

DISCRETE QUANTUM GRAVITY

S. Gudder

Department of Mathematics
University of Denver
Denver, Colorado 80208, U.S.A.
sgudder@du.edu

Abstract

We discuss the causal set approach to discrete quantum gravity. We begin by describing a classical sequential growth process in which the universe grows one element at a time in discrete steps. At each step the process has the form of a causal set and the “completed” universe is given by a path through a discretely growing chain of causal sets. We then introduce a method for quantizing this classical formalism to obtain a quantum sequential growth process which may lead to a viable model for a discrete quantum gravity. We also give a method for quantizing random variables in the classical process to obtain observables in the corresponding quantum process. The paper closes by showing that a discrete isometric process can be employed to construct a quantum sequential growth process.

1 Introduction

This paper explores the causal set approach to discrete quantum gravity [1, 3, 11]. There are many good review articles on this subject [10, 17, 20] and we refer the reader to these works for more details and motivation. The origins of this approach stem from studies of the causal structure $(M, <)$ of a Lorentzian spacetime (M, g) . For $a, b \in M$ we write $a < b$ if b is in the causal future of a . If there are no closed causal curves in (M, g) , then $(M, <)$ is a partially ordered set (poset). That is, the order $<$ satisfies:

- (1) $a \not< a$ for all $a \in M$ (irreflexivity).
- (2) $a < b$ and $b < c$ implies that $a < c$ (transitivity).

It has been shown that $(M, <)$ possesses much of the information contained in (M, g) [2, 12, 13, 22]. In fact, $(M, <)$ determines the topological and even the differential structure of the manifold (M, g) . Moreover, $(M, <)$ can be employed to find the length of line elements and the dimension of (M, g) . Finally, counting arguments on $(M, <)$ can be employed to find volume elements in (M, g) . Because of these properties, it is believed that the order structure $(M, <)$ provides a viable candidate for describing a discrete quantum gravity.

For a poset $(A, <)$, the *past* of $b \in A$ is $\{a \in A: a < b\}$. We say that $(A, <)$ is *past finite* if the past of b has finite cardinality for every $b \in A$. A *causal set* is a past finite countable poset. One of the simplifications of this paper is that the relevant posets considered will be finite so they are automatically causal sets. Another simplification is that we shall only consider unlabeled posets. In the literature, causal sets are usually labeled according to the order of “birth” and this causes complications because covariant properties are independent of labeling [1, 3, 15, 17]. In this way our causal sets are automatically covariant.

Section 2 describes a classical sequential growth process in which the universe grows one element at a time in discrete steps. At each step the process has the form of a causal set and the “completed” universe is given by a path through a discretely growing chain of causal sets. The transition probability at each step is given by an expression due to Rideout-Sorkin [15, 21]. Letting Ω be the set of paths, \mathcal{A} be the σ -algebra generated by cylinder sets and ν the probability measure governed by the transition probabilities, the dynamics is described by a Markov chain on the probability space $(\Omega, \mathcal{A}, \nu)$.

In Section 3 we quantize classical frameworks by forming the Hilbert space $H = L_2(\Omega, \mathcal{A}, \nu)$. The quantum dynamics is given by a sequence of states ρ_n on H that satisfy a consistency condition. We employ ρ_n to construct decoherence functionals and a quantum measure μ on a “quadratic algebra” \mathcal{S} of subsets of Ω . In general, the set \mathcal{S} is strictly between the collection of cylinder sets and \mathcal{A} . We then present $(\Omega, \mathcal{S}, \mu)$ as a candidate model for quantum gravity. We also give a method for quantizing random variables in the classical process to obtain observables in the corresponding quantum process. This quantization is then used to define a quantum integral.

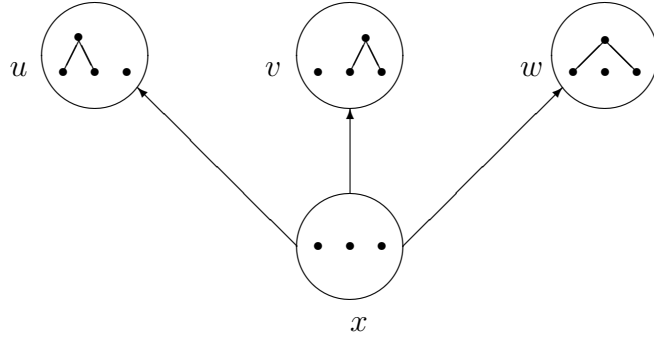


Figure 1

The sequence of states ρ_n discussed in Section 3 is called a quantum sequential growth process. Section 4 shows that a discrete isometric process can be employed to construct a quantum sequential growth process. This work is related to the “sum over histories” approach to quantum mechanics [10].

Of course, much work remains to be done. Of primary importance is to find the specific form of the classical measure ν and the quantum measure μ . One guiding principle is that classical general relativity theory should be a “good approximation” to this quantum counterpart.

2 Quantum Sequential Growth Processes

Let \mathcal{P}_n be the collection of all posets of cardinality n , $n = 0, 1, 2, \dots$, and let $\mathcal{P} = \cup \mathcal{P}_n$ be the collection of all finite posets. An element $a \in x$ for $x \in \mathcal{P}$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then x *produces* y if y is obtained from x by adjoining a single maximal element to x . We also say that x is a *producer* of y and y is an *offspring* of x . If x produces y we write $x \rightarrow y$. Of course, x may produce many offspring and a poset may be the offspring of many producers. Also, x may produce y in various isomorphic ways. For example, in Figure 1 we have that x produces u, v, w . In this paper we identify isomorphic copies of a poset so we identify u, v, w and say that the multiplicity of $x \rightarrow u$ is three and write $m(x \rightarrow u) = 3$. (Strictly speaking, the multiplicity requires a labeling of the elements of a poset and this is the only place that labeling needs to be mentioned.) In Figure 1, notice that within each circle is a Hasse diagram of a poset and a rising line in a diagram represents a *link*.

