

# CAUSAL SET APPROACH TO DISCRETE QUANTUM GRAVITY

S. Gudder  
Department of Mathematics  
University of Denver  
Denver, Colorado 80208, U.S.A.  
sgudder@du.edu

## Abstract

We begin by describing a sequential growth model in which the universe grows one element at a time in discrete time steps. At each step, the process has the form of a causal set and the “completed” universe is given by a path consisting of a discretely growing chain of causal sets. We then introduce a quantum dynamics to obtain a quantum sequential growth process (QSGP) which may lead to a viable model for discrete quantum gravity. A discrete version of Einstein’s field equation is derived and a definition for discrete geodesics is proposed. A type of QSGP called an amplitude process is introduced. An example of an amplitude process called a complex percolation process is studied. This process conforms with general principles of causality and covariance. We end with some detailed quantum measure calculations for a specific percolation constant.

## 1 Introduction

The causal set approach to discrete quantum gravity is an attempt to unify general relativity and quantum mechanics [1, 8, 9]. These two theories are quite different and it is not at all clear how such a unification is possible. This question has been investigated for about 80 years and is probably the greatest unsolved problem in theoretical physics. Briefly speaking, quantum theory is based on the study of self-adjoint and unitary operators on a complex Hilbert space, while general relativity is based on the study of smooth curves and tensors on a 4-dimensional, real, differentiable manifold  $M$  with a Lorentzian metric tensor  $g_{\mu\nu}$ .

We first examine  $M$  more closely. For each  $a \in M$  there is a forward light cone  $C_a^+ \subseteq M$  consisting of points in the future of  $a$  that  $a$  can communicate with via a light signal. If  $b \in C_a^+$  we say that  $b$  is in the *causal future* of  $a$  and write  $a < b$ . Then  $(M, <)$  becomes a *partially ordered set* (poset); that is,  $a \not< a$  (irreflexivity) and  $a < b, b < c$  imply that  $a < c$  (transitivity). We call  $(M, <)$  the *causal structure* on  $M$ . To remind us that we are dealing with causal structures, we call an arbitrary finite poset a *causal set* (or *causet*). Investigators have shown that the causal structure completely determines  $M$  [8, 9]. That is,  $<$  determines the topology, differential structure, smooth functions, dimension, line and volume elements and tensor  $g_{\mu\nu}$  for  $M$ . We can therefore forget about the differential structure of  $M$  and only consider the poset  $(M, <)$  which is clearly a great simplification.

In comparing quantum mechanics (especially quantum field theory) and general relativity, we notice one important similarity. They both contain many singularities and the theories break down at small distances. This indicates that quantum mechanics should be based upon a finite-dimensional complex Hilbert space and that general relativity should be discrete with a minimum distance which we take to be a Planck length  $\ell_p \approx 1.6 \times 10^{-33}$  cm. and a minimum time which we take to be a Planck instant  $t_p \approx 5.4 \times 10^{-44}$  sec.

Beside discreteness, a second motivating feature of the causal set approach is that the universe is expanding both in size and in matter creation. These features suggest that we should consider a discrete sequential growth model. In such a model, the universe grows one element at a time in discrete steps given by Planck instants. At each step, the universe has the form of a causet and the “completed” universe is given by a path consisting of a discretely growing chain of causets. We then introduce a quantum dynamics  $\rho_n, n = 1, 2, \dots$ , to obtain a quantum sequential growth process (QSGP) which may lead to a viable model for discrete quantum gravity. The dynamics  $\rho_n$  is given by a positive operator on the Hilbert space of causet paths of length  $n$ . The operators  $\rho_n$  are required to satisfy certain normalization and consistency conditions.

At this stage of development, the precise form of  $\rho_n$  is not known. However, a discrete version of Einstein’s field equation is derived and it is possible that  $\rho_n$  can be specified by determining whether this discrete equation is approximated by the classical Einstein equation. We also propose a definition for discrete geodesics.

Although various constructions of a QSGP  $\rho_n$  are known [2, 3, 4], we now introduce a particularly simple type called an amplitude process. An example of an amplitude process called a complex percolation process is studied.

This process conforms with general principles of causality and covariance. Some detailed quantum measure calculations for a specific percolation constant are performed and some geodesics are briefly examined.

## 2 Sequential Growth Model

Let  $\mathcal{P}_n$  be the collection of all causets of cardinality  $n$ ,  $n = 1, 2, \dots$ , and let  $\mathcal{P} = \cup \mathcal{P}_n$ . Two isomorphic causets are considered to be identical. If  $x \in \mathcal{P}$  and  $a, b \in x$ , we say that  $a$  is an *ancestor* of  $b$  and  $b$  is a *successor* of  $a$  if  $a < b$ . We say that  $a$  is a *parent* of  $b$  and  $b$  is a *child* of  $a$  if  $a < b$  and there is no  $c \in x$  with  $a < c < b$ . We call  $a$  *maximal* in  $x$  if there is no  $b \in x$  with  $a < b$ . If  $x \in \mathcal{P}_n$ ,  $y \in \mathcal{P}_{n+1}$ , then  $x$  *produces*  $y$  (and  $y$  is a *product* of  $x$ ) if  $y$  is obtained from  $x$  by adjoining a single element  $a$  to  $x$  that is maximal in  $y$ . Thus,  $a$  is not in the causal past of any element of  $y$ . If  $x$  produces  $y$ , we write  $x \rightarrow y$ . The transitive closure of  $\rightarrow$  makes  $\mathcal{P}$  into a poset and we call  $(\mathcal{P}, \rightarrow)$  a *sequential growth model*. A *path* in  $\mathcal{P}$  is a string (sequence)  $\omega = \omega_1 \omega_2 \dots$ ,  $\omega_i \in \mathcal{P}_i$  and  $\omega_i \rightarrow \omega_{i+1}$ ,  $i = 1, 2, \dots$ . An *n-path* is a finite string  $\omega = \omega_1 \omega_2 \dots \omega_n$ , where again  $\omega_i \in \mathcal{P}_i$  and  $\omega_i \rightarrow \omega_{i+1}$ . We denote the set of paths by  $\Omega$  and the set of  $n$ -paths by  $\Omega_n$ . If  $x$  produces  $y$  in  $r$  isomorphic ways, we say that the *multiplicity* of  $x \rightarrow y$  is  $r$  and write  $m(x \rightarrow y) = r$ . For example, in Figure 1,  $m(x_3 \rightarrow x_6) = 2$  and multiplicities greater than 1 are designated. To be precise, the different isomorphic ways requires a labeling of the causets. This is the only place we need to mention labeled causets and we otherwise only consider unlabeled causets.

