THE P-FRAME REFLECTION OF A COMPLETELY REGULAR FRAME

RICHARD N. BALL, JOANNE WALTERS-WAYLAND, AND ERIC ZENK

Dedicated to the memory of our dear friend and colleague, Mel Henriksen.

Abstract. We show that every completely regular frame has a P-frame reflection. The proof is straightforward in the case of a Lindelöf frame, but more complicated in the general case. The chief obstacle to a simple proof is the important fact that a quotient of a P-frame need not be a P-frame, and we give an example of this.

Our proof of the existence of the P-frame reflection in the general case is iterative, freely adding complements at each stage for the cozero elements of the stage before. The argument hinges on the significant fact that frame colimits preserve Lindelöf degree.

We also outline the relationship between the P-frame reflection of a space X and the topology of the P-space coreflection of X. Although the former frame is generally much bigger than the latter, it is always the case that the P-space coreflection of X is the space of points of the P-frame reflection of the topology on X.

Contents

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1. Introduction

Pointfree topology broadens the extent of classical topological ideas, and clarifies the underlying principles. We provide yet another instance of this phenomenon by proving the existence of the P-frame reflection of a completely regular frame, the pointfree counterpart of the well-known P-space coreflection of a Tychonov space. This result, Theorem 7.13, is the culmination of the article.

But pointfree results sometimes diverge from their pointed analogs in important ways, particularly when it comes to products and subspaces, corresponding to frame coproducts and quotients. That phenomenon rears its head in this investigation: although a subspace of a P-space is clearly a P-space, the quotient of a P-frame need not be a P-frame, and we provide an example in Section 6. This fact poses an obstacle to a straightforward proof of the existence of the P-frame reflection, and although such a proof may exist, we have not found it.

Instead we get the P-frame reflection by means of a transfinite construction reminiscent of the famous tower construction. At each step of the iteration we complement only the cozero elements, rather than all of the elements as in the tower construction. Since the tower construction need not terminate, it is a remarkable fact that the P-frame reflection construction does. The termination of this construction depends, in the end, on an important fact of independent interest: colimits preserve Lindelöf degree, Theorem 7.6.

We mention for the record that our results generalize to higher cardinality, giving the P_κ-frame reflection for completely regular frames. This, of course, raises the issue of what the appropriate generalization of cozero element to cardinality κ might be. We defer a discussion of this interesting topic to a forthcoming article [5].

This article is devoted to the following topics. After a preliminary Section 2, we take up P-spaces and P-frames in Section 3, reviewing the main attributes of P-spaces in Subsection 3.1 to motivate the corresponding frame attributes in 3.2. Whereas the aforementioned results are well known, in Subsection 3.3 we give a novel characterization of P-frames L in terms of the epicompleteness of $\mathcal{C}L$ in the category $\mathbf{W}$ of archimedean lattice-ordered groups with weak order unit. Section 4 reviews the P-space coreflection to motivate the P-frame reflection, and Section 5 establishes this reflection in the deceptively simple Lindelöf case.

These first sections emphasize the consonance between the pointed and pointfree formulations. But a direct extension of the proof of Section 5 to the general pointfree setting is confounded by an example in Section 6, a non-P-frame quotient of a P-frame. Since a subspace of a P-space is obviously a P-space, this section points out one of the most important discrepancies between the pointed and pointfree formulations.

The iterative construction of the P-frame reflection constitutes Section 7. In this section, the construction of the canonical extension L' of a frame L in which each cozero of L has a complement, one step in the iterative construction, occupies Subsection 7.1, the iteration problem occupies 7.2, and the iterative construction itself occupies 7.4. Finally, Section 8 is devoted to the relationship between the P-space coreflection of a Tychonov space X and the P-frame reflection of its topology.

The inclusion functor from the full subcategory of P-frames into the category of completely regular frames preserves limits, and so one would expect that the existence of an adjoint, i.e., a P-frame reflection, would be a routine application of the Adjoint Functor Theorem.[1] Indeed, the only real issue is the other hypothesis of this famous theorem, the Solution Set Condition. That this condition holds, however, is by no means obvious, since many completely regular frames have a proper class of pairwise non-isomorphic monic-and-epic embeddings into P-frames (see, e.g., [31]). In fact, one may view the essential content of this article as the verification of the Solution Set Condition for the inclusion functor of P-frames in completely regular frames.

2. Preliminaries

For a general theory of frames we refer to [18], or, for a recent "covariant" account of this subject, to [26]. Here we collect a few facts that will be relevant for our discussion, and fix notation. Recall that a frame is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s)$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top and bottom elements of L by $\top$ and $\bot$, respectively. The pseudocomplement of an element a is the element $a^* = \bigvee \{b : a \wedge b = \bot\}$. In a frame L, we say of elements a and b that a is well below b, and write $a \prec b$, provided that $a^* \vee b = \top$. A scale is a family $\{a_i\}$ indexed by the rational unit interval $(0,1)_{\mathbb{Q}}$, such that $a_i \prec a_j$ whenever $i < j$. We say that a is completely below b, and write $a \prec\!\!\prec b$, if there is a scale $\{a_i\}$ for which $a \leq a_i \leq b$ for all i. A cozero element of L is the join of a scale, i.e., expressible in the form $\bigvee a_i$ for some scale $\{a_i\}$. We refer to the set of cozero elements of a frame L as its cozero part, and denote it by $\mathcal{Q}L$. A frame L is said to be (completely) regular if each of its elements is the join of those well below it (completely below it). Frame morphisms are those functions f between frames which preserve binary meets and arbitrary joins, including empty meets and joins, so that frame maps preserve $\top$ and $\bot$. We denote the category of frames with frame morphisms by **Frm**, and the full subcategories of regular frames and completely regular frames by **rFrm** and **crFrm**, respectively. As far as frames are concerned, our analysis will be confined to the last-mentioned category. *Unless otherwise stipulated, all frames will be assumed to be completely regular.*

When all mentioned joins are restricted to be over countable sets, the resulting constructs are called σ-frames and regular σ-frames, and the categories are designated σ**Frm** and **r**σ**Frm**, respectively.[2] Regular σ-frames appear naturally in the study of frames as their

[1]We would like to thank to Professor Ernest Manes for raising this interesting point when we presented these results at the BLAST conference held in Las Cruces, New Mexico, in August, 2009.

[2]It is an important and nontrivial fact that the notions of regularity and complete regularity coincide for σ-frames. That is because a regular σ-frame is normal, the well-below relation interpolates and therefore

cozero parts. In fact, $\mathcal{Q} : \mathbf{crFrm} \to \mathbf{r\sigma Frm}$ is functorial, which is to say that a frame morphism $g : L \to M$ takes cozero elements of L to cozero elements of M, thereby restricting to a σ-frame morphism $\mathcal{Q}L \to \mathcal{Q}M$, which we denote $\mathcal{Q}g$. Moreover, the inclusion map $\mathcal{Q}L \to UL$, where U is the forgetful functor that regards a frame L as only a σ-frame, is a coreflector. That means that any σ-frame morphism $A \to UL$ out of a regular σ-frame A factors uniquely through the inclusion $\mathcal{Q}L \to UL$, which is to say that $\mathcal{Q}L$ is the largest regular sub-σ-frame of UL. We drop reference to the forgetful functor U in the sequel, trusting the reader to insert it where necessary.

$\mathcal{Q}$ has a left adjoint $\mathcal{H} : \mathbf{r\sigma Frm} \to \mathbf{crFrm}$ which assigns to each $A \in \mathbf{r\sigma Frm}$ the frame $\mathcal{H}A$ of σ-ideals, i.e., down-sets closed under countable joins, of A, and the σ-frame morphism $\eta_A : A \to \mathcal{QH}A$ given by the rule $a \longmapsto\ \downarrow a$, $a \in A$. Then $(\eta_A, \mathcal{H}A)$ is a $\mathcal{Q}$-universal arrow with domain A, meaning that for any $L \in \mathbf{crFrm}$ and σ-frame morphism $f : A \to \mathcal{Q}L$ there is a unique frame morphism $g : \mathcal{H}A \to L$ such that $\mathcal{Q}g \circ \eta_A = f$. If $f : A \to B$ is a σ-frame morphism then the corresponding frame morphism $\mathcal{H}f : \mathcal{H}A \to \mathcal{H}B$ is given by

$$(\mathcal{H}f)(I) = [f(I)]_\sigma, \ I \in \mathcal{H}A,$$

where $[f(I)]_\sigma$ designates the σ-ideal generated by $f(I)$.

More important for our purposes is the co-unit of the adjunction: for each frame $L \in \mathbf{crFrm}$ we have the frame morphism $\lambda_L : \mathcal{HQ}L \to L$ given by the rule $I \longmapsto \bigvee I$, $I \in \mathcal{HQ}L$. Then $(\lambda_L, \mathcal{HQ}L)$ is an $\mathcal{H}$-co-universal arrow with codomain L, meaning that for any $A \in \mathbf{r\sigma Frm}$ and frame morphism $g : \mathcal{H}A \to L$ there exists a unique σ-frame morphism $f : A \to \mathcal{Q}L$ such that $g = \lambda_L \circ \mathcal{H}f$.

$\mathcal{H}$ maps $\mathbf{r\sigma Frm}$ onto the full subcategory $\mathbf{rLFrm}$ of $\mathbf{crFrm}$ consisting of the regular Lindelöf frames.[3] (A frame L is Lindelöf if, for any subset $S \subseteq L$, $\bigvee S = \top$ implies $\bigvee S_0 = \top$ for some countable subset $S_0 \subseteq S$. See Subsection 7.3.) In fact, the restriction of the adjunction

$$\mathbf{rLFrm} \underset{\mathcal{Q}}{\overset{\mathcal{H}}{\leftrightarrows}} \mathbf{r\sigma Frm}$$

is a categorical isomorphism. This means that the η_A's are σ-frame isomorphisms, and that the λ_L's are frame isomorphisms when L is regular Lindelöf, but it also means that $\mathbf{rLFrm}$ is a coreflective subcategory of $\mathbf{crFrm}$. We refer to $\lambda_L : \mathcal{HQ}L \to L$ as the *Lindelöf coreflection of L*. (The existence of this coreflection, and this construction of it, are due to Madden and Vermeer [24, p. 476].)