The transitive closure of \rightarrow makes \mathcal{P} into a poset itself. A *path* in \mathcal{P} is a sequence (string) $x_0 x_1 x_2 \dots$ where $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$, $i = 0, 1, 2, \dots$. An *n-path* in \mathcal{P} is a finite string $x_0 x_1 \dots x_n$ where again $x_i \in \mathcal{P}_i$ and $x_i \rightarrow x_{i+1}$.

We denote the set of paths by Ω and the set of n -paths by Ω_n . If $a, b \in x$ with $x \in \mathcal{P}$, we say that a is an *ancestor* of b and b is a *successor* of a if $a < b$. We say that a is a *parent* of b and b is a *child* of a if $a < b$ and there is no c with $a < c < b$. A link in a Hasse diagram represents a parent-child relationship.

Let $t = (t_0, t_1, \dots)$ be a sequence of nonnegative numbers (called *coupling constants* [15, 17]). For $r, s \in \mathbb{N}$ with $r \leq s$, define

$$\lambda_t(s, r) = \sum_{k=r}^s \binom{s-r}{k-r} t_k = \sum_{k=0}^{s-r} \binom{s-r}{k} t_{r+k}$$

For $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$ with $x \rightarrow y$ we define the *transition probability*

$$p_t(x \rightarrow y) = m(x \rightarrow y) \frac{\lambda_t(\alpha, \pi)}{\lambda_t(n, 0)} \quad (2.1)$$

where α is the number of ancestors and π the number of parents of the adjoined maximal element to x that produces y . The definition of $p_t(x \rightarrow y)$ originally appears in [15]. It is shown there that $p_t(x \rightarrow y)$ is a probability distribution in that it satisfies the Markov-sum rule

$$\sum \{p_t(x \rightarrow y) : y \in \mathcal{P}_{n+1} \text{ with } x \rightarrow y\} = 1$$

The distribution $y \mapsto \mathcal{P}_t(x, y)$ is essentially the most general that is consistent with principles of causality and covariance [15, 17]. It is hoped that other theoretical principles or experimental data will determine the coupling constants. One suggestion is to take $t_k = 1/k!$ [17].

As an illustration, which probably will not work for quantum gravity and cosmology, let $t_k = t^k$ for some $t > 0$. This case has been previously studied and is called a *percolation dynamics* [10, 17]. For this choice we have

$$\lambda_t(s, r) = \sum_{k=0}^{s-r} \binom{s-r}{k} t^{r+k} = t^r \sum_{k=0}^{s-r} \binom{s-r}{k} t^k = t^r (1+t)^{s-r}$$

and as a special case $\lambda_t(n, 0) = (1+t)^n$. Letting β be the number of elements of x not related to the adjoined maximal element, by (2.1) we have

$$p_t(x \rightarrow y) = m(x \rightarrow y) t^\pi \frac{(1+t)^{\alpha-\pi}}{(1+t)^n} = m(x \rightarrow y) \frac{t^\pi}{(1+t)^{\pi+\beta}}$$

Letting $r = t(1+t)^{-1}$ we have that $1-r = (1+t)^{-1}$ and we obtain the more familiar form $p_t(x \rightarrow y) = m(x \rightarrow y)r^\pi(1-r)^\beta$

We call an element $x \in \mathcal{P}$ a *site* and view a site $x \in \mathcal{P}_n$ as a possible universe at step n while a path may be viewed as a possible (evolved) universe. The set \mathcal{P} together with the set of transition probabilities $p_t(x \rightarrow y)$ forms a *classical sequential growth process* (CSGP) which we denote by (\mathcal{P}, p_t) [15, 21]. It is clear that (\mathcal{P}, p_t) is a Markov chain. (In traditional Markov chains, sites are called states but we reserve that term for quantum states to be used later.) As with any Markov chain, the probability of an n -path $\omega = x_0x_1 \cdots x_n$ is

$$p_t^n(\omega) = p_t(x_0 \rightarrow x_1)p_t(x_1 \rightarrow x_2) \cdots p_t(x_{n-1} \rightarrow x_n)$$

and the probability of a site $x \in \mathcal{P}_n$ is

$$p_t^n(x) = \sum \{p_t^n(\omega) : \omega \in \Omega_n, x_n = x\}$$

Of course, $\omega \mapsto p_t^n(\omega)$ is a probability measure on Ω_n .

Example 1. Figure 2 illustrates the first three steps of a CSGP where the 2 indicates the multiplicity $m(x_3 \rightarrow x_6)$. To compute probabilities, we need the values of $\lambda_t(\alpha, \pi)$ given in Table 1.

