# QUANTUM MEASURES and INTEGRALS

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**Abstract**

We show that quantum measures and integrals appear naturally in any $L_2$-Hilbert space $H$. We begin by defining a decoherence operator $D(A,B)$ and it's associated $q$-measure operator $\mu(A)=D(A,A)$ on $H$. We show that these operators have certain positivity, additivity and continuity properties. If $\rho$ is a state on $H$, then $D_\rho(A,B)=\mathrm{tr}\left[\rho D(A,B)\right]$ and $\mu_\rho(A)=D_\rho(A,A)$ have the usual properties of a decoherence functional and $q$-measure, respectively. The quantization of a random variable $f$ is defined to be a certain self-adjoint operator $\widehat{f}$ on $H$. Continuity and additivity properties of the map $f\mapsto\widehat{f}$ are discussed. It is shown that if $f$ is nonnegative, then $\widehat{f}$ is a positive operator. A quantum integral is defined by $\int f d\mu_\rho=\mathrm{tr}(\rho\widehat{f}\,)$. A tail-sum formula is proved for the quantum integral. The paper closes with an example that illustrates some of the theory.

**Keywords:** quantum measures, quantum integrals, decoherence functionals.

## 1 Introduction

Quantum measure theory was introduced by R. Sorkin in his studies of the histories approach to quantum gravity and cosmology [11, 12]. Since 1994 a considerable amount of literature has been devoted to this subject [1, 3, 5, 9, 10, 13, 15] and more recently a quantum integral has been introduced [6, 7]. At first sight this theory appears to be quite specialized and its applicability has been restricted to the investigation of quantum histories and the related coevent interpretation of quantum mechanics [4, 7, 8, 14]. However, this article intends to demonstrate that quantum measure theory may have wider application and that its mathematical structure is already present in the standard quantum formalism. One of our aims is to show that quantum measures are abundant in any $L_2$-Hilbert space $H$. In particular, for any state (density operator) $\rho$ on $H$ there is a naturally associated quantum measure $\mu_\rho$. For an event $A$ we interpret

$\mu_\rho(A)$ as the quantum propensity that $A$ occurs. Moreover corresponding to $\rho$ there is a natural quantum integral $\int f d\mu_\rho$ that can be interpreted as the quantum expectation of the random variable $f$.

The article begins by defining a decoherence operator $D(A,B)$ for events $A,B$ and the associated $q$-measure operator $\mu(A)=D(A,A)$ on $H$. It is shown that these operator-valued functions have certain positivity, additivity and continuity properties. Of particular importance is the fact that although $\mu_\rho(A)$ is not additive, it does satisfy a more general grade-2 additivity condition. If $\rho$ is a state on $H$, then $D_\rho(A,B)=\operatorname{tr}\left[\rho D(A,B)\right]$ and $\mu_\rho(A)=D_\rho(A,A)$ have the usual properties of a decoherence functional and $q$-measure, respectively. The quantization of a random variable $f$ is defined to be a certain self-adjoint operator $\widehat{f}$ on $H$. Continuity and additivity properties of the map $f\mapsto\widehat{f}$ are discussed. It is shown that if $f$ is nonnegative, then $\widehat{f}$ is a positive operator. A quantum integral is defined by $\int f d\mu_\rho=\operatorname{tr}(\rho\widehat{f})$. A tail-sum formula is proved for the quantum integral. It follows that $\int f d\mu_\rho$ coincides with the quantum integral considered in previous works. The paper closes with an example that illustrates some of the theory. The example shows that the usual decoherence functionals and $q$-measures considered before reduce to the form given in Section 2.

## 2 Quantum Measures

A *probability space* is a triple $(\Omega,\mathcal{A},\nu)$ where $\Omega$ is a *sample space* whose elements are *sample points* or *outcomes*, $\mathcal{A}$ is a $\sigma$-algebra of subsets of $\Omega$ called *events* and $\nu$ is a measure on $\mathcal{A}$ satisfying $\nu(\Omega)=1$. For $A\in\mathcal{A}$, $\nu(A)$ is interpreted as the probability that event $A$ occurs. Let $H$ be the Hilbert space

$$H=L_2(\Omega,\mathcal{A},\nu)=\left\{f\colon\Omega\to\mathbb{C},\int\left|f\right|^2d\nu<\infty\right\}$$

with inner product $\langle f,g\rangle=\int\bar{f}gd\nu$. We call real-valued functions $f\in H$ *random variables*. If $f$ is a random variable, then by Schwarz's inequality

$$\left|\int fd\nu\right|\le\int\left|f\right|d\nu\le\|f\| \tag{2.1}$$

so the *expectation* $E(f)=\int fd\nu$ exists and is finite. Of course (2.1) holds for any $f\in H$.

The characteristic function $\chi_A$ of $A\in\mathcal{A}$ is a random variable with $\|\chi_A\|=\nu(A)^{1/2}$ and we write $\chi_\Omega=1$. For $A,B\in\mathcal{A}$ we define the *decoherence operator* $D(A,B)$ as the operator on $H$ defined by $D(A,B)=|\chi_A\rangle\langle\chi_B|$. Thus, for $f\in H$ we have

$$D(A,B)f=\langle\chi_B,f\rangle\chi_A=\int_B fd\nu\chi_A$$

Of course, if $\nu(A)\nu(B)=0$, then $D(A,B)=0$.