Frames, of course, model topologies. Explicitly, we have the functor $\mathcal{O} : \mathbf{Sp} \to \mathbf{Frm}$, where the $\mathbf{Sp}$ is the category of topological spaces with continuous functions, which assigns to each topological space X its frame $\mathcal{O}X$ of open sets, and assigns to each continuous

coincides with the completely below relation, and the σ-frame is consequently completely regular. See Banaschewski [5].

[3]That a regular Lindelöf frame is completely regular is also important and nontrivial. This follows directly from the preceding note, by way of the categorical equivalence between regular σ-frames and regular Lindelöf frames.

function $g : X \to Y$ the frame morphism $\mathcal{O}g : \mathcal{O}Y \to \mathcal{O}X$ given by

$$\mathcal{O}g(U) = g^{-1}(U),\ U \in \mathcal{O}Y.$$

Conversely, to each frame L we assign its space $\mathcal{S}L$ of points, as follows. A point of L is a frame morphism $x : L \to 2$, where 2 designates the two-element frame $\{\bot, \top\}$. The topology on $\mathcal{S}L$ consists of the frame of subsets of the form

$$\tau_L(a) \equiv \{x \in \mathcal{S}L : x(a) = \top\},\ a \in L,$$

and the map $\tau_L : L \to \mathcal{O}\mathcal{S}L$ thus described is a frame surjection which makes $(\tau_L, \mathcal{S}L)$ an $\mathcal{O}$-universal arrow with domain L. L is called spatial when this map is an isomorphism. The other unit of this adjunction is the assignment to a given space X of the $\mathcal{S}$-universal arrow $(\delta_X, \mathcal{S}\mathcal{O}X)$ with domain X, where $\delta_X : X \to \mathcal{S}\mathcal{O}X$ is defined by the rule

$$\delta_X(x)(a) = \begin{cases} \top & \text{if } x \in a \\ \bot & \text{if } x \notin a \end{cases},\ a \in \mathcal{O}X,\ x \in X.$$

X is called sober when δ_X is a homeomorphism.

The frame terminology generally comes from spaces via the $\mathcal{O}$ functor. Thus an element a of a frame L is a cozero iff there is some frame morphism $f : \mathcal{O}\mathbb{R} \to L$ such that $f(R \smallsetminus \{0\}) = a$, a space X is (completely) regular iff $\mathcal{O}X$ is (completely) regular, etc. Therefore, consistent with our running assumption that all frames are completely regular unless otherwise stipulated, *we assume all spaces are Tychonov, i.e., Hausdorff and completely regular, unless otherwise stipulated.* We denote by **crSp** the full subcategory of **Sp** consisting of the Tychonov spaces.

Behind many of the considerations taken up here lies the Baire field of a space X, the smallest σ-field of subsets of X which contains $\mathcal{Q}\mathcal{O}X$. It may be obtained concretely by starting with the family of cozero sets of X and iteratively adding complements and then countable unions. The iteration must be transfinite, taking unions at the limit ordinal stages, but need only be carried out through ω_1 steps. We use $\mathcal{R}X$ to denote the Baire field of X, regarded as a (Boolean) σ-frame.

3. P-spaces and P-frames

3.1. **P-spaces.** A point x in a space X is called a *P-point* if every continuous real-valued function on X is constant in a neighborhood of x. The space X itself is called a *P-space* if all its points are P-points. Discrete spaces are P-spaces, as are the one-point Lindelöfications of infinite discrete spaces.[4] There are even P-spaces without isolated points. A few examples of P-spaces appeared sporadically in the literature, where they were regarded as aberrations, until Gillman and Henriksen undertook a systematic study of P-spaces in [14], which introduced the terms P-point and P-space. Since the appearance of this paper, P-points and P-spaces have emerged in many mathematical contexts, often playing an important role in the analysis. A good introduction to the topic may be found in problems 4J-N of [15], from which Theorem 3.1 is drawn.

[4] The one point Lindelöfication of an infinite discrete space D is formed by adjoining an additional point d_∞ to D. A subset U of the resulting set is declared open if $d_\infty \in U$ implies $D \smallsetminus U$ is at most countable.

Theorem 3.1. *The following are equivalent for a space X.*

1. *X is a P-space, i.e., zero sets are open, i.e., cozero sets are clopen.*
2. *Each cozero set of X is C-embedded.*
3. *$C(X)$ is a regular ring, i.e.,*

$$\forall f \ \exists \ f_0 \ \left(f^2 f_0 = f\right).$$

Proof. We offer a few words of explanation here for purposes of comparison with the corresponding pointfree arguments to come. The implication from (1) to (2) is obvious, since every clopen subset is C-embedded. Assuming (2), we take an arbitrary $f \in C(X)$ and invert it on its cozero set, extending the result to the whole space (because the latter is C-embedded) to get f_0. Assuming (3), we get (1) by observing that the zero set of $f \in C(X)$, $f \geq 0$, is $\operatorname{coz}(1 - f f_0)$. □

3.2. P**-frames.** Theorem 3.1 has a pointfree counterpart, Theorem 3.2 below. It is interesting to see how the arguments used to establish the equivalence of the conditions in the pointfree version are ready generalizations of the pointed arguments, becoming at the same time simpler and broader in scope. The equivalence of conditions (1) and (2) in Theorem 3.2 is due to Ball and Walters-Wayland ([4, 8.4.7]), while the equivalence of (2) and (3) is due to Dube [13].

Theorem 3.2. *The following are equivalent for a frame L.*

1. *The cozero elements of L are complemented.*
2. *Each open quotient of a cozero element of L is a C-quotient.*
3. *$C(L)$ is a regular ring.*

Proof. The argument that the open quotient of a complemented element $a \in L$ is a C-quotient, i.e., the implication from (1) to (2), is straightforward. We outline a proof of the implication from (2) to (3) in order to point out how closely the reasoning follows the spatial argument in the proof of Theorem 3.1. Assume (2), and consider $f \in C(L)$, i.e., f is a frame map from $\mathcal{O}\mathbb{R}$ into L. Let $a \equiv \operatorname{coz} f = f(\mathbb{R} \smallsetminus \{0\}) \in \operatorname{coz} L$. Now $mf : \mathcal{O}\mathbb{R} \to \ \downarrow a \in C(\downarrow a)$ has the feature that $mf(\mathbb{R} \smallsetminus \{0\}) = \top$, so that according to Proposition 3.3.1 of [4] it may be inverted, i.e., there is some $g \in C(\downarrow a)$ such that $fg = 1$. Since the open quotient map $m : L \to \ \downarrow a$, given by $b \longmapsto b \wedge a$, $b \in L$, is a C-quotient, g may be extended over m, i.e., there is some $f_0 \in C(L)$ such that $mf_0 = g$. It is then clear that $f^2 f_0 = f$. The implication from (3) to (1) goes along the same lines as that from (3) to (1) in Theorem 3.1. That is, one shows that

$$\begin{aligned}
&\operatorname{coz} f \vee \operatorname{coz}(1 - f f_0) = \operatorname{coz}(f \vee (1 - f f_0)) = \top, \\
&\operatorname{coz} f \wedge \operatorname{coz}(1 - f f_0) = \bot.
\end{aligned}$$

The computational machinery developed in [4] can be used to establish these equalities. □

The striking parallelism of Theorems 3.1 and 3.2 motivates the central definition of this article.

Definition 3.3. *We say that a frame L is a P*-frame *if it satisfies the conditions of Theorem 3.2.*

The reader should be warned that [4, 8.4.7] contains a serious misstatement of this definition. The condition that $a \in \operatorname{Coz} L$ implies $a^* \in \operatorname{Coz} L$ is *not* equivalent to those of Theorem 3.2, and does *not* define a P-frame. This is an error on our part.

The class of P-frames includes the topologies of the P-spaces, but extends far beyond them. For example, any complete Boolean algebra A is a P-frame, and if A is atomless then its associated space $\mathcal{S}A$ is empty. In the language of locales, a complete atomless Boolean algebra is a pointless P-locale.

Dube has characterized P-frames by means of several interesting and elegant ring theoretic properties of $C(L)$. See [13]. We add several more characterizations of P-spaces and P-frames in terms of $C(X)$ or $C(L)$ regarded as $\mathbf{W}$-objects. To the best of our knowledge, these characterizations are new.

3.3. **In W.** $\mathbf{W}$ is the category whose objects are of the form (G, u), where G is an archimedean lattice-ordered group with weak order unit u. (For general background, see [10], [22], and [12].) There is an adjoint relationship

$$\mathbf{W} \underset{\mathcal{Y}}{\overset{\mathcal{C}}{\leftrightarrows}} \mathbf{rLFrm},$$

where $\mathcal{Y}$ is the functor which assigns to each $\mathbf{W}$-object G its regular Lindelöf frame $\mathcal{Y}G$ of $\mathbf{W}$-kernels, and $\mathcal{C}$ is the functor which assigns to each regular Lindelöf frame L the $\mathbf{W}$-object $\mathcal{C}L$ of frame maps $\mathcal{O}\mathbb{R} \to L$ ([25], [2]). The functor $\mathcal{C}$ maps $\mathbf{rLFrm}$ onto the full subcategory $\mathbf{c}^3\mathbf{W}$ of $\mathbf{W}$ consisting of those objects which are closed under countable composition, an attribute whose definition we omit. The restricted adjunction

$$\mathbf{c}^3\mathbf{W} \underset{\mathcal{Y}}{\overset{\mathcal{C}}{\leftrightarrows}} \mathbf{rLFrm}$$

is a categorical isomorphism.

For $G \in \mathbf{W}$, we denote the positive cone $\{g \in G : g \geq 0\}$ of G by G^+.