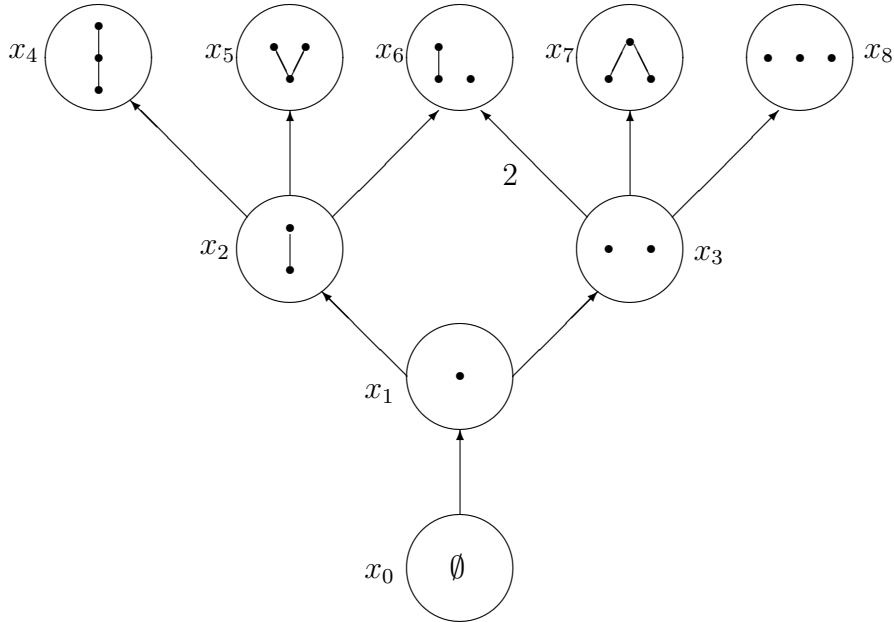


Figure 2

(α, π)	$(0, 0)$	$(1, 0)$	$(1, 1)$	$(2, 0)$	$(2, 1)$	$(2, 2)$
$\lambda_t(\alpha, \pi)$	t_0	$t_0 + t_1$	t_1	$t_0 + 2t_1 + t_2$	$t_1 + t_2$	t_2

Table 1

From Table 1 and (2.1) we obtain the transition probabilities given in Table 2 where $s_0 = t_0 + t_1$, $s_1 = t_0 + 2t_1 + t_2$

$x_i \rightarrow x_j$	$x_0 \rightarrow x_1$	$x_1 \rightarrow x_2$	$x_1 \rightarrow x_3$	$x_2 \rightarrow x_4$	$x_2 \rightarrow x_5$
$p_t(x_i \rightarrow x_j)$	1	t_1/s_0	t_0/s_0	$(t_1 + t_2)/s_1$	t_1/s_1

$x_i \rightarrow x_j$	$x_2 \rightarrow x_6$	$x_3 \rightarrow x_6$	$x_3 \rightarrow x_7$	$x_3 \rightarrow x_8$
$p_t(x_i \rightarrow x_j)$	t_0/s_1	$2t_1/s_1$	t_2/s_1	t_0/s_1

Table 2

Finally, Table 3 lists the probabilities of the various sites, where $s_2 = s_0 s_1$ and $p_t^0(x_0) = 1$ by convention.

x_i	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$p_t^{(n)}(x_i)$	1	1	$\frac{t_1}{s_0}$	$\frac{t_0}{s_0}$	$\frac{t_1(t_1+t_2)}{s_2}$	$\frac{t_1^2}{s_2}$	$\frac{3t_0t_1}{s_2}$	$\frac{t_0t_2}{s_2}$	$\frac{t_0^2}{s_2}$

Table 3

Example 2. Figure 3 illustrates the offspring of x_8 in Figure 2.

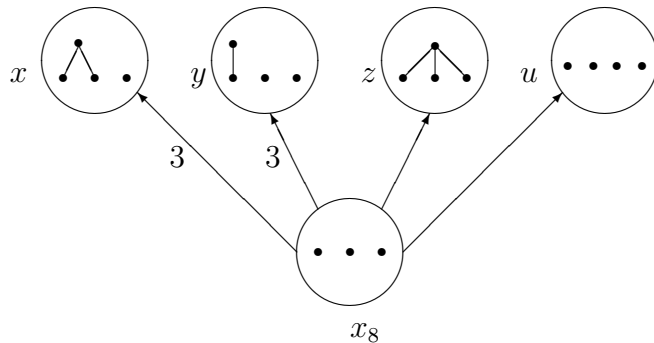


Figure 3

We now compute the transition probabilities.

$$p_t(x_8 \rightarrow x) = \frac{3\lambda_t(2, 2)}{\lambda_t(3, 0)} = \frac{3t_2}{t_0 + 3t_1 + 3t_2 + t_3}$$

$$p_t(x_8 \rightarrow y) = \frac{3\lambda_t(1, 1)}{\lambda_t(3, 0)} = \frac{3t_1}{t_0 + 3t_1 + 3t_2 + t_3}$$

$$p_t(x_8 \rightarrow z) = \frac{3\lambda_t(3, 3)}{\lambda_t(3, 0)} = \frac{t_3}{t_0 + 3t_1 + 3t_2 + t_3}$$

$$p_t(x_8 \rightarrow u) = \frac{3\lambda_t(0, 0)}{\lambda_t(3, 0)} = \frac{t_0}{t_0 + 3t_1 + 3t_2 + t_3}$$

Letting \mathcal{A}_n be the power set 2^{Ω_n} we have that \mathcal{A}_n is an algebra of subsets of Ω_n and $(\Omega_n, \mathcal{A}_n, p_t^n)$ is a probability space. Now we can consider Ω_n to be the product space $\Omega_n = \mathcal{P}_0 \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ and Ω to be the product space $\Omega = \mathcal{P}_0 \times \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots$. (Strictly speaking Ω_n is a subset of $\mathcal{P}_0 \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ because all elements of the latter set do not correspond to n -paths. However, we can define

$$p_t^n(\mathcal{P}_0 \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_n \setminus \Omega_n) = 0$$

and adjoining sets of measure zero is harmless. The same remark holds for Ω and for cylinder sets to be discussed next.) A subset $C \subseteq \Omega$ is a *cylinder set* if

$$C = C_1 \times \mathcal{P}_{n+1} \times \mathcal{P}_{n+2} \times \cdots \tag{2.2}$$

for some $C_1 \in \mathcal{A}_n$. In particular, if $\omega \in \Omega_n$, then the *elementary cylinder set* $\text{cyl}(\omega)$ is defined by

$$\text{cyl}(\omega) = \omega \times \mathcal{P}_{n+1} \times \mathcal{P}_{n+2} \times \cdots$$