We think of a path  $\omega \in \Omega$  as a possible universe (universe history) [2, 3, 4]. For  $\omega = \omega_1 \omega_2 \dots$ ,  $\omega_i \in \mathcal{P}_i$  represents a universe at Planck instant  $i$ . This gives a growth model for the universe [6, 9, 10]. The vertices of the causet  $\omega_i$  represent a space-time framework (scaffolding) at step  $i$  (instant  $i$ ). A vertex may or may not be occupied by a point mass or energy. Figure 1 gives the first four steps of the sequential growth model representing possible universes. The vertical rectangles represent antimatter causets, the horizontal rectangles represent matter causets and the circles represent mixed causets. For a discussion of these types, we refer the reader to [4].

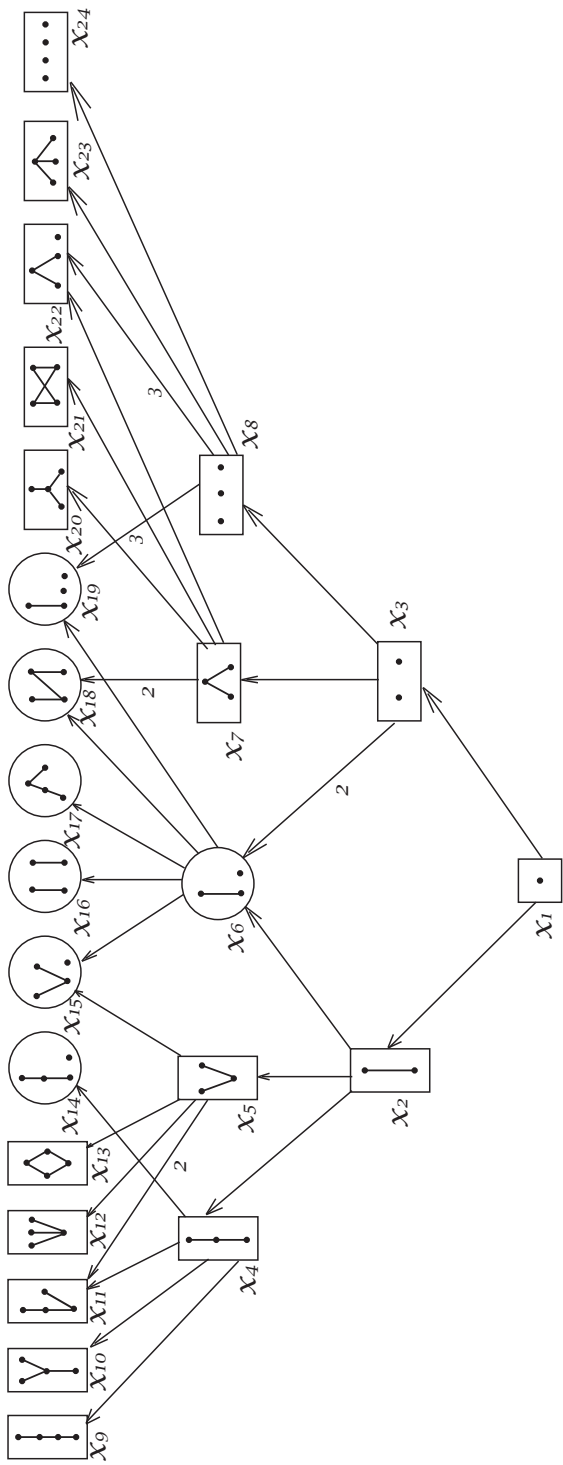


Figure 1

We use the notation  $\mathcal{A}_n$  for the power set  $2^{\Omega_n}$ ,  $n = 1, 2, \dots$ . For  $x \in \mathcal{P}_i$  we use the notation

$$x \rightarrow = \{y \in \mathcal{P}_{i+1} : x \rightarrow y\}$$

and for  $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega_n$  we write

$$\omega \rightarrow = \{\omega_1\omega_2 \cdots \omega_n\omega_{n+1} : \omega_n \rightarrow \omega_{n+1}\} \in \mathcal{A}_{n+1}$$

Finally, for  $A \in \mathcal{A}_n$  we define

$$A \rightarrow = \bigcup_{\omega \in A} (\omega \rightarrow) \in \mathcal{A}_{n+1}$$

The set of paths beginning with  $\omega = \omega_1 \cdots \omega_n \in \Omega_n$  is called an *elementary cylinder set* and is denoted  $\text{cyl}(\omega)$ . If  $A \in \mathcal{A}_n$ , then the *cylinder set*  $\text{cyl}(A)$  is defined by

$$\text{cyl}(A) = \bigcup_{\omega \in A} \text{cyl}(\omega)$$

Using the notation

$$\mathcal{C}(\Omega_n) = \{\text{cyl}(A) : A \in \mathcal{A}_n\}$$

notice that if  $A \in \mathcal{C}(\Omega_n)$ , then  $A = \text{cyl}(A_1)$  for some  $A_1 \in \mathcal{A}_n$  so  $A = \text{cyl}(A_1 \rightarrow) \in \mathcal{C}(\Omega_{n+1})$ . We conclude that

$$\mathcal{C}(\Omega_1) \subseteq \mathcal{C}(\Omega_2) \subseteq \cdots$$

is an increasing sequence of subalgebras of the *cylinder algebra*  $\mathcal{C}(\Omega) = \cup \mathcal{C}(\Omega_n)$ . For  $A \in 2^\Omega$  we define the set  $A^n \in \mathcal{A}_n$  by

$$A^n = \{\omega_1\omega_2 \cdots \omega_n \in \Omega_n : \omega_1\omega_2 \cdots \omega_n\omega_{n+1} \cdots \in A\}$$

We think of  $A^n$  as the step- $n$  approximation to  $A$ . Notice that  $A^n$  is the set of  $n$ -paths whose continuations are in  $A$ .

### 3 Quantum Sequential Growth Processes

Denoting the cardinality of a set  $A$  by  $|A|$ , we define the  $|\Omega_n|$ -dimensional complex Hilbert space  $H_n = L_2(\Omega_n, \mathcal{A}_n, \nu_n)$  where  $\nu_n$  is the counting measure on  $\Omega_n$ . Of course,  $H_n$  is isomorphic to  $\mathbb{C}^{|\Omega_n|}$ . Let  $\chi_A$  denote the characteristic function of a set  $A \in \mathcal{A}_n$  and let  $1_n = \chi_{\Omega_n}$ . A positive operator  $\rho_n$  on  $H_n$  satisfying  $\langle \rho_n 1_n, 1_n \rangle = 1$  is called a *q-probability operator* and the set of *q-probability operators* on  $H_n$  is denoted  $\mathcal{Q}(H_n)$ . Corresponding to