**Lemma 2.1.** *If $\nu(A)\nu(B) \neq 0$, then $D(A,B)$ is a rank 1 operator with $\|D(A,B)\| = \nu(A)^{1/2}\nu(B)^{1/2}$.*

*Proof.* It is clear that $D(A,B)$ is a rank 1 operator with range $\operatorname{span}(\chi_A)$. For $f \in H$ we have

$$\begin{aligned}\|D(A,B)f\| &= \|\langle \chi_B, f\rangle \chi_A\| = |\langle \chi_B, f\rangle|\,\|\chi_A\| \\ &\leq \|\chi_A\|\,\|\chi_B\|\,\|f\| = \nu(A)^{1/2}\nu(B)^{1/2}\,\|f\|\end{aligned}$$

Hence, $\|D(A,B)\| \leq \nu(A)^{1/2}\nu(B)^{1/2}$. Letting $g$ be the unit vector $\chi_B/\|\chi_B\|$ we have

$$\|D(A,B)g\| = |\langle \chi_B, g\rangle|\,\|\chi_A\| = \|\chi_B\|\,\|\chi_A\| = \nu(A)^{1/2}\nu(B)^{1/2}$$

The result now follows. $\square$

For $A \in \mathcal{A}$ we define the *q-measure operator* $\mu(A)$ on $H$ by

$$\mu(A) = D(A,A) = |\chi_A\rangle\langle\chi_A|$$

Hence,

$$\mu(A)f = \langle \chi_A, f\rangle \chi_A = \int_A f d\nu \chi_A$$

If $\nu(A) \neq 0$, then by Lemma 2.1, $\mu(A)$ is a positive (and hence self-adjoint) rank 1 operator with $\|\mu(A)\| = \nu(A)$.

We now show that $A \mapsto \chi_A$ is a *vector-valued measure* on $\mathcal{A}$. Indeed, if $A \cap B = \emptyset$, then $\chi_{A\cup B} = \chi_A + \chi_B$ so $A \mapsto \chi_A$ is *additive*. Moreover, if $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of events, then letting $A = \cup A_i$ we have

$$\|\chi_A - \chi_{A_n}\|^2 = \|\chi_{A\smallsetminus A_n}\|^2 = \nu(A \smallsetminus A_n) = \nu(A) - \nu(A_n) \to 0$$

Hence, $\lim \chi_{A_n} = \chi_{\cup A_i}$. The *countable additivity condition*

$$\chi_{\cup B_i} = \sum \chi_{B_i}$$

follows for mutually disjoint $B_i \in \mathcal{A}$ where the convergence of the sum is in the vector norm topology. Since $\chi_A$ is orthogonal to $\chi_B$ whenever $A \cap B = \emptyset$, we call $A \mapsto \chi_A$ an *orthogonally scattered* vector-valued measure. A similar computation shows that if $A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing sequence on $\mathcal{A}$, then

$$\lim \chi_{A_n} = \chi_{\cap A_i}$$

This also follows from the fact that the complements $A_i'$ form an increasing sequence so by additivity

$$\begin{aligned}\lim \chi_{A_n} &= 1 - \lim \chi_{A_n'} = 1 - \chi_{\cup A_i'} = 1 - \left[\chi_{(\cap A_i)'}\right] \\ &= \chi_{\cap A_i}\end{aligned}$$

The map $D$ from $\mathcal{A} \times \mathcal{A}$ into the set of bounded operators $\mathcal{B}(H)$ on $H$ has some obvious properties:

(1) If $A \cap B = \emptyset$, then $D(A \cup B, C) = D(A, C) + D(B, C)$ for all $C \in \mathcal{A}$ (additivity)

(2) $D(A, B)^* = D(B, A)$ (conjugate symmetry)

(3) $D(A, B)^2 = \nu(A \cap B) D(A, B)$

(4) $D(A, B) D(A, B)^* = \nu(B)\mu(A)$, $D(A, B)^* D(A, B) = \nu(A)\mu(B)$

Less obvious properties are given in the following theorem.

**Theorem 2.2.** (a) *$D\colon \mathcal{A} \times \mathcal{A} \to \mathcal{B}(H)$ is positive semidefinite in the sense that if $A_i \in \mathcal{A}$, $c_i \in \mathbb{C}$, $i = 1, \ldots, n$, then*

$$\sum_{i,j=1}^{n} D(A_i, A_j) c_i \bar{c}_j$$

*is a positive operator.* (b) *If $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence in $\mathcal{A}$, then the continuity condition*

$$\lim D(A_i, B) = D(\cup A_i, B)$$

*holds for every $B \in \mathcal{A}$ where the limit is in the operator norm topology.*

*Proof.* (a) For $A_i \in \mathcal{A}$, $c_i \in \mathbb{C}$, $i = 1, \ldots, n$, we have

$$\sum_{i,j=1}^{n} D(A_i, A_j) c_i \bar{c}_j = \sum_{i,j=1}^{n} |\chi_{A_i}\rangle\langle\chi_{A_j}| c_i \bar{c}_j = \left|\sum_{i=1}^{n} c_i \chi_{A_i}\right\rangle\left\langle\sum_{j=1}^{n} c_j \chi_{A_j}\right| \tag{2.2}$$

Since the right side of (2.2) is a positive operator, the result follows.
(b) For the increasing sequence $A_i$, let $A = \cup A_i$ and let $f \in H$. Then

$$\begin{aligned}
\|[D(A, B) - D(A_i, B)]\, f\| &= \left\|\int_B f d\nu (\chi_A - \chi_{A_i}\right\| = \left|\int_B f d\nu\right| \|\chi_A - \chi_{A_i}\| \\
&= \left|\int_B f d\nu\right| [\nu(A) - \nu(A_i)]^{1/2} \\
&\le \int_B |f|\, d\nu\, [\nu(A) - \nu(A_i)]^{1/2} \\
&\le [\nu(A) - \nu(A_i)]^{1/2} \|f\|
\end{aligned}$$

Hence

$$\lim \|D(A, B) - D(A_i, B)\| \le \lim\, [\nu(A) - \nu(A_i)]^{1/2} = 0 \qquad \square$$

If $A_i$ are mutually disjoint events, the countable additivity condition

$$D\left(\bigcup_{i=1}^{\infty} A_i, B\right) = \sum_{i=1}^{\infty} D(A_i, B)$$

follows from Theorem 2.2(b). We conclude that $A \mapsto D(A, B)$ is an operator-valued measure. By conjugate symmetry, $B \mapsto D(A, B)$ is also an operator-valued measure. As before, it follows that if $A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing sequence in $\mathcal{A}$, then

$$\lim D(A_i, B) = D(\cap A_i, B)$$

for all $B \in \mathcal{A}$.

