COMPATIBILITY FOR PROBABILISTIC THEORIES

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Abstract

We define an index of compatibility for a probabilistic theory (PT). Quantum mechanics with index 0 and classical probability theory with index 1 are at the two extremes. In this way, quantum mechanics is at least as incompatible as any PT. We consider a PT called a concrete quantum logic that may have compatibility index strictly between 0 and 1, but we have not been able to show this yet. Finally, we show that observables in a PT can be represented by positive, vector-valued measures.

1 Observables in Probabilistic Theories

This paper is based on the stimulating article [1] by Busch, Heinosaari and Schultz. The authors should be congratulated for introducing a useful new tool for measuring the compatibility of a probabilistic theory (PT). In this paper, we present a simpler, but coarser, measure of compatibility that we believe will also be useful.

A probabilistic theory is a \( \sigma \)-convex subset \( \mathcal{K} \) of a real Banach space \( \mathcal{V} \). That is, if \( 0 \leq \lambda_i \leq 1 \) with \( \sum \lambda_i = 1 \) and \( v_i \in \mathcal{K}, i = 1, 2, \ldots, \) then \( \sum \lambda_i v_i \) converges in norm to an element of \( \mathcal{K} \). We call the elements of \( \mathcal{K} \) states. There is no loss of generality in assuming that \( \mathcal{K} \) generates \( \mathcal{V} \) in the sense that the closed linear hull of \( \mathcal{K} \) equals \( \mathcal{V} \). Denote the collection of Borel
subsets of $\mathbb{R}^n$ by $\mathcal{B}(\mathbb{R}^n)$ and the set of probability measures on $\mathcal{B}(\mathbb{R}^n)$ by $\mathcal{M}(\mathbb{R}^n)$. If $\mathcal{K}$ is a PT, an $n$-dimensional observable on $\mathcal{K}$ is a $\sigma$-affine map $M: \mathcal{K} \to \mathcal{M}(\mathbb{R}^n)$. We denote the set of $n$-dimensional observables by $\mathcal{O}_n(\mathcal{K})$ and write $\mathcal{O}(\mathcal{K}) = \mathcal{O}_1(\mathcal{K})$. We call the elements of $\mathcal{O}(\mathcal{K})$ observables. For $M \in \mathcal{O}(\mathcal{K})$, $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$, we interpret $M(s)(A)$ as the probability that $M$ has a value in $A$ when the system is in state $s$.

A set of observables $\{M_1, \ldots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$ is compatible or jointly measurable if there exists an $M \in \mathcal{O}_n(\mathcal{K})$ such that for every $A \in \mathcal{B}(\mathbb{R})$ and every $s \in \mathcal{K}$ we have

\[
M(s)(\mathbb{R} \times A \times \ldots \times \mathbb{R}) = M_1(s)(A) \\
M(s)(\mathbb{R} \times \mathbb{R} \times A \times \ldots \times \mathbb{R}) = M_2(s)(A) \\
\vdots \\
M(s)(\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \times A) = M_n(s)(A)
\]

In this case, we call $M$ a joint observable for $\{M_1, \ldots, M_n\}$ and we call $\{M_1, \ldots, M_n\}$ the marginals for $M$. It is clear that if $\{M_1, \ldots, M_n\}$ is compatible, then any proper subset is compatible. However, we suspect that the converse is not true. If a set of observables is not compatible we say it is incompatible.

It is clear that convex combinations of observables give an observable so $\mathcal{O}(\mathcal{K})$ forms a convex set. In the same way, $\mathcal{O}_n(\mathcal{K})$ is a convex set. Another way of forming new observables is by taking functions of an observable. If $f: \mathbb{R} \to \mathbb{R}$ is a Borel function and $M \in \mathcal{O}(\mathcal{K})$, the observable $f(M): \mathcal{K} \to \mathcal{M}(\mathbb{R})$ is defined by $f(M)(s)(A) = M(s)(f^{-1}(A))$ for all $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$.

**Theorem 1.1.** If $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ are functions of a single observable $M$, then $M_1, M_2$ are compatible.

**Proof.** Suppose $M_1 = f(M)$, $M_2 = g(M)$ where $f$ and $g$ are Borel functions. For $A, B \in \mathcal{B}(\mathbb{R})$, $s \in \mathcal{K}$ define $\tilde{M}(s)$ on $A \times B$ by

\[
\tilde{M}(s)(A \times B) = M(s)\left[f^{-1}(A) \cap g^{-1}(B)\right]
\]

By the Hahn extension theorem, $\tilde{M}(s)$ extends to a measure in $\mathcal{M}(\mathbb{R}^2)$. Hence, $\tilde{M} \in \mathcal{O}_2(\mathcal{K})$ and the marginals of $\tilde{M}$ are $f(M)$ and $g(M)$. We conclude that $M_1 = f(M)$ and $M_2 = g(M)$ are compatible. 

\[\blacksquare\]
It follows from Theorem 1.1 that an observable is compatible with any Borel function of itself and in particular with itself. In a similar way we obtain the next result.