Theorem 3.4. *The following are equivalent for a frame L.*

(1) *L is a P-frame.*
(2) *$\mathcal{C}L$ is epicomplete in $\mathbf{W}$ or in $\mathbf{c}^3\mathbf{W}$, i.e., $\mathcal{C}L$ has no proper epimorphic extensions.*
(3) *$\mathcal{C}L$ is*
   (a) *conditionally σ-complete, i.e., every bounded countable subset of $\mathcal{C}L^+$ has a supremum, and*
   (b) *laterally σ-complete, i.e., every countable pairwise disjoint subset of $\mathcal{C}L^+$ has a supremum.*
(4) *$\mathcal{C}L$ is laterally σ-complete.*

If L is replaced by $\mathcal{HQ}L$ in any of these conditions then the resulting condition remains equivalent to those above.

Proof. This theorem is about epicompleteness in three categories: $\mathbf{r\sigma Frm}$, $\mathbf{rLFrm}$, and $\mathbf{c^3W}$. Since all three are isomorphic, we get that a regular σ-frame A is epicomplete in $\mathbf{r\sigma Frm}$ iff $\mathcal{H}A$ is epicomplete in $\mathbf{LFrm}$ iff $\mathcal{CH}A$ is epicomplete in $\mathbf{c^3W}$. But the epicomplete objects in $\mathbf{r\sigma Frm}$ are well-known to be the Boolean ones ([21]), $\mathcal{C}L$ is isomorphic to $\mathcal{CHQ}L$, either as a ring or a $\mathbf{W}$-object since every frame map $\mathcal{O}\mathbb{R} \to L$ extends uniquely over the Lindelöf coreflection map $\mathcal{HQ}L \to L$ because $\mathcal{O}\mathbb{R}$ is Lindelöf, and $\mathbf{c^3W}$ is an epireflective subcategory of $\mathbf{W}$ so that the notions of epimorphism and epicompletion in $\mathbf{c^3W}$ coincide with the same notions in $\mathbf{W}$. Thus the first two conditions and their Lindelöf variations coincide. The third condition is a known internal characterization of epicomplete $\mathbf{W}$-objects ([1]). But, in the presence of divisibility and regular uniform completeness, both attributes of $\mathcal{C}L$, a laterally σ-complete $\mathbf{W}$ object is conditionally σ-complete. (See [28] and Theorem 5.4 of [16]; see also the remark following Theorem 5.2 in [3].) $\square$

When specialized to spaces, Theorem 3.4 becomes Corollary 3.5.

Corollary 3.5. *The following are equivalent for a space X.*

1. *X is a P-space.*
2. *$C(X)$ is epicomplete in $\mathbf{W}$ or in $\mathbf{c^3W}$, i.e., $C(X)$ has no proper epimorphic extensions.*
3. *$C(X)$ is laterally σ-complete.*

Proof. In this case $C(X)$ is $\mathbf{W}$-isomorphic to $\mathcal{CO}X$, and X is a P-space iff $\mathcal{O}X$ is a P-frame. The equivalence of (1) and (3) is due to Buskes [11]. $\square$

By connecting P-spaces with epicompleteness in $\mathbf{W}$, Corollary 3.5 shows that, far from being curiosities, P-spaces arise naturally and unavoidably in general topology. But what is also interesting about Corollary 3.5 is that, while the result itself is about spaces (X and $C(X)$, classical stuff), its proof reduces to a diagram chase in frames.

4. The P-space coreflection

One of the most important properties of P-spaces is that every space has a "nearest" P-space relative. (See [29, Chapter 10].) Put another way, among all the P-space topologies finer than the given topology on a space X, there is a coarsest one. This topology goes by several names, among them being the P-space topology, the G_δ-topology, and the Baire topology. We denote by $\mathcal{P}X$ the space that results from equipping the carrier set X with this finer topology, and we denote by ρ_X the identity map $\mathcal{P}X \to X$, which is continuous. It is an entertaining exercise to establish that

$$\begin{aligned} \mathcal{OP}X &= \{V : V \text{ is a union of cozero sets of } X\} \\ &= \{V : V \text{ is a union of } G_\delta \text{ sets of } X\} \\ &= \{V : V \text{ is a union of sets of } \mathcal{R}X\}. \end{aligned}$$

An informative reference is [27].

Theorem 4.1. *P-spaces are bicoreflective in spaces. In particular, a coreflector for the space X is $\rho_X : \mathcal{P}X \to X$, meaning that for any continuous function $f : Y \to X$ out of a P-space Y there is a unique continuous function $\widehat{f} : Y \to \mathcal{P}X$ such that $\rho_X \widehat{f} = f$.*

The purpose of this article is to extend Theorem 4.1 to the pointfree context, i.e., to prove the existence of the P-frame reflection. This we do in Theorem 7.13. We begin with the special case of Lindelöf frames.

5. The P-frame reflection in the Lindelöf case

Although the proof in this case is straightforward, we will see in Section 6 that it does not readily generalize. Let L be a Lindelöf frame with cozero part A, and let $\beta_A : A \to \mathcal{B}A$ be a Boolean reflector for A. Now $\lambda_L : \mathcal{HQ}L \to L$ is an isomorphism because L is Lindelöf, so that we have the map

$$\rho_L \equiv \mathcal{H}\beta_A \circ \lambda_L^{-1} : L \to \mathcal{HB}A \equiv \mathcal{P}L.$$

(Our use of the same symbol $\mathcal{P}$ to designate both the P-frame reflection and the P-space coreflection (Section 4) is purposeful; see Section 8.) Unwinding these definitions gives

$$\rho_L(a) = \{b \in \mathcal{B}A : b \leq \beta_A(a)\}, \ a \in L.$$

Theorem 5.1. *Every Lindelöf frame L has a P-frame reflector, namely $\rho_L : L \to \mathcal{P}L$. That means that for any frame map $k : L \to M$ into a P-frame M there is a unique frame map $\widehat{k} : \mathcal{P}L \to M$ such that $\widehat{k}\rho_L = k$. And $\mathcal{P}L$ is Lindelöf.*

We emphasize that the codomains M of the test maps k are not required to be Lindelöf, but instead range over all P-frames. Lindelöf P-frames are reflective in Lindelöf frames, and it happens that this reflection is also the P-frame reflection in the category of all (completely regular) frames.

Proof. Let L be a Lindelöf frame with cozero part A. First observe that $\mathcal{P}L$ is a P-frame since its cozero part is isomorphic to the Boolean σ-frame $\mathcal{B}A$ via $\eta_{\mathcal{B}A}$. Now consider a test map k as above, and let $B \equiv \mathcal{Q}M$. Then $\mathcal{Q}k : A \to B$ factors uniquely through β_A since B is Boolean; let $j : \mathcal{B}A \to B$ be the unique map satisfying $j\beta_A = \mathcal{Q}k$. Applying the $\mathcal{H}$ functor to this factorization gives the commuting diagram.

$$\begin{tikzcd}
L \arrow[r, "{\lambda_L}^{-1}"] \arrow[d, "\rho_L"'] & \mathcal{H}A \arrow[r, "\mathcal{HQ}k"] \arrow[d, "\mathcal{H}\beta_A"'] & \mathcal{H}B \arrow[r, "\lambda_M"] & M \\
\mathcal{P}L \arrow[r, phantom, "\equiv"] & \mathcal{HB}A \arrow[ur, "\mathcal{H}j"'] & &
\end{tikzcd}$$

The desired map $\widehat{k}$ is $\lambda_M \circ \mathcal{H}j$; its uniqueness with respect to satisfying $\widehat{k}\rho_L = k$ follows from the fact that β_A is epic and therefore so is $\mathcal{H}\beta_A$ and so is ρ_L. □

Theorem 5.1 permits a relatively concrete description of the P-frame reflection of a compact frame L, Corollary 5.5, for in this case we have a nice characterization of the Boolean reflection of the cozero part of L. For a space X, recall that $\mathcal{R}X$ designates the (Boolean) σ-frame of Baire measurable subsets of X.

Proposition 5.2 ([25]; see also [7]). *For a compact space X with $A = \mathcal{QO}X$, the identical insertion $i_X : A \to \mathcal{R}X$ serves as a Boolean reflector for A.*

Corollary 5.3. *For a compact space X, the map $\mathcal{O}X \to \mathcal{HR}X$ given by the rule*

$$U \longmapsto \{V \in \mathcal{R}X : V \subseteq U\}, \; U \in \mathcal{O}X,$$

serves as a P-frame reflector for $\mathcal{O}X$.

Proof. From Theorem 5.1 we learned that $\rho_{\mathcal{O}X} : \mathcal{O}X \to \mathcal{HBQO}X$ is a P-frame reflector for $\mathcal{O}X$, and from Proposition 5.2 we find that we can replace $\mathcal{BQO}X$ by $\mathcal{R}X$ in this formula. Unwinding the definitions leads to the mapping displayed. □

Example 5.4. *Consider the frame $L = \mathcal{O}[0,1]$, the topology on the closed unit interval. In this case singletons are zero sets, so the Baire field $\mathcal{R}[0,1]$ is $2^{[0,1]}$, the entire power set of $[0,1]$. So the embedding $L \to \mathcal{H}2^{[0,1]}$ given by*

$$U \longmapsto \{V \subseteq [0,1] : V \subseteq U\}, \; U \in \mathcal{O}[0,1],$$

is a P-frame reflector for L.