$\rho_n \in \mathcal{Q}(H_n)$  we have an  $n$ -decoherence functional  $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$  given by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

which gives a measure of the interference between  $A$  and  $B$ . It is easy to show that  $D_n$  has the usual properties of a decoherence functional. That is,  $D_n(\Omega_n \Omega_n) = 1$ ,  $D_n(A, B) = \overline{D_n(B, A)}$ ,  $A \mapsto D_n(A, B)$  is a complex measure on  $\mathcal{A}_n$  for any  $B \in \mathcal{A}_n$  and if  $A_1, \dots, A_m \in \mathcal{A}_n$  then  $D_n(A_i, A_j)$  are the components of a positive semidefinite  $m \times m$  matrix. The map  $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$  given by  $\mu_n(A) = D_n(A, A)$  is called the  $q$ -measure corresponding to  $\rho_n$  [7]. We interpret  $\mu_n(A)$  as the *propensity* of the event  $A$  when the system is described by  $\rho_n$  [2, 3]. Notice that  $\mu_n(\Omega_n) = 1$ . Although  $\mu_n$  is not additive, it does satisfy the *grade-2 additivity* condition: if  $A, B, C \in \mathcal{A}_n$  are mutually disjoint, then

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

We say that a sequence  $\rho_n \in \mathcal{Q}(H_n)$ ,  $n = 1, 2, \dots$ , is *consistent* if  $D_{n+1}(A \rightarrow, B \rightarrow) = D_n(A, B)$  for all  $A, B \in \mathcal{A}_n$ . Of course, it follows that  $\mu_{n+1}(A \rightarrow) = \mu_n(A)$  for all  $A \in \mathcal{A}_n$ . A consistent sequence  $\rho_n \in \mathcal{Q}(H_n)$  provides a quantum dynamics for the growth model  $(\mathcal{P}, \rightarrow)$  and we call  $\rho_n$  a *quantum sequential growth process* (QSGP) [2, 3, 4]. At this stage of development we do not know the specific form of  $\rho_n$  that would describe quantum gravity. It is hoped that further theoretical properties or experimental data will determine  $\rho_n$ . One possible approach is considered in Section 4.

Let  $\rho_n \in \mathcal{Q}(H_n)$  be a QSGP. Although we have a  $q$ -measure  $\mu_n$  on  $\mathcal{A}_n$ ,  $n = 1, 2, \dots$ , it is important to extend  $\mu_n$  to physically relevant subsets of  $\Omega$  in a systematic way. We say that a set  $A \subseteq \Omega$  is *beneficial* if  $\lim \mu_n(A^n)$  exists and is finite in which case we define  $\mu(A)$  to be this limit. We denoted the collection of beneficial sets by  $\mathcal{B}(\rho_n)$ . If  $A \in \mathcal{C}(\Omega)$  is a cylinder set, then  $A \in \mathcal{C}(\Omega_i)$  for some  $i \in \mathbb{N}$ . In this case  $A = \text{cyl}(A_1)$  for some  $A_1 \in \mathcal{A}_i$ . Now  $A^i = A_1$ ,  $A^{i+1} = A_1 \rightarrow$ ,  $A^{i+2} = (A_1 \rightarrow) \rightarrow$ ,  $\dots$ . Hence,

$$\lim \mu_n(A^n) = \mu_i(A^i) = \mu_i(A_1)$$

so  $A$  is beneficial and  $\mu(A) = \mu_i(A_1)$ . We conclude that  $\mathcal{C}(\Omega) \subseteq \mathcal{B}(\rho_n)$  and if  $A \in \mathcal{C}(\Omega)$  then  $\mu(A) = \mu_n(A^n)$  for  $n$  sufficiently large. Simple examples are  $\emptyset, \Omega \in \mathcal{B}(\rho_n)$  with  $\mu(\emptyset) = 0$ ,  $\mu(\Omega) = 1$ . Of course, there are physically relevant subsets of  $\Omega$  that are not cylinder sets. For example, if  $\omega \in \Omega$  then  $\{\omega\} \notin \mathcal{C}(\Omega)$ . Whether  $\{\omega\} \in \mathcal{B}(\rho_n)$  depends on  $\rho_n$  and we shall consider some examples in Section 6.

## 4 Discrete Einstein Equation

Let  $Q_n = \cup_{i=1}^n \mathcal{P}_i$  and let  $K_n$  be the Hilbert space  $\mathbb{C}^{Q_n}$  with the standard inner product

$$\langle f, g \rangle = \sum_{x \in Q_n} \overline{f(x)} g(x)$$

Let  $L_n = K_n \otimes K_n$  which we identify with  $\mathbb{C}^{Q_n \times Q_n}$ . Let  $\rho_n \in \mathcal{Q}(H_n)$  be a QSGP with corresponding decoherence matrices

$$D_n(\omega, \omega') = D_n(\{\omega\}, \{\omega'\}), \quad \omega, \omega' \in \Omega_n$$

If  $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$  and  $\omega_i = x$  for some  $i$ , then  $\omega$  contains  $x$ . For  $x, y \in Q_n$  we define

$$D_n(x, y) = \sum \{D_n(\omega, \omega') : \omega \text{ contains } x, \omega' \text{ contains } y\}$$

Due to the consistency of  $\rho_n$ ,  $D_n(x, y)$  is independent of  $n$  if  $n \geq |x|, |y|$  where  $|x|$  is the cardinality of  $x \in Q_n$ . Also  $D_n(x, y)$ ,  $x, y \in Q_n$ , are the components of a positive semi-definite matrix.

We think of  $Q_n$  as an analogue of a differentiable manifold and  $D_n(x, y)$  as an analogue of a metric tensor. If  $y \in Q_n$  and  $x \rightarrow y$  we think of the pair  $(x, y)$  as a tangent vector at  $y$ . Thus, there are as many tangent vectors at  $y$  as there are producers of  $y$ . Finally, the elements of  $K_n$  are analogous to smooth functions on the manifold.

If  $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$  and  $\omega_i = x$ , then  $i = |x|$  and  $\omega$  contains  $x$  if and only if  $\omega_{|x|} = x$ . An  $n$ -path  $\omega$  containing  $x$  determines a tangent vector  $(\omega_{|x|-1}, x)$  at  $x$  (assuming  $|x| \geq 2$ ). For  $\omega \in \Omega_n$  define the *difference operator*  $\Delta_\omega^n$  on  $K_n$  by

$$\Delta_\omega^n f(x) = [f(x) - f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}}$$

where  $\delta_{x, \omega_{|x|}}$  is the Kronecker delta. It is easy to check that  $\Delta_\omega^n$  satisfies the *discrete Leibnitz rule*:

$$\Delta_\omega^n f g(x) = f(x) \Delta_\omega^n g(x) + g(\omega_{|x|-1}) \Delta_\omega^n f(x)$$