**Theorem 1.2.** If $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ are compatible and $f, g$ are Borel functions, then $f(M_1)$ and $g(M_2)$ are compatible.

**Proof.** Since $M_1, M_2$ are compatible, they have a joint observable $M \in \mathcal{O}_2(\mathcal{K})$. For $A, B \in \mathcal{B}(\mathbb{R}), s \in \mathcal{K}$ define $\tilde{M}(s)$ on $A \times B$ by

$$\tilde{M}(s)(A \times B) = M(s) \left[ f^{-1}(A) \times g^{-1}(B) \right]$$

As in the proof of Theorem 1.1, $\tilde{M}(s)$ extends to a measure in $\mathcal{M}(\mathbb{R}^2)$. Hence, $\tilde{M} \in \mathcal{O}(\mathcal{K})$ and the marginals of $\tilde{M}$ are

$$\tilde{M}(s)(A \times \mathbb{R}) = M(s) \left[ f^{-1}(A) \times \mathbb{R} \right] = M_1(s) \left[ f^{-1}(A) \right] = f(M_1)(s)(A)$$

$$\tilde{M}(s)(\mathbb{R} \times A) = M(s) \left[ \mathbb{R} \times g^{-1}(A) \right] = M_2(s) \left[ g^{-1}(A) \right] = g(M_2)(s)(A)$$

We conclude that $f(M_1)$ and $g(M_2)$ are compatible. □

The next result is quite useful and somewhat surprising.

**Theorem 1.3.** Let $M_i^j \in \mathcal{O}(\mathcal{K})$ for $i = 1, \ldots, n, j = 1, \ldots, m$ and suppose $\{M_i^1, \ldots, M_i^m\}$ is compatible, $i = 1, \ldots, n$. If $\lambda_i \in [0, 1]$ with $\sum \lambda_i = 1$, $i = 1, \ldots, n$, then

$$\left\{ \sum_{i=1}^n \lambda_i M_i^1, \sum_{i=1}^n \lambda_i M_i^2, \ldots, \sum_{i=1}^n \lambda_i M_i^m \right\}$$

is compatible.

**Proof.** Let $\tilde{M}_i \in \mathcal{O}_m(\mathcal{K})$ be the joint observable for $\{M_i^1, \ldots, M_i^m\}$, $i = 1, \ldots, n$. Then $\tilde{M} = \sum_{i=1}^n \lambda_i \tilde{M}_i$ is an $m$-dimensional observable with marginals
\[ \widetilde{M}(s)(A \times \mathbb{R} \times \cdots \times \mathbb{R}) = \sum_{i=1}^{n} \lambda_i \widetilde{M}_i(s)(A \times \mathbb{R} \times \cdots \times \mathbb{R}) = \sum_{i=1}^{n} \lambda_i M_i^1(s)(A) \]

\[ \widetilde{M}(s)(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R}) = \sum_{i=1}^{n} \lambda_i \widetilde{M}_i(s)(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R}) = \sum_{i=1}^{n} \lambda_i M_i^2(s)(A) \]

\[ \vdots \]

\[ \widetilde{M}(s)(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) = \sum_{i=1}^{n} \lambda_i \widetilde{M}_i(s)(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) = \sum_{i=1}^{n} \lambda_i M_i^m(s)(A) \]

The result now follows \(\square\)

**Corollary 1.4.** Let \(M, N, P \in O(K)\) and \(\lambda \in [0, 1]\). If \(M\) is compatible with \(N\) and \(P\), then \(M\) is compatible with \(\lambda N + (1 - \lambda)P\).

**Proof.** Since \(\{M, N\}\) and \(\{M, P\}\) are compatible sets, by Theorem 1.3, we have that \(M = \lambda M + (1 - \lambda)M\) is compatible with \(\lambda N + (1 - \lambda)P\). \(\square\)

## 2 Noisy Observables

If \(p \in \mathcal{M}(\mathbb{R})\), we define the trivial observable \(T_p \in \mathcal{O}(K)\) by \(T_p(s) = p\) for every \(s \in K\). A trivial observable represents noise in the system. We denote the set of trivial observables on \(K\) by \(T(K)\). The set \(T(K)\) is convex with

\[ \lambda T_p + (1 - \lambda)T_q = T_{\lambda p + (1 - \lambda)q} \]

for every \(\lambda \in [0, 1]\) and \(p, q \in \mathcal{M}(\mathbb{R})\). An observable \(M \in \mathcal{O}(K)\) is compatible with any \(T_p \in \mathcal{T}(K)\) and a joint observable \(\widetilde{M} \in \mathcal{O}_2(K)\) is given by

\[ \widetilde{M}(s)(A \times B) = p(A)M(s)(B) \]

If \(M \in \mathcal{O}(K)\), \(T \in \mathcal{T}(K)\) and \(\lambda \in [0, 1]\) we consider \(\lambda M + (1 - \lambda)T\) as the observable \(M\) together with noise. Stated differently, we consider \(\lambda M + \).
(1 − λ)T to be a noisy version of M. The parameter 1 − λ gives a measure of the proportion of noise and is called the noise index. Smaller λ gives a larger proportion of noise. As we shall see, incompatible observables may have compatible noisy versions.

