{1}\bar{4}$	4523617	$\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}$	$34\bar{1}2576$	$\bar{7}\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}\bar{6}$	$237\bar{1}465$
$\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}$	2365174	$\bar{4}\bar{5}\bar{6}\bar{7}\bar{3}\bar{2}\bar{1}$	1254763	$\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}\bar{4}$	7143652	$\bar{6}\bar{7}\bar{3}\bar{2}\bar{1}\bar{4}\bar{5}$	6732541
$\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}$	7543612	$1\bar{4}\bar{5}\bar{6}\bar{7}\bar{3}\bar{2}$	$643257\bar{1}$	$\bar{4}\bar{5}\bar{6}\bar{7}\bar{1}\bar{2}\bar{3}$	5321467	$\bar{5}\bar{6}\bar{7}\bar{3}\bar{2}\bar{1}\bar{4}$	4217356

$4567\bar{1}23$	$\bar{4}\bar{6}\bar{7}\bar{2}1\bar{5}\bar{3}$	$34567\bar{1}2$	$\bar{7}\bar{5}\bar{2}\bar{3}\bar{4}1\bar{6}$	$234567\bar{1}$	$\bar{5}1\bar{6}\bar{7}\bar{3}\bar{4}\bar{2}$
5671234	$\bar{3}\bar{4}\bar{6}\bar{7}\bar{2}\bar{1}\bar{5}$	4567123	$\bar{6}\bar{7}\bar{5}\bar{2}\bar{3}\bar{4}\bar{1}$	3456712	$\bar{2}\bar{5}\bar{1}\bar{6}\bar{7}\bar{3}\bar{4}$
6712345	$\bar{5}\bar{3}\bar{4}\bar{6}\bar{7}\bar{2}\bar{1}$	5671234	$\bar{1}\bar{6}\bar{7}\bar{5}\bar{2}\bar{3}\bar{4}$	4567123	$\bar{4}\bar{2}\bar{5}\bar{1}\bar{6}\bar{7}\bar{3}$
7123456	$\bar{1}\bar{5}\bar{3}\bar{4}\bar{6}\bar{7}\bar{2}$	6712345	$\bar{4}\bar{1}\bar{6}\bar{7}\bar{5}\bar{2}\bar{3}$	5671234	$\bar{3}\bar{4}\bar{2}\bar{5}\bar{1}\bar{6}\bar{7}$
$\bar{2}234567$	$1\bar{1}\bar{5}\bar{3}\bar{4}\bar{6}\bar{7}$	7123456	$\bar{3}\bar{4}1\bar{6}\bar{7}\bar{5}\bar{2}$	6712345	$\bar{7}\bar{3}\bar{4}\bar{2}\bar{5}\bar{1}\bar{6}$
2345671	$\bar{7}\bar{2}\bar{1}\bar{5}\bar{3}\bar{4}\bar{6}$	$\bar{2}234567$	$1\bar{3}\bar{4}\bar{1}\bar{6}\bar{7}\bar{5}$	7123456	$\bar{6}\bar{7}\bar{3}\bar{4}\bar{2}\bar{5}\bar{1}$
3456712	$\bar{6}\bar{7}\bar{2}\bar{1}\bar{5}\bar{3}\bar{4}$	2345671	$\bar{5}\bar{2}\bar{3}\bar{4}\bar{1}\bar{6}\bar{7}$	$\bar{2}234567$	$\bar{1}\bar{6}\bar{7}\bar{3}\bar{4}1\bar{5}$
$\bar{5}\bar{7}\bar{1}\bar{2}\bar{4}\bar{3}\bar{6}$	4513672	$\bar{6}\bar{3}\bar{7}\bar{1}\bar{2}\bar{4}\bar{5}$	3472561	$\bar{7}\bar{6}\bar{2}\bar{3}\bar{1}\bar{5}\bar{4}$	2361457
$\bar{6}\bar{5}\bar{7}\bar{1}\bar{2}\bar{4}\bar{3}$	6452713	$\bar{5}\bar{6}\bar{3}\bar{7}\bar{1}\bar{2}\bar{4}$	5341672	$\bar{4}\bar{7}\bar{6}\bar{2}\bar{3}\bar{1}\bar{5}$	4237561
$\bar{3}\bar{6}\bar{5}\bar{7}1\bar{2}\bar{4}$	$7345\bar{1}26$	$\bar{4}\bar{5}\bar{6}\bar{3}\bar{7}\bar{1}\bar{2}$	$62347\bar{1}5$	$\bar{5}\bar{4}\bar{7}\bar{6}\bar{2}\bar{3}1$	$5\bar{1}23674$
$\bar{4}\bar{3}\bar{6}\bar{5}\bar{7}\bar{1}\bar{2}$	2734561	$1\bar{4}\bar{5}\bar{6}\bar{3}\bar{7}\bar{1}$	$\bar{2}623457$	$\bar{1}\bar{5}\bar{4}\bar{7}\bar{6}\bar{2}\bar{3}$	7512346
$1\bar{4}\bar{3}\bar{6}\bar{5}\bar{7}\bar{1}$	$\bar{2}267354$	$\bar{1}\bar{2}\bar{4}\bar{5}\bar{6}\bar{3}\bar{7}$	7156243	$\bar{3}\bar{1}\bar{5}\bar{4}\bar{7}\bar{6}\bar{2}$	6745132
$\bar{1}\bar{2}\bar{4}\bar{3}\bar{6}\bar{5}\bar{7}$	5621437	$\bar{7}\bar{1}\bar{2}\bar{4}\bar{5}\bar{6}\bar{3}$	4517326	$1\bar{3}\bar{1}\bar{5}\bar{4}\bar{7}\bar{6}$	$34762\bar{2}5$
$\bar{7}\bar{1}\bar{2}\bar{4}\bar{3}\bar{6}\bar{5}$	3176245	$\bar{3}\bar{7}\bar{1}\bar{2}\bar{4}\bar{5}\bar{6}$	2765134	$\bar{6}\bar{2}\bar{3}\bar{1}\bar{5}\bar{4}\bar{7}$	1654723