The map $\mu \colon \mathcal{A} \to \mathcal{B}(H)$ need not be additive. For example, if $A, B \in \mathcal{A}$ are disjoint, then

$$\begin{aligned}
\mu(A \cup B) &= |\chi_{A\cup B}\rangle\langle\chi_{A\cup B}| = |\chi_A + \chi_B\rangle\langle\chi_A + \chi_B| \\
&= |\chi_A\rangle\langle\chi_A| + |\chi_B\rangle\langle\chi_B| + |\chi_A\rangle\langle\chi_B| + |\chi_B\rangle\langle\chi_A| \\
&= \mu(A) + \mu(B) + 2\mathrm{Re}\, D(A, B)
\end{aligned}$$

Notice that additivity is spoiled by the presence of the self-adjoint operator $2\mathrm{Re}\, D(A, B)$. For this reason, we view this operator as measuring the interference between the events $A$ and $B$. Because of this nonadditivity, we have that $\mu(A') \neq \mu(\Omega) - \mu(A)$ in general and $A \subseteq B$ need not imply $\mu(A) \neq \mu(B)$ in the usual order of self-adjoint operators. However, $A \mapsto \mu(A)$ does satisfy the condition given in (a) of the next theorem.

**Theorem 2.3.** (a) *$\mu$ satisfies grade-2 additivity:*

$$\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C)$$

*whenever $A, B, C \in \mathcal{A}$ are mutually disjoint.* (b) *$\mu$ satisfies the continuity conditions*

$$\begin{aligned}
\lim \mu(A_i) &= \mu(\cup A_i) \\
\lim \mu(B_i) &= \mu(\cap A_i)
\end{aligned}$$

*in the operator norm topology for any increasing sequence $A_i$ in $\mathcal{A}$ or decreasing sequence $B_i \in \mathcal{A}$.*

*Proof.* (a) For $A, B, C \in \mathcal{A}$ mutually disjoint, we have

$$\begin{aligned}
&\mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C) \\
&\quad = |\chi_A + \chi_B\rangle\langle\chi_A + \chi_B| + |\chi_A + \chi_C\rangle\langle\chi_A + \chi_C| + |\chi_B + \chi_C\rangle\langle\chi_B + \chi_C| \\
&\qquad - |\chi_A\rangle\langle\chi_A| - |\chi_B\rangle\langle\chi_B| - |\chi_C\rangle\langle\chi_C| \\
&\quad = |\chi_A\rangle\langle\chi_A| + |\chi_B\rangle\langle\chi_B| + |\chi_C\rangle\langle\chi_C| + |\chi_A\rangle\langle\chi_B| + |\chi_B\rangle\langle\chi_A| \\
&\qquad + |\chi_A\rangle\langle\chi_C| + |\chi_C\rangle\langle\chi_A| + |\chi_B\rangle\langle\chi_C| + |\chi_C\rangle\langle\chi_C| \\
&\quad = |\chi_A + \chi_B + \chi_C\rangle\langle\chi_A + \chi_B + \chi_C| = \mu(A \cup B \cup C)
\end{aligned}$$

(b) For an increasing sequence $A_i \in \mathcal{A}$, let $A = \cup A_i$. For $f \in L_2(\Omega, \mathcal{A}, \nu)$ we have

$$
\begin{aligned}
\|[\mu(A_i)-\mu(A)]\,f\| &= \left\|\int_{A_i} f d\nu\chi_{A_i} - \int_A f d\nu\chi_A\right\| \\
&\le \left\|\int_{A_i} f d\nu\chi_{A_i} - \int_{A_i} f d\nu\chi_A\right\| + \left\|\int_{A_i} f d\nu\chi_A - \int_A f d\nu\chi_A\right\| \\
&= \left\|\int_{A_i} f d\nu(\chi_{A_i}-\chi_A)\right\| + \left\|\int f(\chi_{A_i}-\chi_A) d\nu\chi_A\right\| \\
&= \left|\int_{A_i} f d\nu\right| \|\chi_A-\chi_{A_i}\| + \left|\int f(\chi_A-\chi_{A_i})d\nu\right| \|\chi_A\| \\
&\le \int_{A_i} |f|\,d\nu\,[\nu(A)-\nu(A_i)]^{1/2} + \|f\|\,\|\chi_A-\chi_{A_i}\|\,\nu(A)^{1/2} \\
&\le 2\nu(A)^{1/2}\,[\nu(A)-\nu(A_i)]^{1/2}\,\|f\|
\end{aligned}
$$

Hence,

$$\|\mu(A_i)-\mu(A)\| \le 2\nu(A)^{1/2}\,[\nu(A)-\nu(A_i)]^{1/2} \to 0$$

A similar proof holds for a decreasing sequence $B_i \in \mathcal{A}$ □

Additional properties of $\mu$ are given in the next lemma.