Corollary 5.5. *The P-frame reflection of a compact frame L is isomorphic to*

$$L \overset{\lambda_L^{-1}}{\to} \mathcal{HQ}L \overset{\mathcal{H}(i_{\mathcal{S}L} \circ \mathcal{Q}\tau_L)}{\to} \mathcal{HRS}L.$$

Proof. $\tau_L : L \to \mathcal{OS}L$ is a frame isomorphism by the Axiom of Choice. □

Summary 5.6. *We summarize the conclusions of this section in two formulas.*

(1) *For a Lindelöf frame L, $\mathcal{P}L \cong \mathcal{HBQ}L$.*
(2) *For a compact frame L, $\mathcal{P}L \cong \mathcal{HRS}L$.*

6. The quotient of a P-frame need not be a P-frame

In light of the straightforward proof of Theorem 5.1, one might hope to simply push out the diagram

$$\begin{tikzcd} \mathcal{HQ}L \arrow[r, "\rho_{\mathcal{HQ}L}"] \arrow[d, "\lambda_L"'] & \mathcal{HBQ}L \\ L & \end{tikzcd}$$

in order to get the P-frame reflection of an arbitrary frame L. But that would require something very close to the closure of P-frames under quotients. One would certainly expect the class of P-frames to be closed under quotients since a subspace of a P-space is clearly a P-space. But the example presented in the this section shows that this expectation is unfounded.

We construct a frame surjection whose domain is a P-frame and whose codomain is not. Note that the search for an example of this type may be confined to Lindelöf frames. That is because, if $f : L \to M$ is a frame surjection with a domain which is a P-frame and a codomain which is not, so is the composition of f with the Lindelöf coreflection map $\mathcal{HQ}L \to L$. After all, L and $\mathcal{HQ}L$ have isomorphic cozero parts, so that one is a P-frame iff the other is. Moreover, since a regular Lindelöf frame is entirely determined by its cozero part, the search for an example of this type may be understood to be the search for a Boolean σ-frame having certain properties. What are those properties?

One rather simple way in which a frame may fail to be a P-frame is if it has a countable collection of complemented elements whose join is not complemented. For complemented elements are cozeros, and the cozeros are closed under countable joins.

6.1. **Frames having a quotient in which the complemented elements are not closed under countable joins.** Theorem 6.1 characterizes the frames with such quotients. This theorem requires that we recall some well-known machinery for handling quotient maps.

6.1.1. *Prenuclei.* The finest frame congruence identifying two members u and v of a frame L is also the finest congruence identifying $u \wedge v$ and $u \vee v$, so when we speak of pairs identified by a particular congruence, we will assume that the pairs are of the form (u, v) with $u \leq v$. It is well-known and easy to verify that the finest frame congruence $\sim$ identifying such a pair (u, v) is given by

$$a \sim b \Longleftrightarrow (u \wedge (a \vee b) \leq a \wedge b \text{ and } v \vee (a \wedge b) \geq a \vee b)\,.$$

Thus the corresponding nucleus is

$$j\,(a) = \bigvee \{b : u \wedge b \leq a \text{ and } v \vee a \geq b\}\,,\ a \in L.$$

(For if b satisfies only the two inequalities displayed above then $a \vee b \sim a$.) In particular, for $v = \top$ this simplifies to

$$j\,(a) = \bigvee \{b : b \wedge u \leq a\}\,,\ a \in L.$$

This is sometimes expressed in the form

$$j\,(a) = u \to a,\ a \in L.$$

What if we have not one pair, but a set of pairs to be identified by the frame congruence? Then the same sort of considerations apply, except that we get a prenucleus rather than a nucleus [9]. Thus for any subset $S \subseteq L$, the finest frame congruence which identifies the members of S with $\top$ has prenucleus

$$j\,(a) = \bigvee_S \{b : b \wedge u \leq a \text{ for some } u \in S\}\,,\ a \in L.$$

This is sometimes expressed in the form

$$j\,(a) = \bigvee_S (u \to a)\,,\ a \in L.$$

Since complemented elements are cozeros, these frames have a dense quotient which is not a P-frame.

Theorem 6.1. *A frame L has a dense quotient in which the complemented elements are not closed under countable joins iff it contains elements c_n, $n \in \mathbb{N}$, and z with the following properties.*

(1) $c_n \leq z < \top$ *for all* $n \in \mathbb{N}$.
(2) $c_n^* \to z = z$ *for all* $n \in \mathbb{N}$.
(3) $c^* \leq z$, *where* $c = \bigvee_{\mathbb{N}} c_n$.

Proof. Suppose L contains elements c_n, $n \in \mathbb{N}$, and z as specified. Let j be the prenucleus of the finest frame congruence which identifies all the $c_n \vee c_n^*$'s with $\top$. That is,

$$j\left(a\right) = \bigvee \left\{b : b \wedge \left(c_n \vee c_n^*\right) \leq a \text{ for some } n \in \mathbb{N}\right\}, \ a \in A.$$

Let M be the fixed point set of j, regarded as a frame in the order it inherits from L, and let $m : L \to M$ be the frame morphism corresponding to j. Note that $j\left(\bot\right) = \bot$, so that m is dense.

We claim $j\left(z\right) = z$. For if not then there is some $b \in L$ and $n \in \mathbb{N}$ such that $b \wedge \left(c_n \vee c_n^*\right) \leq z$ but $b \not\leq z$. But then $b \wedge c_n^* \leq z$ would imply $b \leq c_n^* \to z$, contrary to (2). And since $m\left(c\right)^* = m\left(c^*\right)$ by virtue of the density of m, we have

$$m\left(c\right) \vee m\left(c\right)^* = m\left(c\right) \vee m\left(c^*\right) = m\left(c \vee c^*\right) = m\left(z\right) < \top,$$

the point being that $\bigvee_{\mathbb{N}} m\left(c_n\right) = m\left(\bigvee_{\mathbb{N}} c_n\right) = m\left(c\right)$ is not complemented in M.

Now suppose $m : L \to M$ is a dense frame surjection such that elements x_n, $n \in \mathbb{N}$, are complemented in M but $x = \bigvee_{\mathbb{N}} x_n$ is not. Let $c_n \equiv m_*\left(x_n\right)$, $n \in \mathbb{N}$, and let $z \equiv m_*\left(x \vee x^*\right)$. These elements clearly satisfy (1) and (3). To see that they satisfy (2), consider $a \in L$ such that $a \wedge c_n^* \leq z$ for some n. Then, since $m\left(c_n^*\right) = m\left(c_n\right)^*$ by the density of m, we get $m\left(a\right) \wedge x_n^* \leq x \vee x^*$. Hence

$$\begin{aligned} m\left(a\right) &= m\left(a\right) \wedge \top = m\left(a\right) \wedge \left(x_n \vee x_n^*\right) \\ &= \left(m\left(a\right) \wedge x_n\right) \vee \left(m\left(a\right) \wedge x_n^*\right) \\ &\leq x \vee x^*, \end{aligned}$$

with the result that $a \leq m_*\left(x \vee x^*\right) = z$. □

6.2. **The Boolean σ-frame A.** We return now to the the discussion at the beginning of the section. In order to find a frame surjection whose domain is a P-frame and whose codomain is not, it is sufficient to find a Boolean σ-frame A with the properties necessary to insure that its frame of σ-ideals satisfies the conditions of Theorem 6.1. We construct A with the aid of two auxiliary Boolean σ-frames, B and D.

6.2.1. *The auxiliary Boolean σ-frame B.* Let E be an uncountable set, and let X designate the set of all finite sequences x of elements of E. For $x \in X$, let $|x|$ designate the length of x, let λ designate the empty sequence of length 0, and for $x, y \in X$ let xy designate the concatenation of x and y. Partially order X by declaring $x \leq y$ iff $x = yz$ for some $z \in X$. For any subset $U \subseteq X$ we denote the set of its lower bounds by

$$\downarrow U \equiv \{y : \exists\, x \in U \ (y \leq x)\},$$

and we abbreviate $\downarrow\{x\}$ to $\downarrow x$. Note that X is a tree, meaning that the set of upper bounds of any element is a finite chain.

Definition 6.2. *B is the Boolean sub-σ-frame of the power set 2^X generated by all subsets of the form $\downarrow x$, $x \in X$.*

The elements of B have a normal form which we now describe. We call a subset $U \subseteq X$ *pairwise incomparable* if no two different elements x and y of U have a common lower bound. For each $x \in X$ and pairwise incomparable countable subset $U \subseteq (\downarrow x) \smallsetminus \{x\}$ we let

$$b\,(x, U) \equiv (\downarrow x) \smallsetminus (\downarrow U)\,.$$

Use of the notation $b\,(x, U)$ is meant to imply that U is a pairwise incomparable countable subset of $\downarrow x \smallsetminus \{x\}$. Figure 1 shows a typical $b\,(x, U)$ visualized as a subset of the tree X.

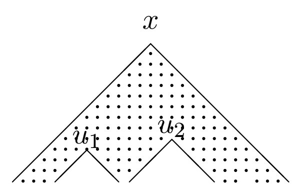

FIGURE 1. $b(x, U)$ shaded

Proposition 6.3. *Each member of B is the union of a unique countable family of pairwise disjoint subsets of the form $b\,(x, U)$, $x \in X$.*

Proof. Let B' designate the collection of subsets which can be expressed as unions of $b\,(x, U)$'s as above. It is clear that each individual $b\,(x, U)$ lies in B, so that the same is true of each element of B'. We must show that B' forms a Boolean sub-σ-frame of 2^X.

We first show that the complement of each $b\,(x, U)$ lies in B'. For if $x = \lambda$ then $b\,(x, U) = X \smallsetminus (\downarrow U)$ and $X \smallsetminus b\,(x, U)$ is $\downarrow U$, which clearly lies in B'. And if $x \neq \lambda$ then

$$X \smallsetminus b\,(x, U) = b\,(\lambda, \{x\}) \cup \left(\bigcup_{y \in U} b\,(y, \emptyset) \right),$$

which also lies in B'.