It is easy to check that the collection of cylinder sets $\mathcal{C}(\Omega)$ forms an algebra of subsets of Ω . Moreover, for $C \in \mathcal{C}(\Omega)$ of the form (2.2) we define $p_t(C) = p_t^n(C_1)$. Then p_t is a well-defined probability measure on the algebra $\mathcal{C}(\Omega)$. It follows from the Kolmogorov extension theorem that p_t has a unique extension to a probability measure ν_t on the σ -algebra \mathcal{A} generated by $\mathcal{C}(\Omega)$. We conclude that $(\Omega, \mathcal{A}, \nu_t)$ is a probability space. We can identify \mathcal{A}_n with the algebra of cylinder sets of the form (2.2) to obtain an increasing sequence of subalgebras $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots$ of \mathcal{A} that generate \mathcal{A} . Also, the restriction $\nu_t | \mathcal{A}_n = p_t^n$.

3 Quantum Sequential Growth Processes

In Section 2 we described a general CSPG (\mathcal{P}, p_t) . We now show how to “quantize” (\mathcal{P}, p_t) to obtain a quantum sequential growth process (QSGP). It is hoped that this formalism can be employed to construct a model for discrete quantum gravity. At the end of Section 2 we formed a path probability space $(\Omega, \mathcal{A}, \nu_t)$ which we interpret as a space of potential universes. Let $H = L_2(\Omega, \mathcal{A}, \nu_t)$ be the *path Hilbert space*. We previously observed that \mathcal{A}_n considered as an algebra of cylinder sets is a subalgebra of \mathcal{A} and $\nu_t | \mathcal{A}_n = p_t^n$. We conclude that the *n-path Hilbert spaces* $H_n = L_2(\Omega, \mathcal{A}_n, p_t^n)$ form an increasing sequence $H_1 \subseteq H_2 \subseteq \dots$ of closed subspaces of H . Assuming that $p_t^n(\omega) \neq 0$ for every $\omega \in \Omega_n$ we have that $\dim(H_n) = |\Omega_n|$ and that

$$\{\chi_{\text{cyl}(\omega)}/p_t^n(\omega)^{1/2} : \omega \in \Omega_n\}$$

forms an orthonormal basis for H_n .

Let ρ be a state (density operator) on H_n . We can and shall assume that ρ is also a state on H by defining $\rho f = 0$ for all $f \in H_n^\perp$. If $A \in \mathcal{A}_n$ then the characteristic function $\chi_A \in H$ and $\|\chi_A\| = p_t^n(A)^{1/2}$. We define the *decoherence functional* $D_\rho: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$ by

$$D_\rho(A, B) = \text{tr}(\rho|\chi_B\rangle\langle\chi_A|)$$

It can be shown [7] that D_ρ has the usual properties of a decoherence functional. Namely, $D_\rho(A, B) = \overline{D_\rho(B, A)}$, $A \mapsto D_\rho(A, B)$ is a complex measure on \mathcal{A}_n and if $A_i \in \mathcal{A}_n$, $i = 1, \dots, r$, then the $r \times r$ matrix with components $D_\rho(A_i, A_j)$ is positive semidefinite. We also define the *q-measure* $\mu_\rho: \mathcal{A}_n \rightarrow \mathbb{R}^+$ by $\mu_\rho(A) = D_\rho(A, A)$. In general, μ_ρ is not additive so μ_ρ is

not a measure on \mathcal{A}_n . However, μ_ρ is *grade-2 additive* [5, 6, 16, 18] in the sense that if $A, B, C \in \mathcal{A}_n$ are mutually disjoint then

$$\begin{aligned} \mu_\rho(A \cup B \cup C) &= \mu_\rho(A \cup B) + \mu_\rho(A \cup C) + \mu_\rho(B \cup C) \\ &\quad - \mu_\rho(A) - \mu_\rho(B) - \mu_\rho(C) \end{aligned} \quad (3.1)$$

A subset $\mathcal{Q} \subseteq \mathcal{A}$ is a *quadratic algebra* if $\emptyset, \Omega \in \mathcal{Q}$ and if $A, B, C \in \mathcal{Q}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{Q}$, then $A \cup B \cup C \in \mathcal{Q}$. A *q-measure* on a quadratic algebra \mathcal{Q} is a map $\mu: \mathcal{Q} \rightarrow \mathbb{R}^+$ satisfying (3.1) whenever $A, B, C \in \mathcal{Q}$ are mutually disjoint with $A \cup B, A \cup C, B \cup C \in \mathcal{Q}$. In particular \mathcal{A}_n is a quadratic algebra and $\mu_\rho: \mathcal{A}_n \rightarrow \mathbb{R}^+$ is a *q-measure* in this sense.

Let ρ_n be a state on H_n , $n = 1, 2, \dots$, which can be viewed as a state on H . We say that the sequence ρ_n is *consistent* if

$$D_{\rho_{n+1}}(A \times \mathcal{P}_{n+1}, B \times \mathcal{P}_{n+1}) = D_{\rho_n}(A, B)$$

for every $A \in \mathcal{A}_n$. We call a consistent sequence ρ_n a *discrete quantum process* and we call the ρ_n the *local states* for the process. If $\lim \rho_n = \rho$ exists in the strong operator topology, we call ρ the *global state* for the process. If the global state ρ exists, then μ_ρ is a (continuous) *q-measure* on \mathcal{A} that extends μ_{ρ_n} , $n = 1, 2, \dots$. Unfortunately the global state does not exist, in general, so we must work with the local states [8, 9, 19]. We contend that there is a discrete quantum process ρ_n on the path Hilbert space H that describes the dynamics for a discrete quantum gravity. As with the probability measures p_t^n , theoretical principles or experimental data will be required to give restrictions on the possible ρ_n . We shall consider one possibility shortly.