Given a function  $f \in \mathbb{C}^{Q_n \times Q_n} = L_n$  of two variables we have the function  $\tilde{f} \in K_n$  of one variable  $\tilde{f}(x) = f(x, x)$  and given a function  $g \in K_n$  we have the functions of two variables  $g_1, g_2 \in L_n$  where  $g_1(x, y) = g(x)$  and  $g_2(x, y) = g(y)$  for all  $x, y \in Q_n$ . For  $\omega, \omega' \in \Omega_n$ , we want  $\Delta_{\omega, \omega'}^n : L_n \rightarrow L_n$  that extends  $\Delta_\omega^n$  and satisfies Leibnitz's rule. That is,

$$\Delta_{\omega, \omega'}^n g_1(x, y) = \Delta_\omega^n g(x) \delta_{y, \omega'_{|y|}}, \Delta_{\omega, \omega'}^n g_2(x, y) = \Delta_{\omega'}^n g(y) \delta_{x, \omega_{|x|}} \quad (4.1)$$

and

$$\Delta_{\omega,\omega'}^n fg(x,y) = f(x,y) \Delta_{\omega,\omega'} g(x,y) + g(\omega_{|x|-1}, \omega'_{|x|-1}) \Delta_{\omega,\omega'}^n f(x,y) \quad (4.2)$$

The next two theorems are proved in [5]

**Theorem 4.1.** *A linear operator  $\Delta_{\omega,\omega'}^n: L_n \rightarrow L_n$  satisfies (4.1) and (4.2) if and only if it has the form*

$$\Delta_{\omega,\omega'}^n f(x,y) = \left[ f(x,y) - f(\omega_{|x|-1}, \omega'_{|y|-1}) \right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}} \quad (4.3)$$

The result (4.3) is not surprising because it is the natural extension of  $\Delta_{\omega}^n$  from  $K_n$  to  $L_n$ . Also,  $\Delta_{\omega,\omega'}^n$  extends  $\Delta_{\omega}^n$  in the sense that

$$\Delta_{\omega,\omega}^n f(x,y) = \Delta_{\omega}^n \tilde{f}(x)$$

**Theorem 4.2.** (a) *A linear operator  $T_{\omega}: K_n \rightarrow K_n$  satisfies the Leibnitz rule and  $T_{\omega}f(x) = 0$  when  $\omega_{|x|} \neq x$  if and only if there exists a function  $\beta_{\omega}: Q_n \rightarrow \mathbb{C}$  such that  $T_{\omega} = \beta_{\omega} \Delta_{\omega}^n$ . (b) *A linear operator  $T_{\omega,\omega'}: L_n \rightarrow L_n$  satisfies the Leibnitz rule and  $T_{\omega,\omega'}f(x,y) = 0$  when  $\omega_{|x|} \neq x$  or  $\omega'_{|y|} \neq y$  if and only if there exists a function  $\beta_{\omega,\omega'}: Q_n \times Q_n \rightarrow \mathbb{C}$  such that  $T_{\omega,\omega'} = \beta_{\omega,\omega'} \Delta_{\omega,\omega'}^n$ .**

It is clear that  $\mu_n(x) = D_n(x,x)$  is not stationary. That is,  $\Delta_{\omega}^n \mu_n(x) \neq 0$  for all  $x \in Q_n$  in general. It is shown in [5] that the simplest nontrivial combination  $\nabla_{\omega}^n = \beta_{\omega} \Delta_{\omega} + \alpha_{\omega}$  satisfying  $\nabla_{\omega}^n \mu_n(x) = 0$  for all  $x \in Q_n$  is given by

$$\nabla_{\omega}^n f(x) = [\mu_n(\omega_{|x|-1})f(x) - \mu_n(x)f(\omega_{|x|-1})] \delta_{x,\omega_{|x|}}$$

We call  $\nabla_{\omega}^n$  the *covariant difference operator*.

Again,  $\Delta_{\omega,\omega'}^n D_n(x,y) \neq 0$  for all  $x,y \in Q_n$ . It is shown in [5] that the simplest nontrivial combination

$$\nabla_{\omega,\omega'}^n = \beta_{\omega,\omega'} \Delta_{\omega,\omega'}^n + \alpha_{\omega,\omega'}$$

satisfying  $\nabla_{\omega,\omega'}^n D_n(x,y) = 0$  for all  $x,y \in Q_n$  is given by

$$\nabla_{\omega,\omega'}^n f(x,y) = \left[ D_n(\omega_{|x|-1}, \omega'_{|y|-1})f(x,y) - D_n(x,y)f(\omega_{|x|-1}, \omega'_{|y|-1}) \right] \delta_{x,\omega_{|x|}} \delta_{y,\omega'_{|y|}}$$

We call  $\nabla_{\omega,\omega'}^n$  the *covariant bidifference operator*.



The *curvature operator* is defined as

$$\mathcal{R}_{\omega, \omega'}^n = \nabla_{\omega, \omega'}^n - \nabla_{\omega', \omega}^n$$

We define the *metric operator*  $\mathcal{D}_{\omega, \omega'}^n$  on  $L_n$  by

$$\begin{aligned} \mathcal{D}_{\omega, \omega'}^n f(x, y) = D_n(x, y) & \left[ f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right. \\ & \left. - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right] \end{aligned}$$

and the *mass-energy operator*  $\mathcal{T}_{\omega, \omega'}^n$  on  $L_n$  by

$$\begin{aligned} \mathcal{T}_{\omega, \omega'}^n f(x, y) = & \left[ D_n(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right. \\ & \left. - D_n(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right] f(x, y) \end{aligned}$$

It is shown in [5] that

$$\mathcal{R}_{\omega, \omega'}^n = \mathcal{D}_{\omega, \omega'}^n + \mathcal{T}_{\omega, \omega'}^n \tag{4.4}$$

We call (4.4) the *discrete Einstein equation* [5, 11]. In this sense, Einstein's equation always holds in this framework no matter what we have for the quantum dynamics  $\rho_n$ . One might argue that we obtained (4.4) just by definition. However, our derivation shows that  $\mathcal{R}_{\omega, \omega'}^n$  is a reasonable counterpart of the classical curvature tensor and  $\mathcal{D}_{\omega, \omega'}^n$  is a discrete counterpart of the metric tensor.

Equation (4.4) does not give information about  $D_n(x, y)$  and  $D_n(\omega, \omega')$  (which after all, are what we wanted to find), but it may give useful indirect information. If we can find  $D_n(\omega, \omega')$  such that the classical Einstein equation is an approximation to (4.4), then this gives information about  $D_n(\omega, \omega')$ . Moreover, an important problem in discrete quantum gravity theory is how to test whether general relativity is a close approximation to the theory. Whether Einstein's equation is an approximation to (4.4) would provide such a test. In order to consider approximations by Einstein's equation, it will be necessary to let  $n \rightarrow \infty$  in (4.4). However, the convergence of the operators depends on  $D_n$  and will be left for later investigations.