We next show that B' is closed under countable intersection. For that purpose consider a countable subset $\{b_i : i \in \mathbb{N}\} \subseteq B'$, where

$$b_i = \bigcup_{n \in \mathbb{N}} b\left(x_n^i, U_n^i\right)$$

for a pairwise disjoint family $\{b(x_n^i, U_n^i) : n \in \mathbb{N}\}$. Then, by virtue of the complete distributivity of 2^X, we have

$$(*) \qquad \bigcap_{i\in\mathbb{N}} b_i = \bigcap_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} b\left(x_n^i, U_n^i\right) = \bigcup_{\theta\in\mathbb{N}^{\mathbb{N}}} b_\theta,$$

where

$$b_\theta \equiv \bigcap_{i\in\mathbb{N}} b\left(x_{\theta(i)}^i, U_{\theta(i)}^i\right).$$

We claim that

(1) all but countably many of the b_θ's are empty, and
(2) the nonempty b_θ's are of the form $b(y, U)$ for y ranging over a countable subset $Y \subseteq X$, and
(3) the $b(y, U)$'s are pairwise disjoint.

We establish the claim by first defining

$$Y \equiv \left\{x_n^i : \forall\, j\, \exists\, m \ \left(x_n^i \in b\left(x_m^j, U_m^j\right)\right)\right\}.$$

Y is clearly countable. Furthermore, for each $y = x_n^i \in Y$ and each j, the pairwise disjointness of the $b(x_m^j, U_m^j)$'s, $m \in \mathbb{N}$, implies that there is, in fact, a unique m_j for which $y \in b(x_{m_j}^j, U_{m_j}^j)$. This allows us to define $\theta_y \in \mathbb{N}^{\mathbb{N}}$ by setting $\theta_y(j) \equiv m_j$, $j \in \mathbb{N}$. Note that, since $y \in b(x_{\theta_y(j)}^j, U_{\theta_y(j)}^j)$ for all j, it follows that $y \le x_{\theta_y(j)}^j$ and $y \notin\ \downarrow U_{\theta_y(j)}^j$ for all j. If we let U_y be the collection of maximal elements of $\bigcup_{j\in\mathbb{N}}(\downarrow y \cap U_{\theta_y(j)}^j)$, then, as the reader will have no difficulty checking, $b_{\theta_y} = b(y, U_y)$.

We next claim that the $b(y, U_y)$'s are pairwise disjoint. Clearly $b(y_1, U_{y_1})$ is disjoint from $b(y_2, U_{y_2})$ if y_1 and y_2 are unrelated elements of Y, for $b\left(y_j, U_{y_j}\right) \subseteq\ \downarrow y_j$. On the other hand, if y_1 and y_2 are related elements of Y, say $y_1 < y_2$, then, abbreviating θ_{y_j} to θ_j, we have

$$x_{n_1}^{i_1} = y_1 < y_2 \le x_{\theta_2(i_1)}^{i_1},$$

which establishes that $\theta_2(i_1) \ne n_1$. But then $b(y_1, U_{y_1})$ and $b(y_2, U_{y_2})$ are contained, respectively, in the disjoint sets $b(x_{n_1}^{i_1}, U_{n_1}^{i_1})$ and $b(x_{\theta_2(i_1)}^{i_1}, U_{\theta_2(i_1)}^{i_1})$.

We complete the proof of the claim by showing that every nonempty b_θ is of the form $b(y, U_y)$ for a unique $y \in Y$. For the fact that $b_\theta \ne \emptyset$ implies that $x_{\theta(i)}^i$ must be related to $x_{\theta(j)}^j$ for all i and j. It also implies that the chain $\{x_{\theta(i)}^i : i \in \mathbb{N}\}$ is finite; let $y \equiv x_n^i$ be its least element. Clearly $y \le x_{\theta(j)}^j$ for all j, and since $\emptyset \ne b_\theta \subseteq\ \downarrow y$, it follows that $y \notin\ \downarrow U_{\theta(j)}^j$ for all j. That is, $y \in b(x_{\theta(j)}^j, U_{\theta(j)}^j)$ for all j, which establishes that $y \in Y$. The reader may readily check that $b_\theta = b(y, U_y)$.

The claim shows that the expression on the right in $(*)$ is of the form required for membership in B', and hence that B' is closed under countable intersection. Combined with the first paragraph, this allows us to conclude that B' is closed under arbitrary complementation. These two facts, in turn, imply that B' is a Boolean sub-σ-frame of 2^X and complete the proof. $\square$

Corollary 6.4. *Each nonempty $b \in B$ is uncountable. Consequently, for $b_i \in B$, $b_1 = b_2$ iff their symmetric difference $b_1 \Delta b_2$ is countable.*

Proof. The first statement is a consequence of Proposition 6.3, since each $b(x, U)$ is uncountable. $\square$

6.2.2. *The Boolean σ-frames D and A.*

Definition 6.5. *Let D be the Boolean σ-frame of all subsets $d \subseteq X$ for which there exists some $b \in B$ such that $d \Delta b$ is countable. (By Corollary 6.4, there can be at most one such b.) Finally, let*

$$A \equiv \left\{(d_1, d_2) \in D^2 : |d_1 \Delta d_2| \leq \omega\right\}.$$

Clearly D is a sub-σ-frame of 2^X, A is a sub-σ-frame of D^2, and both are Boolean. Note that if $(d_1, d_2) \in A$ then there exists a unique $b \in B$ such that $d_1 \Delta b$ and $d_2 \Delta b$ are countable. We refer to (d_1, d_2) as being *small* if $b = \emptyset$, and *large* if $b \neq \emptyset$. Note that the countable join of small elements is small.

6.3. **The example.** Let L designate $\mathcal{H}A$, the frame of σ-ideals of A. Define in L

$$\begin{aligned} I_n &\equiv \{(d_1, d_2) \in A : d_2 = \emptyset \text{ and } |x| \leq n \text{ for all } x \in d_1\}, \ n \in \mathbb{N}, \\ I_n^* &\equiv \{a \in A : \forall\, b \in I_n \ (a \wedge b = 0)\}, \ n \in \mathbb{N}, \\ J &\equiv \{a \in A : a \text{ is small}\}. \end{aligned}$$

Note that the elements of I_n are small, whereas those of I_n^* need not be. For example,

$$(b(x, U), b(x, U)) \in I_n^*$$

for all x such that $|x| \geq n + 1$.

Lemma 6.6. *For every large element $a \in A$ and every $n \in \mathbb{N}$ there is a large $a' \in A$ such that $a \geq a' \in I_n^*$.*

Proof. Since a is large it is of the form (d_1, d_2) for a unique $\emptyset \neq b \in B$ such that $d_1 \Delta b$ and $d_2 \Delta b$ are countable. In turn, b is the union of a unique countable family of pairwise disjoint subsets of the form $b(x, U)$. Fix any one of these $b(x, U)$'s, and let y be any element of $b(x, U) \smallsetminus (d_1 \Delta b \cup d_2 \Delta b)$ of length at least $n + 1$. Such an element must exist because X has uncountable branching at each point. Then $(b(y, \emptyset), b(y, \emptyset))$ has the properties of the element a' we seek. $\square$

Our discussion of the example is completed by showing that L satisfies the hypotheses of Theorem 6.1.

Proposition 6.7. *The following hold in L.*

1. $I_n \subseteq J \subsetneq A$ *for all n.*
2. $I_n^* \rightarrow J = J$ *for all n.*
3. $I^* \subseteq J$, *where $I \equiv \bigvee_{\mathbb{N}} I_n$ in L.*

Proof. We have already remarked on the truth of (1), and (2) follows from Lemma 6.6. (3) follows from the observation that

$$I = \bigvee_{\mathbb{N}} I_n = \{(d_1, d_2) : |d_i| \leq \omega\},$$

so that $I^* = \{(\emptyset, \emptyset)\} = 0$. □

7. The P-frame reflection

We construct the P-frame reflection $\rho_L : L \to \mathcal{P}L$ of a frame L iteratively, at each step freely complementing the cozero elements. We begin with the first step.

7.1. **One step: freely complementing the cozeros of L.** It is well known that for any frame L and subset $S \subseteq L$ there is a frame injection $f : L \to L_S$ which is universal with respect to complementing the elements of S ([19], see also [31], [20]). That means that $f(s)$ is complemented in L_S for each $s \in S$, and that any frame morphism $g : L \to M$ such that $g(s)$ is complemented in M for each $s \in S$ factors through f, i.e., there is a unique frame morphism $h : L_S \to M$ such that $g = hf$. This property characterizes f and L_S up to isomorphism over L. Of the several known constructions of this extension, perhaps the most accessible is Wilson's. We record that construction here, specialized to $S = \mathcal{Q}L$, in order to familiarize the reader with the extension and to make a couple of elementary remarks about it. We then return to Joyal and Tierney's original construction, and elaborate upon it in order to draw the conclusions necessary for our purposes.

Recall that a frame L may be regarded as a subframe of its frame NL of nuclei by means of the embedding $c : L \to NL$ which maps beach $a \in L$ to the closed nucleus $c(a)$ defined by $c(a)(b) = a \vee b$, $b \in L$ ([18]). Recall also that each $c(a)$ has a complement in NL, namely the open nucleus $u(a)$ defined by

$$u(a)(b) = a \to b \equiv \bigvee_{d \wedge a \leq b} d, \; b \in L.$$

In fact, the embedding $c : L \to NL$ may be characterized as the result of freely complementing all of the elements of L.

Proposition 7.1 ([31, 16.2])**.** *For a frame L, let L' designate the subframe of NL generated by $c(L) \cup u(\mathcal{Q}L)$, and let $c_L : L \to L'$.designate the codomain restriction of c. Then $c_L : L \to L'$ is universal with respect to complementing the cozeros of L.*

Corollary 7.2. *For a frame L, let $f : L \to L'$ be the result of freely complementing the cozero elements of L, no matter how constructed. Then each element of L' is the join of differences of cozero elements of L.*

Proof. This is true of Wilson' construction in NL. □

7.2. **The iteration problem.** Although each $a \in \mathcal{Q}L$ has a complemented image in $\mathcal{Q}L'$, we have no assurance that every member of $\mathcal{Q}L'$ is complemented, i.e., that L' is a P-frame. A natural strategy is, therefore, to iterate the passage from L to L'

$$L \to L' \to L'' \to L''' \to \cdots ,$$

taking colimits at limit ordinal stages. If this process terminates, or stabilizes, then this extension is a likely candidate for the P-frame reflection of L.