Let ρ_n be a discrete quantum process on $H = L_2(\Omega, \mathcal{A}, \nu_t)$. If $C \in \mathcal{C}(\Omega)$ has the form (2.2) we define $\mu(C) = \mu_{\rho_n}(C_1)$. It is easy to check that μ is well-defined and gives a *q-measure* on algebra $\mathcal{C}(\Omega)$. In general, μ cannot be extended to a *q-measure* on \mathcal{A} , but it is important to extend μ to other physically relevant sets [8, 14, 19]. We say that a set $A \in \mathcal{A}$ is *suitable* if $\lim \text{tr}(e_n |\chi_A\rangle \langle \chi_A|)$ exists and is finite and if this is the case we define $\tilde{\mu}(A)$ to be the limit. We denote the collection of suitable sets by $\mathcal{S}(\Omega)$. The proof of the next theorem is similar to a proof given in [8].

Theorem 3.1. *$\mathcal{S}(\Omega)$ is a quadratic algebra and $\tilde{\mu}$ is a q-measure on $\mathcal{S}(\Omega)$ that extends μ from $\mathcal{C}(\Omega)$.*

We call a real-valued function $f \in H$ a *random variable* (Actually, we are considering random variables with finite second moment but this restriction is convenient here.) We now give a method for “quantizing” f to obtain a bounded self-adjoint operator (*observable*) \widehat{f} on H . Although we employ \widehat{f} to define a quantum integral of f , there may be another important use for \widehat{f} . The map $f \mapsto \widehat{f}$ transforms classical observables to quantum observables. If a discrete quantum process ρ_n governs the dynamics for a discrete quantum gravity, then in some sense, Einstein’s field equation should be an approximation to the sequence ρ_n which gives a strong restriction on ρ_n . In this respect, the map $f \mapsto \widehat{f}$ may be useful in transforming the observables of classical relativity to quantum relativity.

The *quantization* of a nonnegative random variable f is the operator \widehat{f} on H defined by

$$(\widehat{f}g)(y) = \int \min [f(x), f(y)] g(x) d\nu_t(x)$$

It easily follows that $\|\widehat{f}\| \leq \|f\|$ so \widehat{f} is a bounded self-adjoint operator on H . If f is an arbitrary random variable, we have that $f = f^+ - f^-$ where $f^+(x) = \max [f(x), 0]$ and $f^- = -\min [f(x), 0]$. We define the bounded self-adjoint operator \widehat{f} on H by $\widehat{f} = f^{+\wedge} - f^{-\wedge}$. It can be shown that $\|\widehat{f}\| \leq \max (\|\widehat{f}^+\|, \|\widehat{f}^-\|)$ [8]. The next result summarizes some of the important properties of \widehat{f} [8].

Theorem 3.2. (a) For every $A \in \mathcal{A}$, $\widehat{\chi}_A = |\chi_A\rangle\langle\chi_A|$. (b) For every $\alpha \in \mathbb{R}$, $(\alpha f)^\wedge = \alpha \widehat{f}$. (c) If $f \geq 0$, then \widehat{f} is a positive operator. (d) If $0 \leq f_1 \leq f_2 \leq \dots$ is an increasing sequence of random variables converging in norm to a random variable f , then $\widehat{f}_i \rightarrow \widehat{f}$ in the operator norm topology. (e) If f, g, h are random variables with disjoint supports, then

$$(f + g + h)^\wedge = (f + g)^\wedge + (f + h)^\wedge + (g + h)^\wedge - \widehat{f} - \widehat{g} - \widehat{h}$$

Let ρ be a state on H and let μ_ρ be the corresponding q -measure on \mathcal{A}_n or \mathcal{A} . If f is a random variable, we define the q -integral (or q -expectation) of f with respect to μ_ρ as

$$\int f d\mu_\rho = \text{tr}(\rho \widehat{f})$$

The next corollary follows from Theorem 3.2

Corollary 3.3. (a) For every $A \in \mathcal{A}$, $\int \chi_A d\mu_\rho = \mu_\rho(A)$. (b) For every $\alpha \in \mathbb{R}$, $\int \alpha f d\mu_\rho = \alpha \int f d\mu_\rho$. (c) If $f \geq 0$, then $\int f d\mu_\rho \geq 0$. (d) If $f_i \geq 0$ is an increasing sequence of random variables converging in norm to a random variable f , then $\lim \int f_i d\mu_\rho = \int f d\mu_\rho$. (e) If f, g, h are random variables with disjoint supports, then

$$\begin{aligned} \int (f + g + h) d\mu_\rho &= \int (f + g) d\mu_\rho + \int (f + h) d\mu_\rho + \int (g + h) d\mu_\rho \\ &\quad - \int f d\mu_\rho - \int g d\mu_\rho - \int h d\mu_\rho \end{aligned}$$

The next result is called the *tail-sum* formula and gives a justification for calling $\int f d\mu_\rho$ a *q-integral* [6, 7].

Theorem 3.4. If $f \geq 0$ is a random variable, then

$$\int f d\mu_\rho = \int_0^\infty \mu_\rho(\{x: f(x) > \lambda\}) d\lambda$$

where $d\lambda$ denotes Lebesgue measure on \mathbb{R} .