We now propose a definition for discrete geodesics. For  $A \subseteq Q_n$  we define the  $q$ -measure

$$\mu_n(A) = \mu_n(\{\omega \in \Omega_n : \omega_{|x|} = x \text{ for some } x \in A\})$$

and for  $x, y \in Q_n$  let

$$\mu_n(x \cap y) = \mu_n(\{\omega \in \Omega_n : \omega_{|x|} = x \text{ and } \omega_{|y|} = y\})$$

For  $x, y \in Q_n$  it is natural to define the *conditional q-measure*

$$\mu_n(x | y) = \frac{\mu_n(x \cap y)}{\mu_n(y)} \quad \text{if } \mu_n(y) \neq 0$$

and  $\mu_n(x | y) = 0$  if  $\mu_n(y) = 0$ . For  $\omega \in \Omega_n$  define the function  $\widehat{\omega}: Q_n \rightarrow \mathbb{R}^+$  by

$$\widehat{\omega}(x) = \mu_n(x | \omega_{|x|-1}) \delta_{x, \omega_{|x|}} \quad (4.5)$$

Of course,  $\widehat{\omega} \in K_n$ . We say that  $\omega \in \Omega_n$  is a *discrete geodesic* if there is an  $a \in \mathbb{R}$  such that  $\Delta_\omega^n \widehat{\omega} = a \widehat{\omega}$ ; that is  $\widehat{\omega}$  is an eigenvector of  $\Delta_\omega^n$ .

**Theorem 4.3.** *An  $n$ -path  $\omega \in \Omega_n$  is a discrete geodesic if and only if there exists a  $c \in \mathbb{R}$  such that whenever  $\omega_{|x|} = x$  for  $|x| \geq 3$  we have*

$$\mu_n(x | \omega_{|x|-1}) = c \mu_n(\omega_{|x|-1} | \omega_{|x|-2}) \quad (4.6)$$

*Proof.* By definition,  $\omega$  is a discrete geodesic if and only if there is an  $a \in \mathbb{R}$  such that for all  $x \in Q_n$  we have

$$a \widehat{\omega}(x) = \Delta_\omega^n \widehat{\omega}(x) = [\widehat{\omega}(x) - \widehat{\omega}(\omega_{|x|-1})] \delta_{x, \omega_{|x|}}$$

Letting  $c = 1 - a$  this last statement is equivalent to

$$\widehat{\omega}(x) = c \widehat{\omega}(\omega_{|x|-1}) \delta_{x, \omega_{|x|}}$$

Applying (4.5) we conclude that

$$\mu_n(x | \omega_{|x|-1}) \delta_{x, \omega_{|x|}} = c \mu_n(\omega_{|x|-1} | \omega_{|x|-2}) \delta_{x, \omega_{|x|}}$$

If  $\omega_{|x|} \neq x$ , both sides of this equation vanish so the equation holds. If  $\omega_{|x|} = x$  we obtain (4.6).  $\square$

This definition of a discrete geodesic is very restrictive and it seems desirable to have a more general concept. If  $\omega = \omega_j \omega_{j+1} \cdots \omega_n$  with  $\omega_i \in \mathcal{P}_i$ ,  $\omega_i \rightarrow \omega_{i+1}$ ,  $i = j, \dots, n-1$ , we call  $\omega$  an  *$n$ -path starting at  $\omega_j$* . Motivated by Theorem 4.3, we say that  $\omega = \omega_j \omega_{j+1} \cdots \omega_n$  is a *discrete geodesic starting at  $\omega_j$*  if  $\omega$  is a maximal  $n$ -path starting at  $\omega_j$  satisfying

$$\mu_n(\omega_k | \omega_{k+1}) = c \mu_n(\omega_{k-1} | \omega_{k-2}), \quad k = j+2, \dots, n$$

for some  $c \in \mathbb{R}$ .

## 5 Amplitude Processes

Various constructions of a QSGP have been investigated [2, 3, 4]. In this section we introduce a simple type of QSGP that we call an amplitude process. If nothing else, this might serve as a toy model for discrete quantum gravity.

For  $x \in \mathcal{P}_n$ ,  $y \in \mathcal{P}_{n+1}$  with  $x \rightarrow y$ , let  $a(x \rightarrow y) \in \mathbb{C}$  satisfy

$$\sum \{a(x \rightarrow y) : y \in x \rightarrow\} = 1 \quad (5.1)$$

We call  $a(x \rightarrow y)$  a *transition amplitude* from  $x$  to  $y$ . By convention we define  $a(x \rightarrow y) = 0$  if  $x \not\rightarrow y$ . For  $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega_n$  we define the *amplitude* of  $\omega$  by

$$a_n(\omega) = a(\omega_1 \rightarrow \omega_2)a(\omega_2 \rightarrow \omega_3) \cdots a(\omega_{n-1} \rightarrow \omega_n)$$

and we call the vector  $a_n \in H_n$  an *amplitude vector*. For  $\omega, \omega' \in \Omega_n$  define the *decoherence matrix* as

$$D_n(\omega, \omega') = a_n(\omega)\overline{a_n(\omega')}$$

Let  $\rho_n$  be the operator on  $H_n$  given by the matrix  $D_n(\omega, \omega')$ . We call the sequence of operators  $\rho_n$ ,  $n = 1, 2, \dots$ , an *amplitude process* (AP).

**Theorem 5.1.** *An AP  $\rho_n$  is a QSGP.*

*Proof.* It is clear that  $\rho_n$  is a positive operator on  $H_n$ . Moreover, we have

$$\begin{aligned} \langle \rho_n 1_n, 1_n \rangle &= \left\langle \sum_{\omega' \in \Omega_n} D_n(\omega, \omega'), 1_n \right\rangle = \sum_{\omega, \omega' \in \Omega_n} D_n(\omega, \omega') \\ &= \sum_{\omega, \omega' \in \Omega_n} a_n(\omega)\overline{a_n(\omega')} = \left| \sum_{\omega \in \Omega_n} a_n(\omega) \right|^2 \end{aligned} \quad (5.2)$$

Applying (5.1) we obtain

$$\begin{aligned} \sum_{\omega \in \Omega_n} a_n(\omega) &= \sum a(\omega_1 \rightarrow \omega_2)a(\omega_2 \rightarrow \omega_3) \cdots a(\omega_{n-1} \rightarrow \omega_n) \\ &= \sum a(\omega_1 \rightarrow \omega_2) \cdots a(\omega_{n-2} \rightarrow \omega_{n-1}) \sum_{\omega_{n-1} \rightarrow} a(\omega_{n-1} \rightarrow \omega_n) \\ &= \sum a(\omega_1 \rightarrow \omega_2) \cdots a(\omega_{n-2} \rightarrow \omega_{n-1}) \\ &\quad \vdots \\ &= \sum_{\omega_1 \rightarrow} a(\omega_1 \rightarrow \omega_2) = 1 \end{aligned} \quad (5.3)$$