**Lemma 2.4.** (a) *If $A \cap B = \emptyset$, then $\mu(A)\mu(B) = 0$.* (b) *$\mu(A) = 0$ if and only if $\nu(A) = 0$.* (c) *If $A \cap B = \emptyset$ and $\mu(A) = 0$, then $\mu(A \cup B) = \mu(B)$.* (d) *If $\mu(A \cup B) = 0$, then $\mu(A) = \mu(B) = 0$.*

*Proof.* That (a) holds is clear. (b) If $\mu(A) = 0$, then for any $f \in H$ we have

$$\int_A f d\nu\chi_A = \mu(A)f = 0$$

Letting $f = 1$ gives $\nu(A)\chi_A = 0$ a.e. $[\nu]$. Hence, $\nu(A) = 0$. The converse clearly holds. (c) If $A \cap B = \emptyset$ and $\mu(A) = 0$, then by (b) we have that $\nu(A) = 0$. Hence, for $f \in H$ we have

$$|\chi_A\rangle\langle\chi_B|f = \int_B f d\nu\chi_A = 0$$

We conclude that

$$\mu(A \cup B) = 2\mathrm{Re}\,|\chi_A\rangle\langle\chi_B| + |\chi_B\rangle\langle\chi_B| = \mu(B)$$

(d) If $\mu(A \cap B) = 0$, then by (b) we have that $\nu(A \cap B) = 0$. Since $\nu$ is additive, $\nu(A) = \nu(B) = 0$ so by (b), $\mu(A) = \mu(B) = 0$. □

If $\rho$ is a density operator (state) on $H$, we define the *decoherence functional* $D_\rho\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$D_\rho(A,B) = \mathrm{tr}\,[\rho D(A,B)] = \langle \rho\chi_A, \chi_B\rangle$$

Decoherence functionals have been extensively studied in the literature [1, 3, 9, 12, 13] where $D_\rho(A,B)$ is used to describe the interference between $A$ and $B$ for the state $\rho$. Concrete examples of $D_\rho(A,B)$ are given in Section 3. Notice that

$$|D_\rho(A,B)| \le \|\rho\| \, \|\chi_A\| \, \|\chi_B\| \le \nu(A)^{1/2}\nu(B)^{1/2} \tag{2.3}$$

The next result, which follows from Theorem 2.2, shows that $D_\rho(A,B)$ has the usual properties of a decoherence functional.

**Corollary 2.5.** (a) *$A \mapsto D_\rho(A,B)$ is a complex measure on $\mathcal{A}$ for any $B \in \mathcal{A}$.* (b) *If $A_1,\ldots,A_n \in \mathcal{A}$, then the $n\times n$ matrix $D_\rho(A_i,A_j)$ is positive semidefinite.*

For a density operator $\rho$ on $H$, we define the *q-measure* $\mu_\rho\colon \mathcal{A} \to \mathbb{R}^+$ by

$$\mu_\rho(A) = \operatorname{tr}\left[\rho\mu(A)\right] = \langle \rho\chi_A, \chi_A\rangle$$

We interpret $\mu_\rho(A)$ as the quantum propensity that the event $A$ occurs [5, 11, 12]. It follows from (2.3) that $\mu_\rho(A) \le \nu(A)$. Theorem 2.3 holds with $\mu$ replaced by $\mu_\rho$. This shows that $\mu_\rho$ has the usual properties of a $q$-measure.

# 3 Quantum Integrals

Let $f \in H$ be a nonnegative random variable. The *quantization* of $f$ is the operator $\widehat{f}$ on $H$ defined by

$$(\widehat{f}g)(y) = \int \min\left[f(x), f(y)\right] g(x) d\nu(x) \tag{3.1}$$

We can write (3.1) as

$$(\widehat{f}g)(y) = \int_{\{x\colon f(x)\le f(y)\}} fgd\nu + f(y)\int_{\{x\colon f(x) > f(y)\}} gd\nu$$

Since

$$\begin{aligned}\left|(\widehat{f}g)(y)\right| &\le \int \min\left[f(x), f(y)\right] |g(x)|\, d\nu(x) \le \int f\, |g|\, d\nu\\ &\le \|f\|\, \|g\|\end{aligned}$$

we have $\left\|\widehat{f}g\right\| \le \|f\|\,\|g\|$. We conclude that $\widehat{f}$ is bounded with $\left\|\widehat{f}\right\| \le \|f\|$. Since $\widehat{f}$ is bounded and symmetric, it follows that $\widehat{f}$ is a self-adjoint operator. If $f$ is an arbitrary random variable we can write $f = f^+ - f^-$ where $f^+(x) = \max\left[f(x), 0\right]$ and $f^-(x) = -\min\left[f(x), 0\right]$. Then we have that $f^+, f^- \ge 0$ and we define the *quantization* $\widehat{f} = f^{+\wedge} - f^{-\wedge}$. Again, $\widehat{f}$ is a bounded self-adjoint operator on $H$. According to the usual formalism, we can interpret $\widehat{f}$ as an observable for a quantum system.

**Theorem 3.1.** (a) *For any* $A \in \mathcal{A}$, $\widehat{\chi}_A = |\chi_A\rangle\langle\chi_A| = \mu(A)$. (b) *For any* $\alpha \in \mathbb{R}$, $(\alpha f)^\wedge = \alpha \widehat{f}$. (c) *If* $0 \le f_1 \le f_2 \le \cdots$ *is an increasing sequence of random variables converging pointwise to a random variable* $f$, *then* $\widehat{f}_i$ *converges to* $\widehat{f}$ *in the operator norm topology.* (d) *If* $f, g, h$ *are random variables with mutually disjoint support, then*

$$(f+g+h)^\wedge = (f+g)^\wedge + (f+h)^\wedge + (g+h)^\wedge - \widehat{f} - \widehat{g} - \widehat{h} \tag{3.2}$$

*Proof.* (a) For $A \in \mathcal{A}$ we have that $\min[\chi_A(x), \chi_A(y)] = \chi_A(x)\chi_A(y)$. Hence, for $g \in H$ we obtain

$$(\widehat{\chi}_A g)(y) = \int \chi_A(x)\chi_A(y)g(x)d\nu(x) = \int_A g d\nu \chi_A(y) = [\mu(A)g]\,(y)$$

(b) If $\alpha \ge 0$ and $f \ge 0$, then clearly $(\alpha f)^\wedge = \alpha \widehat{f}$. If $\alpha \ge 0$ and $f$ is a random variable, then

$$\alpha f = (\alpha f)^+ - (\alpha f)^- = \alpha f^+ - \alpha f^-$$

Hence,

$$(\alpha f)^\wedge = (\alpha f^+)^\wedge - (\alpha f^-)^\wedge = \alpha f^{+\wedge} - \alpha f^{-\wedge} = \alpha f^\wedge$$