It follows from Theorem 3.4 that if f is an arbitrary random variable, then

$$\int f d\mu_\rho = \int_0^\infty \mu_\rho(\{x: f(x) > \lambda\}) d\lambda - \int_0^\infty \mu_\rho(\{x: f(x) < -\lambda\}) d\lambda$$

Let ρ_n be a discrete quantum process on H . We say that a random variable f is *integrable* for ρ_n if $\lim \text{tr}(\rho_n \widehat{f})$ exists and is finite and in this case we define $\int f d\tilde{\mu}$ to be this limit. Notice that if $A \in \mathcal{S}(\Omega)$, then χ_A is integrable and $\int \chi_A d\tilde{\mu} = \tilde{\mu}(A)$. The next result follows from Corollary 3.3.

Theorem 3.5. (a) If f is integrable and $\alpha \in \mathbb{R}$, then αf is integrable and $\int \alpha f d\tilde{\mu} = \alpha \int f d\tilde{\mu}$. (b) If f is integrable with $f \geq 0$, then $\int f d\tilde{\mu} \geq 0$. (c) If f, g, h are integrable with mutually disjoint supports and $f + g, f + h, g + h$ are integrable, then $f + g + h$ is integrable and

$$\begin{aligned} \int (f + g + h) d\tilde{\mu} &= \int (f + g) d\tilde{\mu} + \int (f + h) d\tilde{\mu} + \int (g + h) d\tilde{\mu} \\ &\quad - \int f d\tilde{\mu} - \int g d\tilde{\mu} - \int h d\tilde{\mu} \end{aligned}$$

4 Discrete Isometric Processes

Section 3 discussed a quantum gravity model in terms of a discrete quantum process ρ_n , $n = 1, 2, \dots$, on $H = L_2(\Omega, \mathcal{A}, \nu_t)$. It may be that ρ_n is determined by a system of isometries (there is some controversy about whether this is possible [17]). This would provide a restriction on the possible ρ_n . Moreover, we are familiar with dynamics governed by isometries so this might aid our intuition. The reader should note that such a formalism is motivated by and related to the sum over histories approach to quantum mechanics. The results in this section are similar to results in [8] taken in a different context.

Let K_n be the Hilbert space of complex-valued function on \mathcal{P}_n with the usual inner product

$$\langle f, g \rangle = \sum_{x \in \mathcal{P}_n} \overline{f(x)} g(x)$$

We call K_n the *n-site Hilbert space* and we denote the standard basis $\chi_{\{x\}}$, $x \in \mathcal{P}_n$, of K_n by e_x^n . The projection operator $P_n(x) = |e_x^n\rangle\langle e_x^n|$, $x \in \mathcal{P}_n$, describe the site at step n . In our context, a *discrete isometric system* is a collection of isometries $U(s, r)$, $r \leq s \in \mathbb{N}$, such that $U(s, r): K_r \rightarrow K_s$, $U(r, r) = I_r$ and $U(t, r) = U(t, s)U(s, r)$ for every $r \leq s \leq t \in \mathbb{N}$. Recall that $U(s, r)$ is an isometry means that $U(s, r)$ is an operator satisfying $U(s, r)^*U(s, r) = I_r$ and $U(s, r)U(s, r)^* = P_s$ where I_r is the identity on K_r and P_s is the projection onto the range of $U(s, r)$ in K_s .

Let $U(s, r)$, $r \leq s \in \mathbb{N}$ be a discrete isometric system and let $\omega \in \Omega_n$ be an n -path. Since all n -paths go through x_1 of Figure 1 we can and shall assume that all n -paths begin at x_1 . Then ω has the form $\omega = x_1\omega_2\omega_3 \dots \omega_n$, $\omega_i \in \mathcal{P}_i$, $i = 1, 2, \dots, n$. We describe ω by the operator $C_n(\omega): K_1 \rightarrow K_n$ given by

$$C_n(\omega) = P_n(\omega_n)U(n, n-1)P_{n-1}(\omega_{n-1})U(n-1, n-2) \cdots P_2(\omega_2)U(2, 1) \quad (4.1)$$

Defining $a(\omega)$ by

$$\begin{aligned} a(\omega) &= \langle e_{\omega_n}^n, U(n, n-1)e_{\omega_{n-1}}^{n-1} \rangle \langle e_{\omega_{n-1}}^{n-1}, U(n-1, n-2)e_{\omega_{n-2}}^{n-2} \rangle \\ &\quad \cdots \langle e_{\omega_2}^2, U(2, 1)e_{x_1}^1 \rangle \end{aligned} \quad (4.2)$$

(4.1) becomes

$$C_n(\omega) = a(\omega) |e_{\omega_n}^n\rangle \langle e_{x_1}^1| \quad (4.3)$$

Of course, we can identify K_1 with \mathbb{C} so $C_n(\omega)$ is the operator given by $C_n(\omega)\alpha = \alpha a(\omega) |e_{\omega_n}^n\rangle$ for every $\alpha \in \mathbb{C}$. We call $a(\omega)$ the *amplitude* of

$\omega \in \Omega_n$ and interpret $|a(\omega)|^2$ as the probability of the path ω according to the dynamics $U(s, r)$. The next result shows that $\omega \mapsto |a(\omega)|^2$ is indeed a probability distribution.