By (5.2) and (5.3) we conclude that  $\langle \rho_n 1_n, 1_n \rangle = 1$ . To show that  $\rho_n$  is a consistent sequence, let  $\omega, \omega' \in \Omega_n$  with  $\omega = \omega_1 \omega_2 \cdots \omega_n$ ,  $\omega' = \omega'_1 \omega'_2 \cdots \omega'_n$ . By (5.1) we have

$$\begin{aligned}
D_{n+1}(\omega \rightarrow, \omega' \rightarrow) &= \langle \rho_{n+1} \chi_{\omega \rightarrow}, \chi_{\omega' \rightarrow} \rangle \\
&= \sum \left\{ a_n(\omega) a(\omega_n \rightarrow x) \overline{a_n(\omega')} \overline{a(\omega'_n \rightarrow y)} : \omega_n \rightarrow x, \omega'_n \rightarrow y \right\} \\
&= a_n(\omega) \overline{a_n(\omega')} \sum \{ a(\omega_n \rightarrow x) : \omega_n \rightarrow x \} \sum \{ \overline{a(\omega'_n \rightarrow y)} : \omega'_n \rightarrow y \} \\
&= a_n(\omega) \overline{a_n(\omega')} = D_n(\omega, \omega') \tag{5.4}
\end{aligned}$$

For  $A, B \in \mathcal{A}_n$ , by (5.4) we have

$$\begin{aligned}
D_{n+1}(A \rightarrow, B \rightarrow) &= \sum \{ D_{n+1}(\omega \rightarrow, \omega' \rightarrow) : \omega \in A, \omega' \in B \} \\
&= \sum \{ D_n(\omega, \omega') : \omega \in A, \omega' \in B \} \\
&= D_n(A, B)
\end{aligned}$$

□

Since the operator  $\rho_n$  on  $H_n$  has the form  $\rho_n = |a_n\rangle\langle a_n|$  we not only see that  $\rho_n$  is a positive operator but that it has rank 1 with norm

$$\|\rho_n\| = \||a_n\rangle\langle a_n|\| = \|a_n\|^2 = \sum |a_n(\omega)|^2 = \text{tr}(\rho_n)$$

The decoherence functional corresponding to  $\rho_n$  becomes

$$\begin{aligned}
D_n(A, B) &= \langle \rho_n \chi_B, \chi_A \rangle = \langle |a_n\rangle\langle a_n| \chi_B, \chi_A \rangle \\
&= \langle a_n, \chi_A \rangle \langle \chi_B, a_n \rangle = \sum_{\omega \in A} a_n(\omega) \sum_{\omega' \in B} \overline{a_n(\omega')} \\
&= \sum \{ D_n(\omega, \omega') : \omega \in A, \omega' \in B \}
\end{aligned}$$

for all  $A, B \in \mathcal{A}_n$  which is what we expect. The corresponding  $q$ -measure is given by

$$\mu_n(A) = D_n(A, A) = |\langle a_n, \chi_A \rangle|^2 = \left| \sum_{\omega \in A} a_n(\omega) \right|^2 \tag{5.5}$$

for all  $A \in \mathcal{A}_n$ . In particular, for  $\omega \in \Omega_n$  we have  $\mu_n(\{\omega\}) = |a_n(\omega)|^2$ . We conclude that  $A \in \mathcal{B}(\rho_n)$  if and only if

$$\lim \mu_n(A^n) = \lim \left| \sum_{\omega \in A^n} a_n(\omega) \right|^2$$

exists and is finite in which case  $\mu(A)$  is the limit.

## 6 Complex Percolation Process

This section introduces a particular type of AP that still has physical relevance. We use the notation  $y = x \uparrow a$  if  $x \rightarrow y$  and  $y$  is obtained from  $x$  by adjoining the maximal element  $a$  to  $x$ . Let  $r \in \mathbb{C}$  with  $r \neq 0, 1$ . For  $x, y \in \mathcal{P}$  with  $y = x \uparrow a$  define

$$a(x \rightarrow y) = m(x \rightarrow y)r^p(1-r)^u \quad (6.1)$$

where  $p$  is the number of parents of  $a$  and  $u$  is the number of unrelated (non ancestors, not equal to  $a$ ) elements of  $a$ .

**Theorem 6.1.** *If  $a(x \rightarrow y)$  is given by (6.1), then  $a(x \rightarrow y)$  satisfies (5.1) and hence is a transition amplitude.*

*Proof.* We prove the result by strong induction on  $|x|$ . If  $|x| = 1$ , then  $x = x_1$  and  $(x \rightarrow) = \{x_2, x_3\}$  in Figure 1. Hence,

$$\sum_{x_1 \rightarrow} a(x \rightarrow y) = a(x_1 \rightarrow x_2) + a(x_1 \rightarrow x_3) = r + (1-r) = 1$$

If  $|x| = 2$ , then  $x = x_2$  or  $x = x_3$  and  $(x_2 \rightarrow) = \{x_4, x_5, x_6\}$ ,  $(x_3 \rightarrow) = \{x_6, x_7, x_8\}$  in Figure 1. Hence,

$$\begin{aligned} \sum_{x_2 \rightarrow} a(x_2 \rightarrow y) &= r + r(1-r) + (1-r)^2 = 1 \\ \sum_{x_3 \rightarrow} a(x_3 \rightarrow y) &= 2r(1-r) + r^2 + (1-r)^2 = 1 \end{aligned}$$

Now assume the result holds for  $|x| \leq n$  and suppose that  $|x| = n+1 \geq 3$ . We have that  $x = x' \uparrow a$  for some  $x' \in \mathcal{P}_n$  and if  $z \in x \rightarrow$ , then  $z = x \uparrow b$ . Let

$$\begin{aligned} A &= \{z \in x \rightarrow : b \not\geq a\} \\ B &= \{z \in x \rightarrow : b > a\} \\ v &= \{c \in x : c \not\leq a\} \in \mathcal{P} \end{aligned}$$

We then have

$$\begin{aligned} \sum \{a(x \rightarrow z) : z \in A\} &= (1-r) \sum_{x' \rightarrow} a(x' \rightarrow y') = 1-r \\ \sum \{a(x \rightarrow z) : z \in B\} &= r \sum_{v \rightarrow} a(v \rightarrow v') = r \end{aligned}$$

Hence,

$$\sum_{x \rightarrow} a(x \rightarrow y) = (1 - r) + r = 1$$

This completes the induction proof.  $\square$

It follows from Theorems 5.1 and 6.1 that if  $a(x \rightarrow y)$  is given by (6.1), then the operator  $\rho_n$  corresponding to the matrix  $D_n(\omega, \omega') = a_n(\omega)a(\omega')$  forms an AP and hence a QSGP. We then call  $\rho_n$   $n = 1, 2, \dots$ , a *complex percolation process* (CPP) with *percolation constant*  $r$ . The form of (6.1) was chosen because it conforms with general principles of causality and covariance [6, 10].