If $\alpha < 0$ and $f$ is a random variable, then

$$\alpha f = (\alpha f)^+ - (\alpha f)^- = |\alpha|\, f^- - |\alpha|\, f^+$$

Hence,

$$(\alpha f)^\wedge = \left(|\alpha|\, f^-\right)^\wedge - \left(|\alpha|\, f^+\right)^\wedge = |\alpha|\, f^{-\wedge} - |\alpha|\, f^{+\wedge} = -|\alpha|\, \widehat{f} = \alpha \widehat{f}$$

(c) For any $g \in H$ we have

$$\begin{aligned}
\left|(\widehat{f} - \widehat{f}_i)g(y)\right| &= \left|\int \{\min[f(x), f(y)] - \min[f_i(x), f_i(y)]\}\, g(x)d\nu(x)\right| \\
&\le \int |\min[f(x), f(y)] - \min[f_i(x), f_i(y)]|\, |g(x)|\, d\nu(x) \\
&\le \int [|f(y) - f_i(y)| + |f(x) - f_i(x)|]\, |g(x)|\, d\nu(x) \\
&\le [f(y) - f_i(y)]\, \|g\| + \|f - f_i\|\, \|g\|
\end{aligned}$$

Squaring, we obtain

$$\begin{aligned}
\left|(\widehat{f} - \widehat{f}_i)g(y)\right|^2 &\le ([f(y) - f_i(y)] + \|f - f_i\|)^2\, \|g\|^2 \\
&= \Big([f(y)-f_i(y)]^2 + \|f-f_i\|^2 + 2\,[f(y)-f_i(y)]\, \|f-f_i\|\Big)\, \|g\|^2
\end{aligned}$$

Hence,

$$\left\|\widehat{f} - \widehat{f}_i)g\right\|^2 \le 4\, \|f - f_i\|^2\, \|g\|^2$$

We conclude that

$$\left\|\widehat{f} - \widehat{f}_i\right\| \leq 2\left\|f - f_i\right\|$$

By the monotone convergence theorem, $\lim\left\|f - f_i\right\| = 0$ and the result follows.
(d) If $f \geq 0$ is a simple function $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $\alpha_i > 0$, then

$$\min\left[f(x), f(y)\right] = \sum_{i,j=1}^{n} \min(\alpha_i, \alpha_j)\chi_{A_i}(x)\chi_{A_j}(y)$$

Hence, for any $u \in H$ we have

$$\begin{aligned}(\widehat{f}u)(y) &= \int \sum_{i,j=1}^{n} \min(\alpha_i, \alpha_j)\chi_{A_i}(x)\chi_{A_j}(y)u(x)d\nu(x)\\ &= \sum_{i,j=1}^{n} \min(\alpha_i, \alpha_j)\int_{A_i} ud\nu\chi_{A_j}(y) \qquad (3.3)\end{aligned}$$

Suppose $g \geq 0$ and $h \geq 0$ are simple functions with $g = \sum \beta_i \chi_{B_i}$ and $h = \sum \gamma_i \chi_{C_i}$, $\alpha_i, \beta_i, \gamma_i > 0$. Assuming that $f, g, h$ have disjoint support, it follows that $\cup A_i$, $\cup B_i$, $\cup C_i$ are mutually disjoint. Employing the notation

$$I(a,b) = \sum_{i,j} \min(a_i, b_j)\int_{A_i} ud\nu\chi_{A_j}$$

As in (3.3) we obtain

$$\begin{aligned}&(f+g)^\wedge u + (f+h)^\wedge u + (g+h)^\wedge u - \widehat{f}u - \widehat{g}u - \widehat{h}u\\ &\quad = I(\alpha,\alpha) + I(\beta,\beta) + I(\gamma,\gamma) + I(\alpha,\beta) + I(\beta,\alpha) + I(\alpha,\gamma)\\ &\qquad + I(\gamma,\alpha) + I(\beta,\gamma) + I(\gamma,\beta)\\ &\quad = (f+g+h)^\wedge u\end{aligned}$$

We conclude that (3.2) holds for simple nonnegative random variables with disjoint support. Now suppose $f, g, h$ are arbitrary nonnegative random variables with disjoint support. Then there exist increasing sequences $f_i, g_i, h_i$ of nonnegative simple random variables converging pointwise to $f$, $g$ and $h$, respectively. Since (3.2) holds for $f_i, g_i, h_i$, applying (c) shows that (3.2) holds for $f, g, h$. Finally, let $f, g, h$ be arbitrary random variables with disjoint support. It is easy to check that $(f+g)^+ = f^+ + g^+$, $(f+g)^- = f^- + g^-$, $(f+g+h)^+ = f^+ + g^+ + h^+$, etc. Then (3.2) becomes

$$\begin{aligned}&(f^+ + g^+ + h^+)^\wedge - (f^- + g^- + h^-)^\wedge\\ &\quad = (f^+ + g^+)^\wedge - (f^- + g^-)^\wedge + (f^+ + h^+)^\wedge - (f^- + h^-)^\wedge + (g^+ + h^+)^\wedge\\ &\qquad - (g^- + h^-)^\wedge - f^{+\wedge} + f^{-\wedge} - g^{+\wedge} + g^{-\wedge} - h^{+\wedge} + h^{-\wedge}\end{aligned}$$

But this follows from our previous work because $f^+, g^+, h^+$ and $f^-, g^-, h^-$ are nonnegative and have disjoint support so (3.2) holds for $f^+, g^+, h^+$ and also for $f^-, g^-, h^-$. $\square$

The next result shows that $f \mapsto \widehat{f}$ preserves positivity.