Lemma 4.1. *For the n -path space Ω_n we have*

$$\sum_{\omega \in \Omega_n} |a(\omega)|^2 = 1$$

Proof. By (4.2) we have

$$\begin{aligned} & \sum_{\omega \in \Omega_n} |a(\omega)|^2 \\ &= \sum_{\omega \in \Omega_n} |\langle e_{\omega_n}^n, U(n, n-1)e_{\omega_{n-1}}^{n-1} \rangle|^2 |\langle e_{\omega_{n-1}}^{n-1}, U(n-1, n-2)e_{\omega_{n-2}}^{n-2} \rangle|^2 \\ & \quad \cdots |\langle e_{\omega_2}^2, U(2, 1)e_{x_1}^1 \rangle|^2 \\ &= \sum_{\omega \in \Omega_{n-1}} |\langle e_{\omega_{n-1}}^{n-1}, U(n-1, n-2)e_{\omega_{n-2}}^{n-2} \rangle|^2 \cdots |\langle e_{\omega_2}^2, U(2, 1)e_{x_1}^1 \rangle|^2 \\ & \quad \vdots \\ &= \sum_{\omega \in \Omega_2} |\langle e_{\omega_2}^2, U(2, 1)e_{x_1}^1 \rangle|^2 = 1 \quad \square \end{aligned}$$

The quantity $C_n(\omega')^* C_n(\omega)$ describes the interference between the two paths $\omega, \omega' \in \Omega_n$. Applying (4.3) we see that

$$C_n(\omega')^* C_n(\omega) = \overline{a(\omega')} a(\omega) \delta_{\omega_n, \omega'_n} I_1 \quad (4.4)$$

which we can identify with the complex number $\overline{a(\omega')} a(\omega) \delta_{\omega_n, \omega'_n}$. For $A \in \mathcal{A}_n$ the *class operator* $C_n(A)$ is

$$C_n(A) = \sum_{\omega \in A} C_n(\omega)$$

It is clear that $A \mapsto C_n(A)$ is an operator-valued measure on the algebra \mathcal{A}_n . Moreover, $C_n(\Omega_n) = U(n, 1)$ because by (4.2) and (4.3) we have

$$\begin{aligned} C_n(\Omega_n) &= \sum_{\omega \in \Omega_n} C_n(\omega) = \sum_{\omega \in \Omega_n} \langle e_{\omega_n}^n, U(n, 1)e_{x_1}^1 \rangle |e_{\omega_n}^n\rangle \langle e_{x_1}^1| \\ &= U(n, 1) \end{aligned}$$

It is well-known that $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$ defined by

$$D_n(A, B) = \langle C_n(A)^* C_n(B) e_{x_1}^1, e_{x_1}^1 \rangle$$

is a decoherence functional and we see that $D_n(\Omega_n, \Omega_n) = 1$. Defining the q -measure $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$ by $\mu_n(A) = D_n(A, A)$, we have that $\mu_n(\Omega_n) = 1$.

The n -distribution on \mathcal{P}_n given by

$$p_n(x) = \mu_n(\{\omega \in \Omega_n: \omega_n = x\})$$

is interpreted as the probability that site x is visited at step n . The next result shows that p_n gives the usual quantum distribution.

Theorem 4.2. *For $x \in \mathcal{P}_n$ we have*

$$p_n(x) = \left| \sum_{\omega_n=y} a(\omega) \right|^2 = \left| \langle e_n^n, U(n, 1) e_{x_1}^1 \rangle \right|^2$$

Proof. Letting $A = \{\omega \in \Omega_n: \omega_n = x\}$ we have by (4.4) that

$$\begin{aligned} p_n(x) &= D_n(A, A) = \langle C_n(A)^* C_n(A) e_{x_1}^1, e_{x_1}^1 \rangle \\ &= \sum \{ \langle C_n(\omega')^* C_n(\omega) e_{x_1}^1, e_{x_1}^1 \rangle: \omega'_n = \omega_n = x \} \\ &= \sum \{ \overline{a(\omega')} a(\omega): \omega'_n = \omega_n = x \} \\ &= \left| \sum_{\omega_n=x} a(\omega) \right|^2 \end{aligned}$$

By (4.2) we have that

$$\sum_{\omega_n=x} a(\omega) = \langle e_x^n, U(n, 1) e_{x_1}^1 \rangle$$

and the result follows. \square

We define the *decoherence matrix* as the matrix \widehat{D}_n with components

$$\widehat{D}_n(\omega, \omega') = D_n(\{\omega\}, \{\omega'\})$$

$\omega, \omega' \in \Omega_n$. We have by (4.4) that

$$\widehat{D}_n(\omega, \omega') = \langle C_n(\omega')^* C_n(\omega) e_{x_1}^1, e_{x_1}^1 \rangle = a(\omega) \overline{a(\omega')} \delta_{\omega_n, \omega'_n} \quad (4.5)$$

Notice that $\mu_n(\omega) = \widehat{D}_n(\omega, \omega) = |a(\omega)|^2$ and by Lemma 4.1 that $\sum_{\omega \in \Omega_n} \mu_n(\omega) = 1$. Finally, notice that

$$\begin{aligned} D_n(A, B) &= \sum \left\{ \widehat{D}_n(\omega, \omega') : \omega \in A, \omega' \in B \right\} \\ &= \sum \left\{ a(\omega) \overline{a(\omega')} \delta_{\omega_n, \omega'_n} : \omega \in A, \omega' \in B \right\} \end{aligned}$$

and hence

$$\mu_n(A) = D_n(A, A) = \sum_{\omega, \omega' \in A} \widehat{D}_n(\omega, \omega') = \sum_{\omega, \omega' \in A} a(\omega) \overline{a(\omega')} \delta_{\omega_n, \omega'_n} \quad (4.6)$$

Define the Hilbert space H'_n as the set of complex-valued functions on Ω_n with the usual inner product

$$\langle f, g \rangle = \sum_{\omega \in \Omega_n} \overline{f(\omega)} g(\omega)$$

Then \widehat{D}_n corresponds to the operator (also denoted by \widehat{D}_n) given by

$$(\widehat{D}_n f)(\omega) = \sum_{\omega' \in \Omega_n} \widehat{D}_n(\omega, \omega') f(\omega')$$

Theorem 4.3. *The operator \widehat{D}_n is a state on H'_n .*

Proof. It follows from (4.5) that \widehat{D}_n is a positive operator [9, 14]. By Lemma 4.1 we have

$$\text{tr}(\widehat{D}) = \sum_{\omega \in \Omega_n} D_n(\omega, \omega) = \sum_{\omega \in \Omega_n} |a(\omega)|^2 = 1$$

Hence, \widehat{D}_n is a trace 1 positive operator so \widehat{D}_n is a state on H'_n . \square

Each $\omega \in \Omega_n$ corresponds to a unit vector $\chi_{\{x\}}$ in H'_n and for every $A \in \mathcal{A}_n$ we have the vector $|\chi_A\rangle = \sum \{\chi_{\{x\}} : \omega \in A\}$.