The termination issue is a serious one, since if we replace L' with NL in the definition above, that is, if we complement all of the elements of L at each step instead of just the cozero elements, we get the famous tower construction

$$L \to NL \to NNL \to NNNL \to \cdots ,$$

which does not stabilize in many cases ([18], [31]). In fact, characterizing those frames for which the tower construction stabilizes is one of the most fundamental open problems in pointfree topology. We resolve this issue in the sequel by showing that the tower of extensions $L \to L' \to L'' \cdots$ stabilizes because the Lindelöf degree does not grow. (We review the notion of Lindelöf degree in Subsection 7.3.)

What that means, of course, is that the Lindelöf degree does grow in the tower of extensions $L \to NL \to NNL \to \cdots$. That is indeed the case; the Lindelöf degree of NL may strictly exceed that of L. For it is known (from the equivalence of **rLFrm** with **W**, for instance) that the epicomplete objects in the category of regular Lindelöf frames are the P-frames. If, for a Lindelöf P-frame L, NL were also Lindelöf, then, as an epimorphic extension of L, it would have to coincide with it. That is, every Lindelöf P-frame would be Boolean. Such, however, is not the case.

7.3. **Lindelöf degree.** From this point on, κ stands for a regular cardinal. A *κ-set* is any set of cardinality strictly less than κ, and in any set A, a *κ-subset* is a subset $B \subseteq A$ such that $|B| < \kappa$; we sometimes write $B \subseteq_\kappa A$ for emphasis. Recall that a frame L is said to be *κ-Lindelöf* if for every subset $A \subseteq L$ such that $\bigvee A = \top$ there is κ-subset $B \subseteq_\kappa A$ such that $\bigvee B = \top$. The *Lindelöf degree of* L, written $\operatorname{lind} L$, is the least regular cardinal κ such that L is κ-Lindelöf. For instance, L is compact iff $\operatorname{lind} L = \omega$. When used without the hyphenated cardinal, the term Lindelöf means ω_1-Lindelöf. We record the elementary properties of Lindelöf degree.

Proposition 7.3. (1) *If L is a subframe of M then* $\operatorname{lind} L \leq \operatorname{lind} M$*.*

(2) *For a finite family $\{L_i : 1 \leq i \leq n\}$ of frames,*

$$\operatorname{lind}\left(\prod_{1\leq i\leq n} L_i\right) \leq \max\left\{\operatorname{lind} L_i : 1 \leq i \leq n\right\}.$$

(3) *For an element a in a frame L, the closed quotient frame $\uparrow a = \{a' \in L : a' \geq a\}$ satisfies*

$$\operatorname{lind} \uparrow a \leq \operatorname{lind} L.$$

(4) *For a cozero element a in a frame L, the open quotient frame $\downarrow a = \{a' : a' \leq a\}$ satisfies*

$$\operatorname{lind} \downarrow a \leq \max \{\operatorname{lind} L, \omega_1\}.$$

Proof. Only (4) requires explanation. Suppose L is κ-Lindelöf, and consider $a \in \mathcal{Q}L$, say $a = \bigvee_{(0,1)_{\mathbb{Q}}} a_i$ for some scale $\left\{a_i : i \in (0,1)_{\mathbb{Q}}\right\}$. For $i < j$ in $(0,1)_{\mathbb{Q}}$ let b_{ij} be a separating element, i.e., $b_{ij} \wedge a_i = \bot$ and $b_{ij} \vee a_j = \top$. Suppose $\bigvee S = \top$ in $\downarrow a$, which is to say that $\bigvee S = a$ in L. Then for $i < j$ we have

$$\bigvee_S (b_{ij} \vee s) = b_{ij} \vee a \geq b_{ij} \vee a_j = \top.$$

Since L is κ-Lindelöf there must be a κ-subset $S_{ij} \subseteq_\kappa S$ such that $\bigvee_{S_{ij}} (b_{ij} \vee s) = \top$, and since $b_{ij} \wedge a_i = \bot$ and $b_{ij} \vee \bigvee S_{ij} = \bigvee_{S_{ij}} (b_{ij} \vee s) = \top$, it follows that $\bigvee S_{ij} \geq a_i$. Let $S' \equiv \bigcup_{i<j} S_{ij}$, a subset of S of cardinality strictly less than $\max \{\kappa, \omega_1\}$. Clearly $\bigvee S' = a$. $\square$

The most penetrating characterization of Lindelöf degree is by means of κ-frames. Theorems 7.4 and 7.5 extract the relevant facts from Section 4 of Madden's fundamental article [23] on κ-frames; we refer the reader to that source for further explanation and for the definition of terms undefined here. A frame L is said to be *κ-free* provided that there is a universal κ-frame morphism $f : M \to L$, i.e., such that any κ-frame morphism from M into a frame factors uniquely through f. An element a of a frame L is called *κ-Lindelöf* if its open quotient $\downarrow a$ is κ-Lindelöf. We denote by $E^\kappa(L)$ the set of κ-Lindelöf elements of L. Now $E^\kappa(L)$ is evidently closed under joins of κ-subsets, but is not generally a sub-κ-frame. When $E^\kappa(L)$ is a sub-κ-frame of L and generates L, we say that L is *κ-coherent*. That is, L is κ-coherent if $E^\kappa(L)$

- is closed under binary meets,
- contains $\top$, and
- generates L as a frame.

Theorem 7.4 (Madden)**.** *Let $\kappa > \omega$. Then the following are equivalent for a frame L.*

(1) *L is κ-Lindelöf.*
(2) *L is κ-free.*
(3) *L is κ-coherent.*
(4) *L is isomorphic to the frame of κ-ideals of $E^\kappa(L)$.*

More is true.

Theorem 7.5 (Madden)**.** *Let F^κ be the functor which assigns to a regular κ-frame its frame of κ-ideals. Then F^κ and E^κ form a categorical equivalence between the categories of regular κ-frames and κ-Lindelöf frames. Furthermore,*

$$L \cong F^\kappa E^\kappa(L) \text{ and } E^\kappa F^\kappa(L) = \{\downarrow a : a \in L\}$$

for all κ-Lindelöf frames L.

A fact which is crucial for our purposes is that frame colimits preserve Lindelöf degree.

Theorem 7.6. *Let κ be a regular cardinal, and let $\{f_{ij} : L_i \to L_j : i \leq j \text{ in } I\}$ be a directed family of frame maps such that $\operatorname{lind} L_i \leq \kappa$ for all $i \in I$, and let $\{f_i : L_i \to L : i \in I\}$ be the colimit of the family. Then*

$$\operatorname{lind} L \leq \max\{\kappa, \omega_1\}.$$

Proof. Let $\lambda = \max\{\kappa, \omega_1\}$. By Theorem 7.4, each L_i is λ-free, meaning that L_i is the free frame over its sub-λ-frame E_i^λ of λ-Lindelöf elements. By Lemma 4.2 of [23], each f_{ij} restricts to $f_{ij}^\lambda : E_i^\lambda \to E_j^\lambda$, so that we have the directed family $\left\{f_{ij}^\lambda : E_i^\lambda \to E_j^\lambda : i \leq j \text{ in } I\right\}$ of morphisms in the category of regular λ-frames. Let $\left\{f_i^\lambda : E_i^\lambda \to E\right\}$ be its colimit in that category, and then apply the functor F^λ to these maps. It is easy to check that the result gives the colimit of the frame maps $\{f_{ij}\}$. Since the colimit object $F^\lambda(E)$ is λ-Lindelöf by Theorem 7.4, the result is proven. □

We prove in Proposition 7.10 that $\operatorname{lind} L = \operatorname{lind} L'$ for a frame L of Lindelöf degree $\kappa > \omega$. The proof involves a concrete construction of L' based on an insight of Joyal and Tierney ([19]); see also [8]. They showed that freely complementing a single element $a \in L$ can be done by the embedding $L \to \downarrow a \times \uparrow a$ given by the rule

$$x \longmapsto (a \wedge x,\ a \vee x),\ x \in L.$$

If a is a cozero element then, since $\operatorname{lind} \downarrow a = \operatorname{lind} \uparrow a = \kappa$ by Proposition 7.3, clearly $\operatorname{lind}(\downarrow a \times \uparrow a) = \kappa$ as well. So we may freely complement a single cozero element of L without raising the Lindelöf degree. By an elaboration of this argument, we first show that we may freely complement finitely many cozero elements of L all at once without raising the Lindelöf degree. This gives a directed system of κ-Lindelöf extensions of L whose colimit is also κ-Lindelöf by Theorem 7.6. The proof of Proposition 7.10 then consists of observing that this colimit coincides with L'.

Fix a completely regular frame L and a finite subset $R \subseteq \mathcal{Q}L$. Define

$$a_R \equiv \bigvee R \text{ and } b_R \equiv \bigwedge R.$$

For disjoint finite subsets $R, S \subseteq \mathcal{Q}L$, define the interval

$$I(R,S) \equiv [a_S \wedge b_R,\ b_R] = \{x \in L : a_S \wedge b_R \leq x \leq b_R\}.$$

Fix a finite subset $W \subseteq \mathcal{Q}L$, and set

$$L_W \equiv \prod_{R \uplus S = W} I_{R,S},$$

with projection map $p(R,S) : L_W \to I(R,S)$. Here the notation $R \uplus S = W$ means that R and S partition W, i.e., $R \cup S = W$ and $R \cap S = \emptyset$.