**Theorem 3.2.** *If $f \geq 0$ is a random variable, then $\widehat{f}$ is a positive operator.*

*Proof.* Suppose $f \geq 0$ is a simple function with $f = \sum_{i=1}^{n} \alpha_i, \chi_{A_i}$ where $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\cup A_i = \Omega$. If $u \in H$ and $B_j = \bigcup_{i=j}^{n} A_i$, then by (3.3) we have

$$
\begin{aligned}
\left\langle \widehat{f}u, u \right\rangle &= \sum_{i,j=1}^{n} \min(\alpha_i, \alpha_j) \int_{A_i} u d\nu \int_{A_j} \bar{u} d\nu \\
&= \alpha_1 \left[ \int_{A_1} u d\nu \int_{B_1} \bar{u} d\nu + \int_{B_2} u d\nu \int_{A_1} \bar{u} \right] \\
&\quad + 2\alpha_2 \mathrm{Re} \int_{A_2} u d\nu \int_{B_2} \bar{u} d\nu + 2\alpha_3 \mathrm{Re} \int_{A_3} u d\nu \int_{B_3} \bar{u} d\nu \\
&\quad + \cdots + 2\alpha_n \int_{A_n} u d\nu \int_{B_n} \bar{u} d\nu \\
&= \alpha_1 \left[ \left| \int_{B_1} u d\nu \right|^2 - \left| \int_{B_2} u d\nu \right|^2 \right] + \alpha_2 \left[ \left| \int_{B_2} u d\nu \right|^2 - \left| \int_{B_3} u d\nu \right|^2 \right] \\
&\quad + \cdots + \alpha_n \left| \int_{B_n} u d\nu \right|^2 \\
&= \alpha_1 \left| \int_{B_1} u d\nu \right|^2 + (\alpha_2 - \alpha_1) \left| \int_{B_2} u d\nu \right|^2 + (\alpha_3 - \alpha_2) \left| \int_{B_3} u d\nu \right|^2 \\
&\quad + \cdots + (\alpha_n - \alpha_{n-1}) \left| \int_{B_n} u d\nu \right|^2 \geq 0
\end{aligned}
\tag{3.4}
$$

Hence, $\widehat{f}$ is a positive operator. If $f \geq 0$ is an arbitrary random variable, then there exists an increasing sequence of simple functions $f_i \geq 0$ converging pointwise to $f$. Since $\widehat{f_i}$ are positive, it follows from Theorem 3.1(c) that $\widehat{f}$ is positive. □

A random variable $f$ satisfying $0 \leq f \leq 1$ is called a *fuzzy* (or *unsharp*) *event* while functions $\chi_A$, $A \in \mathcal{A}$, are called *sharp events* [2]. If $\nu(A) \neq 0$, denote the projection onto $\mathrm{span}(\chi_A)$ by $P(A)$. Theorem 3.1(a) shows that $\widehat{\chi}_A = \nu(A)P(A)$. Thus, quantization takes sharp events to constants times projections. An operator $T$ on $H$ satisfying $0 \leq T \leq I$ is called an *effect* [2]. Since $\left\| \widehat{f} \right\| \leq \|f\|$, it follows from Theorem 3.2 that quantization takes fuzzy events into effects.

Let $\rho$ be a density operator on $H$ and let $\mu_\rho(A) = \mathrm{tr}\left[\rho\mu(A)\right]$ be the corresponding $q$-measure. If $f$ is a random variable, we define the *$q$-integral* (or *$q$-expectation*) of $f$ with respect to $\mu_\rho$ as

$$\int f d\mu_\rho = \mathrm{tr}(\rho \widehat{f})$$

As usual, for $A \in \mathcal{A}$ we define

$$\int_A f d\mu_\rho = \int \chi_A f d\mu_\rho$$

The next result follows from Theorems 3.1 and 3.2.

**Corollary 3.3.** (a) *For all $A \in \mathcal{A}$, we have $\int \chi_A d\mu_\rho = \mu_\rho(A)$.* (b) *For all $\alpha \in \mathbb{R}$, we have $\int \alpha f d\mu_\rho = \alpha \int f d\mu_\rho$.* (c) *If $f_i \geq 0$, is an increasing sequence of random variables converging to a random variable $f$, then $\lim \int f_i d\mu_\rho = \int f d\mu_\rho$.* (d) *If $f \geq 0$, then $\int f d\mu_\rho \geq 0$.* (e) *If $f, g, h$ are random variables with disjoint support, then*

$$\begin{aligned}\int (f+g+h)d\mu_\rho = \int (f+g)d\mu_\rho + \int (f+h)d\mu_\rho + \int (g+h)d\mu_\rho \\ - \int f d\mu_\rho - \int g d\mu_\rho - \int h d\mu_\rho\end{aligned}$$

(f) *If $A, B, C \in \mathcal{A}$ are mutually disjoint, then*

$$\begin{aligned}\int_{A\cup B\cup C} f d\nu = \int_{A\cup B} f d\mu_\rho + \int_{A\cup C} f d\mu_\rho + \int_{B\cup C} f d\mu_\rho \\ - \int_A f d\mu_\rho - \int_B f d\mu_\rho - \int_C f d\mu_\rho\end{aligned}$$

The following result is called the *tail-sum formula*

**Theorem 3.4.** *If $f \geq 0$ is a random variable, then*

$$\int f d\mu_\rho = \int_0^\infty \mu_\rho \left\{x \colon f(x) > \lambda\right\} d\lambda$$

*where $d\lambda$ denotes Lebesgue measure on $\mathbb{R}$.*

*Proof.* Let $f \geq 0$ be a simple function with $f = \sum \alpha_i \chi_{A_i}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n$. Let $u \in H$ with $\|u\| = 1$ and let $\rho$ be the density operator given by $\rho = |u\rangle\langle u|$. Of course, $\rho$ is a pure state. Then

$$\mu_\rho(A) = \left|\langle u, \chi_A\rangle\right|^2 = \left|\int_A u d\nu\right|^2$$