Lemma 4.4. *The decoherence functional satisfies*

$$D_n(A, B) = \text{tr} \left(|\chi_B\rangle \langle \chi_A| \widehat{D}_n \right)$$

Proof. For every $A, \in \mathcal{A}_n$ we have

$$\begin{aligned}
\text{tr} \left(|\chi_B\rangle\langle\chi_A| \widehat{D}_n \right) &= \sum_{\omega \in \Omega_n} \left\langle |\chi_B\rangle\langle\chi_A| \widehat{D}_n \chi_{\{\omega\}}, \chi_{\{\omega\}} \right\rangle \\
&= \sum_{\omega \in \Omega_n} \left\langle \widehat{D}_n \chi_{\{\omega\}}, |\chi_A\rangle\langle\chi_B| \chi_{\{\omega\}} \right\rangle \\
&= \sum \left\{ \left\langle \widehat{D}_n \chi_{\{\omega\}}, \chi_A \right\rangle : \omega \in B \right\} \\
&= \sum \left\{ \left\langle \widehat{D}_n \chi_{\{\omega\}}, \chi_{\{\omega'\}} \right\rangle : \omega \in B, \omega' \in A \right\} \\
&= \sum \{ D_n(\omega', \omega) : \omega' \in A, \omega \in B \} = D_n(A, B) \quad \square
\end{aligned}$$

We now transfer the states \widehat{D}_n on H'_n to states on H_n , $n = 1, 2, \dots$. The set $\{\chi_{\{\omega\}} : \omega \in \Omega_n\}$ forms an orthonormal basis for H'_n and assuming that $p_t^n(\{\omega\}) \neq 0$ for all $\omega \in \Omega_n$, we have that $\{v_\omega : \omega \in \Omega_n\}$ is an orthonormal basis for H_n where

$$v_\omega = p_t^n(\{\omega\})^{-1/2} \chi_{\text{cyl}(\omega)}$$

Defining $U_n \chi_{\{\omega\}} = v_\omega$ and extending U_n by linearity, $U_n : H'_n \rightarrow H_n$ becomes a unitary operator and $U_n : H'_n \rightarrow H$ is an isometry from H'_n into H . Letting P_n be the projection of H onto the subspace H_n we have

$$P_n f = \sum_{\omega \in \Omega_n} \langle v_\omega, f \rangle v_\omega = \sum_{\omega \in \Omega_n} p_t^n(\{\omega\})^{-1} \int f \chi_{\text{cyl}(\omega)} d\nu_t \chi_{\text{cyl}(\omega)}$$

In particular, for $A \in \mathcal{A}$ we obtain

$$P_n \chi_A = \sum_{\omega \in \Omega_n} p_t^n(\{\omega\})^{-1} \nu_t(A \cap \text{cyl}(\omega)) \chi_{\text{cyl}(\omega)}$$

Hence,

$$P_n 1 = \sum_{\omega \in \Omega_n} \chi_{\text{cyl}(\omega)} = 1$$

To transfer \widehat{D}_n from H'_n to H_n we define $\rho_n = U_n \widehat{D}_n U_n^* P_n$. Then ρ_n is a state on H_n and also on H as before.

Theorem 4.5. *The sequence of states ρ_n , $n = 1, 2, \dots$, is consistent.*

Proof. To show that ρ_n is consistent is equivalent to showing that

$$D_n(A \times \mathcal{P}_{n+1}, B \times \mathcal{P}_{n+1}) = D_n(A, B) \quad (4.7)$$

for all $A, B \in \mathcal{A}_n$. Using the notation $\omega x = \omega_1 \omega_2 \cdots \omega_n x$ for $\omega \in \Omega_n$ and $x \in \mathcal{P}_{n+1}$, (4.7) is equivalent to

$$D_n(\omega, \omega') = \sum_{x, y \in \mathcal{P}_{n+1}} D_{n+1}(\omega x, \omega' x) = \sum_{x \in \mathcal{P}_{n+1}} D_{n+1}(\omega x, \omega' x)$$

for every $\omega, \omega' \in \Omega_n$. Since

$$\sum_{x \in \mathcal{P}_{n+1}} \langle U(n+1, n) e_{\omega'_n}^n, e_x^{n+1} \rangle \langle e_n^{n+1}, U(n+1, n) e_{\omega_n}^n \rangle = \delta_{\omega_n, \omega'_n}$$

it follows that

$$\sum_{x \in \mathcal{P}_{n+1}} a(\omega x) \overline{a(\omega' x)} = a(\omega) \overline{a(\omega')} \delta_{\omega_n, \omega'_n}$$

for all $\omega, \omega' \in \Omega_n$. Hence

$$\begin{aligned} \sum_{x \in \mathcal{P}_{n+1}} D_{n+1}(\omega x, \omega' x) &= \sum_{x \in \mathcal{P}_{n+1}} a(\omega x) \overline{a(\omega' x)} = a(\omega) \overline{a(\omega')} \delta_{\omega_n, \omega'_n} \\ &= D_n(\omega, \omega') \end{aligned} \quad \square$$

We conclude from Theorem 4.5 that ρ_n is a discrete quantum process on $H = L_2(\Omega, \mathcal{A}, \nu_t)$ that was constructed from a discrete isometric process.

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