Lemma 7.7. *Assuming the foregoing notation, if $\operatorname{lind} L = \kappa > \omega$ then $\operatorname{lind} L_W = \kappa$.*

Proof. L_W is a finite product of intervals of the form $I(R,S)$, $R \uplus S = Q$, and each such interval is bounded by cozero elements $a_S \wedge b_R$ and b_R. By 7.3 each of these intervals is κ-Lindelöf, and therefore so is L_W. □

Now we construct bonding maps $f_V^W : L_V \to L_W$ for finite $V \subseteq W \subseteq \mathcal{Q}L$. For that purpose consider a partition $R \uplus S = W$, with corresponding restriction partition $(R \cap V) \uplus (S \cap V) = V$. Let $f_V(R, S) : I(R \cap V, S \cap V) \to I(R, S)$ be the map

$$x \longmapsto (x \vee a_S) \wedge b_R,\ x \in I(R \cap V, S \cap V).$$

The maps

$$f_V(R, S)\, p(R \cap V, S \cap V) : L_V \to I(R, S),\ R \biguplus S = W,$$

induce a map $f_V^W : L_V \to L_W$ such that

$$p(R, S)\, f_V^W = f_V(R, S)\, p(R \cap V, S \cap V),\ R \biguplus S = W.$$

Note that if $V = W$ then f_V^W is the identity map on $L_V = L_W$.

To show these maps consistent, consider finite subsets $U \subseteq V \subseteq W \subseteq \mathcal{Q}L$. Since, for any partition $R \uplus S = W$,

$$b_R \le b_{R\cap V} \le b_{R\cap U} \text{ and } a_S \ge a_{S\cap V} \ge a_{S\cap U},$$

it follows that for any $x \in I(R \cap U, S \cap U)$

$$\begin{aligned}
f_V(R, S)\, f_U(R \cap V, S \cap V)(x) &= (f_U(R \cap V, S \cap V)(x) \vee a_S) \wedge b_R \\
&= (((x \vee a_{S\cap V}) \wedge b_{R\cap V}) \vee a_S) \wedge b_R \\
&= (x \vee a_{S\cap V} \vee a_S) \wedge (b_{R\cap V} \vee a_S) \wedge b_R \\
&= (x \vee a_S) \wedge b_R \\
&= f_U(R, S)(x).
\end{aligned}$$

Therefore for all partitions $R \uplus S = W$ we have

$$\begin{aligned}
p(R, S)\, f_V^W f_U^V &= f_V(R, S)\, p(R \cap V, S \cap V)\, f_U^V \\
&= f_V(R, S)\, f_U(R \cap V, S \cap V)\, p(R \cap U, S \cap U) \\
&= f_U(R, S)\, p(R \cap U, S \cap U).
\end{aligned}$$

From this it follows that $f_V^W f_U^V = f_U^W$, which is to say that the bonding maps form a consistent directed family.

Since $L_\emptyset = I(\emptyset, \emptyset)$ is isomorphic to L, we drop the subscript $\emptyset$ and write $L_\emptyset$ as L, $f_\emptyset(R, S)$ as $f(R, S)$, and $f_\emptyset^W$ as f^W.

Lemma 7.8. *$f^W : L \to L_W$ is universal with respect to complementing the elements of W.*

Proof. Let us first investigate the structure of L_W. For $a \in W$,

$$p(R, S)\, f^W(a) = f(R, S)(a) = \begin{cases} b_R = \top(R, S) & \text{if } a \in R \\ a_S \wedge b_R = \bot(R, S) & \text{if } a \in S \end{cases},\ R \biguplus S = W.$$

Each $f^W(a)$ is complemented in L_W; if we denote this complement by c_a, then it satisfies

$$p(R, S)(c_a) = \begin{cases} a_S \wedge b_R = \bot(R, S) & \text{if } a \in R \\ b_R = \top(R, S) & \text{if } a \in S \end{cases},\ R \biguplus S = W.$$

Furthermore, the c_a's, together with $f^W(L)$, generate all of L_W. To see this, consider a particular partition $R \uplus S = W$ and a particular $x \in L$ such that $a_S \wedge b_R \leq x \leq b_R$. Put

$$y(R,S,x) \equiv \bigwedge_{a \in R} f^W(a) \wedge f^W(x) \wedge \bigwedge_{a \in S} c_a \in L_W.$$

Then for any other partition $T \uplus U = W$ we get

$$\begin{aligned} p(T,U)(y(R,S,x)) &= \bigwedge_{a \in R} p(T,U) f^W(a) \wedge p(T,U) f^W(x) \wedge \bigwedge_{a \in S} p(T,U)(c_a) \\ &= \bigwedge_{a \in R} f(T,U)(a) \wedge f(T,U)(x) \wedge \bigwedge_{a \in S} p(T,U)(c_a) \\ &= \begin{cases} x & \text{if} \quad T = S \\ a_U \wedge b_T = \bot(T,U) & \text{if} \quad T \neq S \end{cases}. \end{aligned}$$

Thus any $y \in L_W$ can be uniquely expressed in the form

$$y = \bigvee_{R \uplus S = W} y(R,S,p(R,S)(y)) = \bigvee_{R \uplus S = W} \left(\bigwedge_{a \in R} f^W(a) \wedge f^W p(R,S)(y) \wedge \bigwedge_{a \in S} c_a \right).$$

Consider a frame morphism $g : L \to K$ such that each $g(a)$, $a \in W$, has a complement d_a in K. Then define $\widehat{g} : L_W \to K$ by the rule

$$\widehat{g}(y) \equiv \bigvee_{R \uplus S = W} \left(\bigwedge_{a \in R} g(a) \wedge gp(R,S)(y) \wedge \bigwedge_{a \in S} d_a \right) \text{ for}$$

$$y = \bigvee_{R \uplus S = W} \left(\bigwedge_{a \in R} f^W(a) \wedge f^W p(R,S)(y) \wedge \bigwedge_{a \in S} c_a \right) \text{ in } L_W.$$

The reader may readily check that $\widehat{g}$ is the unique frame morphism such that $\widehat{g} f^W = g$. □

Let

$$f_W : L_W \to L^*, \text{ finite } W \subseteq QL,$$

be the colimit of the directed family $\left\{ f_V^W : L_V \to L_W : \text{finite } V \subseteq W \subseteq \operatorname{coz} L \right\}$. As usual, we abbreviate $f_\emptyset$ to f.

Lemma 7.9. *L^* and L' are isomorphic over L. That is, there is a frame isomorphism $h : L^* \to L'$ such that $hf = c_L$.*

Proof. It is sufficient to observe that f is universal with respect to complementing the cozero elements of L. For if $g : L \to K$ is a frame map such that $g(a)$ is complemented in K for each $a \in QL$ then g factors through each f^W for each finite $W \subseteq QL$ by Lemma 7.8, so g must also factor through f. □

Proposition 7.10. *If* $\operatorname{lind} L > \omega$ *then* $\operatorname{lind} L' = \operatorname{lind} L$.

Proof. Let $\operatorname{lind} L = \kappa > \omega$, so that $\operatorname{lind} L_W = \kappa$ for each finite subset $W \subseteq QL$ by Lemma 7.7, hence $\operatorname{lind} L^* = \kappa$ by Theorem 7.6. □

7.4. **Iteration.** Armed with Proposition 7.10, we can now show that the iteration of Subsection 7.2 stabilizes. This requires a technical result, Proposition 7.11, which requires some terminology with which the reader may not already be familiar. Let κ be a regular cardinal. In a frame L, a *κ-directed family of subframes* is a family of subframes of L such that every κ-subset of the family has an upper bound (in the inclusion order on subframes) in the family. For a subset $A \subseteq \operatorname{Coz} L$ and element $b \in L$, we denote $\{a \in A : a \leq b\}$ by $\downarrow_A b$.

Proposition 7.11. *Let κ be an uncountable regular cardinal, and let L be a κ-Lindelöf frame having a κ-directed family F of subframes such that $A \equiv \bigcup_F \mathcal{Q}M$ generates L as a frame, i.e., $b = \bigvee \downarrow_A b$ for all $b \in L$. Then $A = \mathcal{Q}L$.*

Proof. For each $M \in F$ let $A_M \equiv \mathcal{Q}M$. Consider a cozero element b in L, say $b = \bigvee_I b_i$ for a scale $\{b_i\}$ in L. For $i < j$ in I, fix $c_{ij} \in L$ such that $b_i \wedge c_{ij} = \bot$ and $b_j \vee c_{ij} = \top$. Then the fact that $b_j \vee c_{ij} = \top$ implies that

$$\bigvee (\downarrow_A b_j \cup \downarrow_A c_{ij}) = \top,$$

and, since L is κ-Lindelöf, there is some $S_{ij} \subseteq_\kappa (\downarrow_A b_j \cup \downarrow_A c_{ij})$ with $\bigvee S_{ij} = \top$. Consequently $S_{ij} \subseteq \mathcal{Q}L_{\lambda(i,j)}$ for some $\lambda(i,j) \in \Lambda$. Let $\lambda_0 \in \Lambda$ be such that $\lambda_0 \geq \lambda(i,j)$ for all $i < j$ in I, and let $S \equiv \bigcup_{i<j} S_{ij}$. Note that $S \subseteq_\kappa \mathcal{Q}L_{\lambda_0}$.

For each $i < j$ in I let

$$b_i' \equiv \bigvee \downarrow_S b_i, \text{ and } c_{ij}' \equiv \bigvee \downarrow_S c_{ij}.$$

Then $b_i \geq b_i' \in L_{\lambda_0}$ and $c_{ij} \geq c_{ij}' \in L_{\lambda_0}$, and

$$b_j' \vee c_{ij}' = \bigvee (\downarrow_S b_i \cup \downarrow_S c_{ij}) \geq \bigvee S_{ij} = \top.$$

Thus the c_{ij}''s witness the fact that $\{b_i'\}$ is a scale in L_{λ_0}; let b' designate the cozero element $\bigvee b_i'$ in L_{λ_0}. We claim that $b' = b$. For it is quite clear that $b' \leq b$ since $b_i' \leq b_i$ for all $i \in I$. But for $i < j$ in I, the facts that

$$c_{ij}' \wedge b_i \leq c_{ij} \wedge b_i = \bot \text{ and } c_{ij}' \vee b_j' = \top$$

imply that $b_i \leq b_j'$. The claim follows, and the proof is complete. $\square$

We define an ordinal sequence of extensions of a frame L as follows.

$$\begin{aligned} L^0 &\equiv L, \\ L^{\alpha+1} &\equiv (L^\alpha)', \ g_L^{\alpha,\alpha+1} \equiv c_{L^\alpha} : L^\alpha \to L^{\alpha+1}, \\ L^\beta &\equiv \operatorname*{colim}_\beta L^\alpha, \ g_L^{\alpha,\beta} : L^\alpha \to L^\beta \equiv \text{ the colimit map, } \alpha < \beta, \beta \text{ a limit ordinal.} \end{aligned}$$

Morphisms $g_L^{\alpha,\alpha} : L^\alpha \to L^\alpha$ are defined to be the identity map for all α, and morphisms $g_L^{\alpha,\beta}$, $\alpha \leq \beta$, not already defined are defined by composition. A straightforward induction establishes that $g_L^{\beta,\gamma} g_L^{\alpha,\beta} = g_L^{\alpha,\gamma}$ for $\alpha \leq \beta \leq \gamma$.