As an illustration of a CPP suppose the percolation constant is

$$r = \frac{1}{\sqrt{2}} e^{i\pi/4} = \frac{1}{2} + \frac{i}{2}$$

This example may have physical relevance because  $r$  is the unique complex number satisfying  $|r|^2 = |1 - r|^2 = 1/2$ . Notice that  $1 - r = \bar{r}$  and we have

$$a(x \rightarrow y) = m(x \rightarrow y)r^p \bar{r}^u = \frac{m(x \rightarrow y)}{2^{(p+u)/2}} e^{i(p-u)\pi/4}$$

For  $n = 2$ , letting  $\gamma_1 = x_1x_2$ ,  $\gamma_2 = x_1x_3$  from Figure 1, the amplitude vector becomes

$$a_2 = \frac{1}{\sqrt{2}} (e^{i\pi/4}, e^{-i\pi/4})$$

and the decoherence matrix is

$$D_2 = \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

We then have that  $\|\rho_2\| = \text{tr}(\rho_2) = 1$ .

For further computations, it is useful to list the transition amplitudes for the causet of Figure 1.

$(i, j)$	(1, 2)	(1, 3)	(2, 4)	(2, 5)	(2, 6)	(3, 7)	(3, 8)
$a(x_i \rightarrow x_j)$	$r$	$1 - r$	$r$	$r(1 - r)$	$2r(1 - r)$	$r^2$	$(1 - r)^2$
(4, 9)	(4, 10)	(4, 11)	(4, 14)	(5, 11)	(5, 12)	(5, 13)	(5, 15)
$r$	$r(1 - r)^2$	$r(1 - r)^2$	$(1 - r)^3$	$2r(1 - r)$	$r(1 - r)^2$	$r^2$	$(1 - r)^3$

(6, 14)	(6, 15)	(6, 16)	(6, 17)	(6, 18)	(6, 19)	(7, 18)
$r(1-r)$	$r(1-r)^2$	$r(1-r)^2$	$r^2$	$r^2(1-r)$	$(1-r)^3$	$2r(1-r)^2$
(7, 20)	(7, 21)	(7, 22)	(8, 19)	(8, 22)	(8, 23)	(8, 24)
$r$	$r^2(1-r)$	$(1-r)^3$	$3r(1-r)^2$	$3r^2(1-r)$	$r^3$	$(1-r)^3$

**Table 1**

Letting  $\gamma_1 = x_1x_2x_4$ ,  $\gamma_2 = x_1x_2x_5$ ,  $\gamma_3 = x_1x_2x_6$ ,  $\gamma_4 = x_1x_3x_6$ ,  $\gamma_5 = x_1x_3x_7$ ,  $\gamma_6 = x_1x_3x_8$  in Figure 1, we have that  $\Omega_6 = \{\gamma_1, \dots, \gamma_6\}$  with amplitude vector

$$a_3 = 2^{-3/2}(\sqrt{2}i, e^{i\pi/4}, e^{-i\pi/4}, 2e^{-i\pi/4}, e^{i\pi/4}e^{-i3\pi/4})$$

We conclude that  $\|\rho_3\| = \text{tr}(\rho_3) = 5/4$ . The decoherence matrix can be computed from

$$D_3(\omega, \omega') = [a_3(\omega) \overline{a_3(\omega')}]$$

the following table gives an ordering of the paths  $\gamma_j \in \Omega_4$ .

$j$	1	2	3	4	5	6	7
$\gamma_j$	$x_1x_2x_4x_9$	$x_1x_2x_4x_{10}$	$x_1x_2x_4x_{11}$	$x_1x_2x_4x_{14}$	$x_1x_2x_5x_{11}$	$x_1x_2x_5x_{12}$	$x_1x_2x_5x_{13}$
	8	9	10	11	12	13	14
	$x_1x_2x_5x_{15}$	$x_1x_2x_6x_{14}$	$x_1x_2x_6x_{15}$	$x_1x_2x_6x_{16}$	$x_1x_2x_6x_{17}$	$x_1x_2x_6x_{18}$	$x_1x_2x_6x_{19}$
	15	16	17	18	19	20	21
	$x_1x_3x_6x_{14}$	$x_1x_3x_6x_{15}$	$x_1x_3x_6x_{16}$	$x_1x_3x_6x_{17}$	$x_1x_3x_6x_{18}$	$x_1x_3x_6x_{19}$	$x_1x_3x_7x_{18}$
	22	23	24	25	26	27	28
	$x_1x_3x_7x_{20}$	$x_1x_3x_7x_{21}$	$x_1x_3x_7x_{22}$	$x_1x_3x_8x_{19}$	$x_1x_3x_8x_{22}$	$x_1x_3x_8x_{23}$	$x_1x_3x_8x_{24}$

**Table 2**

Then  $\Omega_4 = \{\gamma_1, \gamma_2, \dots, \gamma_{28}\}$  with amplitude vector

$$a_4 = \frac{1}{8}(2^{3/2}e^{i3\pi/4}, 2i, \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{-i\pi/4}, 2^{3/2}e^{i\pi/4}, 1, 2i, -i, \sqrt{2}e^{-i\pi/4}, \\ -i, -i, \sqrt{2}e^{i\pi/4}, 1, -1, 2^{3/2}e^{-i\pi/4}, -2i, -2i, 2^{3/2}e^{i\pi/4}, \\ 2, -2, 2, 2i, i, -i, -3, -3i, 1, i)$$

We then have  $\|\rho_4\| = \text{tr}(\rho_4) = 25/16$ . Although we have not been able to show this, we conjecture that in general  $\|\rho_n\| = (5/4)^{n-2}$ .

We now compute some  $q$ -measures. For  $\Omega_2 = \{\gamma_1, \gamma_2\}$  we have  $\mu_2(\gamma_1) = \mu_2(\gamma_2) = 1/2$ ,  $\mu_2(\Omega_2) = 1$ . Also,  $\mu_2(x_2) = \mu_2(x_3) = 1/2$ ,  $\mu_2(\mathcal{P}_2) = 1$ . In this case there is no interference and  $\mu_2$  is a measure.