Using the notation of Theorem 3.2, we have

$$\begin{aligned}&\mu_\rho\left\{x \colon f(x) > \lambda\right\} = \mu_\rho(B_{i+1}) \qquad \text{for } \alpha_i < \lambda \leq \alpha_{i+1} \\ &\mu_\rho\left\{x \colon f(x) > \lambda\right\} = \mu_\rho(B_1) \qquad \text{for } 0 \leq \lambda \leq \alpha_1 \\ &\mu_\rho\left\{x \colon f(x) > \lambda\right\} = 0 \qquad \text{for } \alpha_n < \lambda\end{aligned}$$

Moreover, we have

$$\mu_\rho(B_{i+1}) = \left|\left\langle u, \sum_{j=i+1}^n \chi_{A_j}\right\rangle\right|^2 = \left|\sum_{j=i+1}^n \int_{A_j} u d\nu\right|^2$$

We conclude that

$$
\begin{aligned}
&\int_0^\infty \mu_\rho\left\{x\colon f(x)>\lambda\right\}d\lambda\\
&\qquad=\int_0^{\alpha_1}\mu_\rho\left\{x\colon f(x)>\lambda\right\}d\lambda+\int_{\alpha_1}^{\alpha_2}\mu_\rho\left\{x\colon f(x)>\lambda\right\}d\lambda\\
&\qquad\quad+\cdots+\int_{\alpha_{n-1}}^{\alpha_n}\mu_\rho\left\{x\colon f(x)>\lambda\right\}d\lambda\\
&\qquad=\alpha_1\mu_\rho(B_1)+(\alpha_2-\alpha_1)\mu_\rho(B_2)+(\alpha_3-\alpha_2)\mu_\rho(B_3)\\
&\qquad\quad+\cdots+(\alpha_n-\alpha_{n-1})\mu_\rho(B_n)\\
&\qquad=\alpha\left|\int_{B_1}ud\nu\right|^2+(\alpha_2-\alpha_1)\left|\int_{B_2}ud\nu\right|^2\\
&\qquad\quad+\cdots+(\alpha_n-\alpha_{n-1})\left|\int_{B_n}ud\nu\right|^2 \qquad (3.5)
\end{aligned}
$$

Comparing (3.4) and (3.5) gives

$$\int fd\mu_\rho=\left\langle\widehat{f}u,u\right\rangle=\int_0^\infty\mu_\rho\left\{x\colon f(x)>\lambda\right\}d\lambda$$

Applying Theorem 2.3(b) and Corollary 3.3(c) we conclude that the result holds because a mixed state is a convex combination of pure states. □

Applying Theorem 3.4 we have for an arbitrary random variable $f$ that

$$\int fd\mu_\rho=\int_0^\infty\mu_\rho\left\{x\colon f(x)>\lambda\right\}d\lambda-\int_0^\infty\mu_\rho\left\{x\colon f(x)<-\lambda\right\}d\lambda$$

This result shows that the present definition of a $q$-integral reduces to the definition studied previously [6, 7].

# 4 Finite Unitary Systems

This section discusses a physical example that illustrates the theory of Section 2. A *finite unitary system* is a collection of unitary operators $U(s,r)$, $r\le s\in\mathbb{N}$, on $\mathbb{C}^m$ such that $U(r,r)=I$ and $U(t,r)=U(t,s)U(s,r)$ for all $r\le s\le t\in\mathbb{N}$. These operators describe the evolution of a finite-dimensional quantum system in discrete steps from time $r$ to time $s$. If $\{U(s,r)\colon r\le s\}$ is a finite unitary system, then we have the unitary operators $U(n+1,n)$, $n\in\mathbb{N}$, such that

$$U(s,r)=U(s,s-1)U(s-1,s-2)\cdots U(r+1,r) \tag{4.1}$$

Conversely, if $U(n+1,n)$, $n\in\mathbb{N}$, are unitary operators on $\mathbb{C}^m$, then defining $U(r,r)=I$ and for $r<s$ defining $U(s,r)$ by (4.1) we have the finite unitary system $\{U(s,r)\colon r\le s\}$.

In the sequel, $\{U(s,r)\colon r\le s\}$ will be a fixed finite unitary system on $\mathbb{C}^m$. Suppose the evolution of a particle is governed by $U(s,r)$ and the particle's position is at one of the points $0,1,\dots,m-1$. We call the elements of $S=\{0,1,\dots,m-1\}$ *sites* and the infinite strings $\gamma=\gamma_1\gamma_1\gamma_2\cdots$, $\gamma_i\in S$ are called *paths*. The paths represent particle trajectories and the *path* or *sample space* is

$$\Omega=\{\gamma\colon\gamma\text{ a path}\}$$

The finite strings $\gamma=\gamma_0\gamma_1\cdots\gamma_n$ are *n-paths* and

$$\Omega_n=\{\gamma\colon\gamma\text{ an }n\text{-path}\}$$

is the *n-path* or *n-sample space*. The $n$-paths represent time-$n$ truncated particle trajectories. Notice that the cardinality $|\Omega_n|=m^{n+1}$. The power set $\mathcal{A}_n=2^{\Omega_n}$ is the set of *n-events*. Letting $\nu_n$ be the uniform distribution $\nu_n(\gamma)=1/m^{n+1}$, $\gamma\in\Omega_n$, $(\Omega_n,\mathcal{A}_n,\nu_n)$ becomes a probability space.