Let us address functoriality. The passage from L to L' is certainly functorial: a frame morphism $f : L \to M$ has a unique extension $f' : L' \to M'$ such that $f' c_L = c_M f$, simply by applying the universality of c_L with respect to complementing the cozero elements of L

to the test map $c_M f$. And a straightforward induction yields, for each ordinal α, a unique frame map $f^\alpha : L^\alpha \to M^\alpha$ satisfying $f^\beta g_L^{\alpha,\beta} = g_M^{\alpha,\beta} f^\alpha$ for all $\alpha \leq \beta$.

Definition 7.12. *Let $\mathcal{P}$ be the functor which takes each frame L to L^κ, where $\kappa = \max\{\operatorname{lind} L, \omega_1\}$, and which takes each frame morphism $f : L \to M$ to f^κ, where $\kappa = \max(\operatorname{lind} L, \operatorname{lind} M, \omega_1)$. Designate by $\rho_L : L \to \mathcal{P}L$ the unit $g^{0,\kappa} : L \to L^\kappa$.*

Theorem 7.13. *P-frames are bireflective in frames, and, in particular, $\rho_L : L \to \mathcal{P}L$ serves as a reflector for the frame L. Moreover,*

$$\operatorname{lind} \mathcal{P}L \leq \max\{\operatorname{lind} L, \omega_1\}.$$

Proof. Let L be a frame with $\max\{\operatorname{lind} L, \omega_1\} = \kappa$. Let us first show that L^κ is a P-frame. A simple induction, based on Proposition 7.10 and Theorem 7.6, establishes that all L^α's, $0 < \alpha \leq \kappa$, are κ-Lindelöf. Furthermore, if we let $K^\alpha \equiv g_L^{\alpha,\kappa}(L^\alpha)$, $\alpha < \kappa$, then L^κ has $\{K^\alpha : \alpha < \kappa\}$ as a κ-directed family of subframes. Therefore $A \equiv \bigcup_\kappa \mathcal{Q}K^\alpha$ generates L and $A = \mathcal{Q}L$ by Proposition 7.11. But every member of each $\mathcal{Q}K^\alpha$ is complemented in $K^{\alpha+1}$ by construction, hence A is a Boolean algebra and L is a P-frame.

Now consider an arbitrary frame homomorphism f from L into a P-frame M. Then a simple induction, based only on Proposition 7.1 and the definition of colimit, establishes that, for all α, f extends uniquely to a morphism $f^\alpha : L^\alpha \to M$ such that $f^\beta g_L^{\alpha,\beta} = f^\alpha$ for all $\alpha \leq \beta \leq \kappa$. □

8. The relationship between the P-space coreflection and the P-frame reflection

The existence of the P-frame reflection raises a number of questions which are beyond the scope of this article. But we close by addressing three unavoidable queries.

(1) For a space X, is the P-frame reflection of the topology on X just the topology on the P-space coreflection of X? In other words, is
$$\mathcal{P}\mathcal{O}X \cong \mathcal{O}\mathcal{P}X?$$
(2) Is iteration really necessary? Is it possible, for example, that
$$\mathcal{P}L = L^\kappa = L^1 = L'?$$
(3) For a space X, is the P-space coreflection of X just the space of points of the P-frame reflection of the topology on X? In other words, is
$$\mathcal{P}X \cong \mathcal{S}\mathcal{P}\mathcal{O}X?$$

Let us take up the first question. There is a unique frame morphism $g : \mathcal{P}\mathcal{O}X \to \mathcal{O}\mathcal{P}X$ such that $g\rho_{\mathcal{O}X} = \mathcal{O}\rho_X$ because $\mathcal{O}\mathcal{P}X$ is a P-frame. This morphism is necessarily surjective, and a weaker form of question 1 is to ask whether it is also injective. This question is answered in the negative by Example 5.4. In this instance X is the unit interval $[0,1]$ in its standard topology and $\mathcal{P}X$ is $[0,1]_d$, the unit interval with discrete topology, and the P-space coreflection map $\rho_{[0,1]}$ is the identity $[0,1]_d \to [0,1]$. This gives $\mathcal{O}\rho_{[0,1]} : \mathcal{O}[0,1] \to \mathcal{O}[0,1]_d$ as the embedding of the frame of open subsets of $[0,1]$ into the full power set $2^{[0,1]}$. If, following

Corollary 5.3, we take $\mathcal{POX}$ to be $\mathcal{HRX}$ then g is just the map which sends a σ-ideal on $\mathcal{R}X$ to its union in $2^{[0,1]}$. This map is far from injective; for instance, there are many σ-ideals of Baire measurable subsets of $[0,1]$ whose union is all of $[0,1]$. This answers the weaker form of question 1. But question 1 itself is settled in the negative by the observation that $\mathcal{PO}[0,1]$ must be Lindelöf by Theorem 7.13, whereas $\mathcal{OP}X = 2^{[0,1]}$ is not Lindelöf.

Reasoning along the same lines as in the foregoing paragraph leads to the following conclusion. We omit the details.

Proposition 8.1. *For a compact space X, the unique frame map $g : \mathcal{POX} \to \mathcal{OPX}$ such that $g\rho_{\mathcal{O}X} = \mathcal{O}\rho_X$ is an isomorphism iff $\mathcal{P}X$ is Lindelöf.*

Let us now take up the second question. Again, Example 5.4 is instructive. Let $\mathcal{R}^\alpha$ designate the α-th stage in the formation of the Baire field $\mathcal{R}[0,1]$. Explicitly, set

$$\mathcal{R}^0 \equiv \mathcal{Q}[0,1] = \mathcal{O}[0,1],$$
$$\mathcal{R}^{\alpha+1} \equiv \left\{ \bigcup_{\mathbb{N}} (U_n \smallsetminus V_n) : U_n, V_n \in \mathcal{R}^\alpha \right\},$$
$$\mathcal{R}^\beta \equiv \bigcup_{\alpha<\beta} R^\alpha,\ \beta \text{ a limit ordinal.}$$

The Baire field $\mathcal{R}[0,1]$ is $\mathcal{R}^{\omega_1}$. By Corollary 5.3, $\mathcal{HR}[0,1]$ may be taken to be the P-frame reflection of $\mathcal{O}[0,1]$, with reflector map

$$U \longmapsto \{V \in \mathcal{R}[0,1] : V \subseteq U\},\ U \in \mathcal{O}[0,1].$$

A simple induction, based on the fact that each element of L' is the join of differences of (images of) elements of $\mathcal{Q}L$ (Corollary 7.2), shows that this map lifts to a unique map from $\mathcal{O}[0,1]^\alpha$ onto the subframe of $\mathcal{HR}[0,1]$ generated by

$$\{V \in \mathcal{R}[0,1] : V \subseteq U\},\ U \in \mathcal{R}^\alpha,\ \alpha < \omega_1.$$

Since $\mathcal{R}^\alpha$ is properly contained in $\mathcal{R}^{\alpha+1}$ for $\alpha < \omega_1$, we see that the full iteration called for in the proof of our main Theorem 7.13 is necessary in this example.

In contrast to the first two, the answer to the third question is positive.

Proposition 8.2. *For a space X, $\delta_X^{-1} \circ \mathcal{S}\rho_{\mathcal{O}X} : \mathcal{SPO}X \to X$ is a P-space coreflector for X.*

Proof. Consider a continuous function $f : Y \to X$ for some P-space Y. Since $\mathcal{O}f$ is a frame map from $\mathcal{O}X$ into the P-frame $\mathcal{O}Y$, there is a unique frame map $g : \mathcal{PO}X \to \mathcal{O}Y$ such that $g\rho_{\mathcal{O}X} = \mathcal{O}f$. Then

$$f = \delta_X^{-1} \circ \mathcal{SO}f \circ \delta_Y = \left(\delta_X^{-1} \circ \mathcal{S}\rho_{\mathcal{O}X}\right)(\mathcal{S}g \circ \delta_Y)$$

is the desired factorization. □

More is true. The P-frame reflection $\mathcal{PO}X$ of $\mathcal{O}X$ is $(\mathcal{O}X)^\kappa$, the result of freely complementing the cozero elements through κ iterations, where $\kappa = \max\{\operatorname{lind} L, \omega_1\}$. If we carry out the iteration just once, say $f : \mathcal{O}X \to (\mathcal{O}X)'$, then one may show that already

$$\delta_X^{-1} \circ \mathcal{S}f : \mathcal{S}(\mathcal{O}X)' \to X$$

is a P-space coreflection for X. We omit the details.

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(Ball) Department of Mathematics, University of Denver, Denver, Colorado 80208, U.S.A.
E-mail address, Ball: `rball@du.edu`
URL: `http://www.math.du.edu/~rball`

(Walters-Wayland) OCCTAL,Orange, California 92630, U.S.A.
E-mail address, Walters-Wayland: `joanne@waylands.com`

Spatial Corporation, Broomfield, Colorado 80021, U.S.A.
E-mail address, Zenk: `eric.zenk@3ds.com`