For  $\Omega_3 = \{\gamma_1, \dots, \gamma_6\}$  we have

$$\begin{aligned} \mu_3(\gamma_1) &= 1/4, & \mu_3(\gamma_2) &= \mu_3(\gamma_3) = \mu(\gamma_5) = \mu_3(\gamma_6) = 1/8 \\ \mu_3(\gamma_4) &= 1/2, & \mu_3(\Omega_3) &= 1 \end{aligned}$$

Moreover, by (5.5) we have

$$\begin{aligned} \mu_3(\{\gamma_1, \gamma_2\}) &= \mu_3(\gamma_1) + \mu_3(\gamma_2) + 2\text{Re } a_3(\gamma_1)\overline{a_3(\gamma_2)} \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{4}\text{Re} \left( \sqrt{2}ie^{-i\pi/4} \right) = \frac{5}{8} \end{aligned}$$

The  $q$ -measures of the other doubleton sets are computed in a similar way. These are summarized in Table 3.

$(j, k)$	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(2, 3)	(2, 4)	(2, 5)
$\mu_3(\{\gamma_j, \gamma_k\})$	1/8	1/4	5/8	1/8	1/4	5/8	1/2
$(2, 6)$	(3, 4)	(3, 5)	(3, 6)	(4, 5)	(4, 6)	(5, 6)	
0	9/8	1/4	1/4	5/8	5/8	0	

**Table 3**

The other  $q$ -measure for  $\Omega_3$  can be found using our previous results and grade-2 additivity. For example,

$$\begin{aligned} \mu_3(\{\gamma_2, \gamma_5, \gamma_6\}) &= \mu_3(\{\gamma_2, \gamma_5\}) + \mu_3(\{\gamma_2, \gamma_6\}) + \mu_3(\{\gamma_5, \gamma_6\}) \\ &\quad - \mu_3(\gamma_2) - \mu_3(\gamma_5) - \mu_3(\gamma_6) = 1/8 \end{aligned}$$



In a similar way we obtain

$$\begin{aligned}\mu_3(\{\gamma_1, \gamma_2, \gamma_3\}) &= 1/2, & \mu_3(\{\gamma_1, \gamma_2, \gamma_6\}) &= 1/4 \\ \mu_3(\{\gamma_1, \gamma_2, \gamma_4\}) &= 5/8, & \mu_3(\{\gamma_2, \gamma_3, \gamma_4\}) &= 5/4\end{aligned}$$

An example of a 4-element set is given by

$$\begin{aligned}\mu_3(\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}) &= \mu_3(\{\gamma_1, \gamma_2\}) + \mu_3(\{\gamma_1, \gamma_3\}) + \mu_3(\{\gamma_1, \gamma_4\}) \\ &\quad + \mu_3(\{\gamma_2, \gamma_3\}) + \mu_3(\{\gamma_2, \gamma_4\}) + \mu_3(\{\gamma_3, \gamma_4\}) \\ &\quad - 2[\mu_3(\gamma_1) + \mu_3(\gamma_2) + \mu_3(\gamma_3) + \mu_3(\gamma_4)] = 1\end{aligned}$$

The  $q$ -measure of paths in  $\Omega_4 = \{\gamma_1, \dots, \gamma_{28}\}$  are given in Table 4.

$j$	1	2	3	4	5	6	7	8	9	10
$\mu_4(\gamma_j)$	1/8	1/16	1/32	1/32	1/8	1/64	1/16	1/64	1/32	1/64
11	12	13	14	15	16	17	18	19	20	
1/64	1/32	1/64	1/64	1/8	1/16	1/16	1/8	1/16	1/16	
21	22	23	24	25	26	27	28			
1/16	1/16	1/64	1/64	9/64	9/64	1/64	1/64			

**Table 4**

Instead of finding  $\mu_4(A)$  for arbitrary  $A \in \mathcal{A}_4$ , we compute  $\mu_4(x)$ ,  $x \in \mathcal{P}_4$  which are given in Table 5.

$j$	9	10	11	12	13	14	15	16	17	18
$\mu_4(x_j)$	1/8	1/16	9/32	1/16	1/16	1/4	1/4	9/64	9/32	25/64

**Table 5**

We now briefly consider the  $q$ -measure of some sets in  $\mathcal{B}(\rho_n) \setminus \mathcal{C}(\Omega)$ . If  $\omega = \omega_1\omega_2 \cdots \in \Omega$ , then  $\{\omega\} \notin \mathcal{C}(\Omega)$  and  $\{\omega\}^n = \{\omega_1\omega_2 \cdots \omega_n\}$ . We define

the *multiplicity*  $m(\omega)$  by

$$m(\omega) = \prod_{j=1}^{\infty} m(\omega_j \rightarrow \omega_{j+1})$$

As suggested by Figure 1, most  $\omega \in \Omega$  have finite multiplicity, although there are a few with  $m(\omega) = \infty$ . If  $m(\omega) < \infty$  then it is easy to verify that  $\{\omega\} \in \mathcal{B}(\rho_n)$  and

$$\mu(\{\omega\}) = \lim_{n \rightarrow \infty} |a_n(\omega_1 \omega_2 \cdots \omega_n)|^2 = 0$$

Moreover, it follows by (5.1) that if  $m(\omega) < \infty$ , then  $\{\omega\}' \in \mathcal{B}(\rho_n)$  and

$$\mu(\{\omega\}') = \lim_{n \rightarrow \infty} |1 - a_n(\omega_1 \omega_2 \cdots \omega_n)|^2 = 1$$

It would be interesting to investigate whether there exist  $\omega \in \Omega$  with  $m(\omega) = \infty$  and  $\omega \in \mathcal{B}(\rho_n)$ .

Finally, we briefly discuss discrete geodesics. Let  $\rho_n$  be an AP. If  $x \rightarrow y$  with  $x \in \mathcal{P}_n$ , then

$$\mu_n(x \cap y) = \mu_n(\{\omega: \omega = \omega_1 \omega_2 \cdots xy\}) = \mu_n(x) |a(x \rightarrow y)|^2$$

It follows that  $\mu_n(y | x) = |a(x \rightarrow y)|^2$ . We conclude that  $\omega = \omega_j \omega_{j+1} \cdots \omega_n$  is a discrete geodesic starting with  $\omega_j$  if and only if  $\omega$  is maximal and satisfies

$$|a(\omega_{k-1} \rightarrow \omega_k)| = c |a(\omega_{k-2} \rightarrow \omega_{k-1})| \quad (6.2)$$

$k = j + 2, \dots, n$ , for some  $c \in \mathbb{R}$ . In particular, suppose  $\rho_n$  is a CPP with percolation constant  $r$ . Letting  $n = 4$  and employing the notation of Figure 1, the only discrete geodesics starting at  $x_1$  are  $x_1 x_2 x_4 x_9$  and  $x_1 x_3 x_8 x_{24}$ . The constant  $c$  in (6.2) for  $x_1 x_2 x_4 x_9$  is 1 and for  $x_1 x_2 x_8 x_{24}$  is  $|1 - r|$ . A generic causet is contained in at least two discrete geodesics with  $c = 1$  or  $c = |1 - r|$ . However, there are exceptional causets that are contained in only one discrete geodesic. It would be interesting to classify causets according to their geodesic structure.

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