We call $\mathbb{C}^m$ the *position Hilbert space* Let $e_0,\dots,e_{m-1}$ be the standard basis for $\mathbb{C}^m$ and let $P(i)=|e_i\rangle\langle e_i|$, $i=0,1,\dots,m-1$ be the corresponding projection operators. For $\gamma\in\Omega_n$ the operator $C_n(\gamma)$ on $\mathbb{C}^m$ that describes this trajectory is

$$C_n(\gamma)=P(\gamma_n)U(n,n-1)P(\gamma_{n-1})U(n-1,n-2)\cdots P(\gamma_1)U(1,0)P(\gamma_0)$$

Letting

$$b(\gamma)=\left\langle e_{\gamma_n},U(n,n-1)e_{\gamma_{n-1}}\right\rangle\cdots\langle e_{\gamma_2},U(2,1)e_{\gamma_1}\rangle\langle e_{\gamma_1},U(1,0)e_{\gamma_0}\rangle \tag{4.2}$$

we have that

$$C_n(\gamma)=b(\gamma)|e_{\gamma_n}\rangle\langle e_{\gamma_0}| \tag{4.3}$$

If $\psi\in\mathbb{C}^m$ is a unit vector, the complex number $a_\psi(\gamma)=b(\gamma)\psi(\gamma_0)$ is the *amplitude* of $\gamma$ with initial distribution (state) $\psi$. It is easy to show that

$$\sum_{\gamma\in\Omega_n}|a_\psi(\gamma)|^2=1 \tag{4.4}$$

Moreover, for all $\gamma,\gamma'\in\Omega_n$ we have

$$C_n(\gamma')^*C_n(\gamma)=\overline{b(\gamma')}b(\gamma)\left|e_{\gamma'_0}\right\rangle\langle e_{\gamma_0}|\delta_{\gamma_n,\gamma'_n} \tag{4.5}$$

The operator $C_n(\gamma')^*C_n(\gamma)$ describes the interference between the paths $\gamma$ and $\gamma'$.

For $A\in\mathcal{A}_n$, the *class operator* $C_n(A)$ is defined by

$$C_n(A)=\sum_{\gamma\in A}C_n(\gamma)$$

It is clear that $A\mapsto C_n(A)$ is an operator-valued measure on the algebra $\mathcal{A}_n$ satisfying $C_n(\Omega_n)=U(n,0)$. The *decoherence functional* $\Delta_n\colon\mathcal{A}_n\times\mathcal{A}_n\to\mathbb{C}$

is given by $\Delta_n(A, B) = \langle C_n^*(A) C_n(B)\psi, \psi\rangle$ where $\psi \in \mathbb{C}^m$ is the initial state. It is clear that $A \mapsto \Delta_n(A, B)$ is a complex measure on $\mathcal{A}_n$ for any $B \in \mathcal{A}_n$, $\Delta_n(\Omega_n, \Omega_n) = 1$ and it is well-known that if $A_1, \ldots, A_r \in \mathcal{A}$, then $\Delta_n(A_i, A_j)$ is an $r \times r$ positive semidefinite matrix [3, 11, 12, 13]. Corresponding to an initial state $\psi \in \mathbb{C}^m$, $\|\psi\| = 1$, the $n$-decoherence matrix is

$$\Delta_n(\gamma, \gamma') = \Delta_n(\{\gamma\}, \{\gamma'\})$$

Applying (4.5) we have

$$\begin{aligned} \Delta_n(\gamma, \gamma') &= \langle C_n(\gamma')^* C_n(\gamma)\psi, \psi\rangle = \overline{b(\gamma')} b(\gamma) \overline{\psi(\gamma_0')} \psi(\gamma_0) \delta_{\gamma_n, \gamma_n'} \\ &= \overline{a_\psi(\gamma')} a_\psi(\gamma) \delta_{\gamma_n, \gamma_n'} \end{aligned} \tag{4.6}$$

Define the *n-path Hilbert space* $H_n = (\mathbb{C}^m)^{\otimes(n+1)}$. For $\gamma \in \Omega_n$ we associate the unit vector in $H_n$ given by

$$e_{\gamma_n} \otimes e_{\gamma_{n-1}} \otimes \cdots \otimes e_{\gamma_0}$$

We can think of $H_n$ as $\{\phi\colon \Omega_n \to \mathbb{C}\}$ with the usual inner product and $\Delta_n(\gamma, \gamma')$ corresponds to the operator

$$(\Delta_n \phi)(\gamma) = \sum_{\gamma'} \Delta_n(\gamma, \gamma')\psi(\gamma')$$

Since $\Delta_n(\gamma, \gamma')$ is a positive semidefinite matrix, $\Delta_n$ is a positive operator on $H_n$ and by (4.4) and (4.6) we have

$$\operatorname{tr}(\Delta_n) = \sum_{\gamma} \Delta_n(\gamma, \gamma) = \sum_{\gamma} |a_\psi(\gamma)|^2 = 1$$

We conclude that $\Delta_n$ is a state on $H_n$.

**Lemma 4.1.** *The decoherence functional satisfies*

$$\Delta_n(A, B) = \operatorname{tr}(|\chi_A\rangle\langle\chi_B|\Delta_n)$$

*for all* $A, B \in \mathcal{A}$.

*Proof.* By the definition of the trace we have

$$\begin{aligned} \operatorname{tr}(|\chi_A\rangle\langle\chi_B|\Delta_n) &= \sum_{\gamma} \langle |\chi_A\rangle\langle\chi_B|\Delta_n\gamma, \gamma\rangle \\ &= \sum_{\gamma} \langle \Delta_n\gamma, \chi_B\rangle\langle\chi_A, \gamma\rangle = \sum_{\gamma \in A} \langle \Delta_n\gamma, \chi_B\rangle \\ &= \sum \{\langle \Delta_n\gamma, \gamma'\rangle \colon \gamma \in A, \gamma' \in B\} \\ &= \sum \{\Delta_n(\gamma, \gamma') \colon \gamma \in A, \gamma' \in B\} \\ &= \Delta_n(A, B) \end{aligned}$$

□

Lemma 4.1 shows that the decoherence functional as it is usually defined coincides with the decoherence functional of Section 2. Moreover, the usual $q$-measure $\mu_{\Delta_n}(A) = \Delta_n(A, A)$ coincides with the $q$-measure of Section 2. We have only discussed the time-$n$ truncated path space $\Omega_n$. The infinite time path space $\Omega$ is of primary interest, but its study is blocked by mathematical difficulties. It is hoped that the present structure will help to make progress in overcoming these difficulties.

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