The next lemma follows directly from Corollary 1.4. It shows that if M is compatible with N, then M is compatible with any noisy version of N.

**Lemma 2.1.** If $M \in \mathcal{O}(\mathcal{K})$ is compatible with $N \in \mathcal{O}(\mathcal{K})$, then $M$ is compatible with $\lambda N + (1 − \lambda) T$ for any $\lambda \in [0, 1]$ and $T \in \mathcal{T}(\mathcal{K})$.

The following lemma shows that for any $M, N \in \mathcal{O}(\mathcal{K})$ a noisy version of $N$ with noise index $\lambda$ is compatible with any noisy version of $M$ with noise index $1 − \lambda$. The lemma also shows that if $M$ is compatible with a noisy version of $N$, then $M$ is compatible with a still noisier version of $N$.

**Lemma 2.2.** Let $M, N \in \mathcal{O}(\mathcal{K})$ and $S, T \in \mathcal{T}(\mathcal{K})$. (a) If $\lambda \in [0, 1]$, then $\lambda M + (1 − \lambda) T$ and $(1 − \lambda) N + \lambda S$ are compatible. (b) If $M$ is compatible with $\lambda N + (1 − \lambda) T$, then $M$ is compatible with $\mu N + (1 − \mu) T$ where $0 \leq \mu \leq \lambda \leq 1$.

**Proof.** (a) Since $\{M, S\}$ and $\{T, N\}$ are compatible sets, by Theorem 1.3 $\lambda M + (1 − \lambda) T$ is compatible with $\lambda S + (1 − \lambda) N$. (b) We can assume that $\lambda > 0$ and we let $\alpha = \mu / \lambda$ so $0 \leq \alpha \leq 1$. Since $\{M, \lambda N + (1 − \lambda) T\}$ and $\{M, T\}$ are compatible sets, by Theorem 1.3, $M = \alpha M + (1 − \alpha) M$ is compatible with

$$
\alpha [\lambda N + (1 − \lambda) T] + (1 − \alpha) T = \alpha \lambda N + [\alpha(1 − \lambda) + (1 − \alpha)] T
$$

$$
= \mu N + (1 − \mu) T \quad \Box
$$

The compatibility region $J(M_1, M_2, \ldots, M_n)$ of observables $M_i \in \mathcal{O}(\mathcal{K})$, $i = 1, \ldots, n$, is the set of points $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in [0, 1]^n$ for which there exist $T_i \in \mathcal{T}(\mathcal{K})$, $i = 1, 2, \ldots, n$, such that

$$
\{\lambda_i M_i + (1 − \lambda_i) T_i\}_{i=1}^n
$$

form a compatible set. Thus, $J(M_1, M_2, \ldots, M_n)$ gives parameters for which there exist compatible noisy versions of $M_1, M_2, \ldots, M_n$. It is clear that $0 = (0, \ldots, 0) \in J(M_1, M_2, \ldots, M_n)$ and we shall show that $J(M_1, M_2, \ldots, M_n)$ contains many points. We do not know whether $J(M_1, M_2, \ldots, M_n)$ is symmetric under permutations of the $M_i$. For example, is $J(M_1, M_2) = J(M_2, M_1)$?
**Theorem 2.3.** \( J(M_1, M_2, \ldots M_n) \) is a convex subset of \([0, 1]^n\).

**Proof.** Suppose \((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in J(M_1, \ldots, M_n)\). We must show that

\[
\lambda(\lambda_1, \ldots, \lambda_n) + (1 - \lambda)(\mu_1, \ldots, \mu_n) = (\lambda\lambda_1 + (1 - \lambda)\mu_1, \ldots, \lambda\lambda_n + (1 - \lambda)\mu_n) \in J(M_1, \ldots, M_n)
\]

for all \(\lambda \in [0, 1]\). Now there exist \(S_1, \ldots, S_n, T_1, \ldots, T_n \in T(\mathcal{K})\) such that \(\{\lambda_iM_i + (1 - \lambda_i)S_i\}_{i=1}^n\) and \(\{\mu_iM_i + (1 - \mu_i)T_i\}_{i=1}^n\) are compatible. By Theorem 1.3 the set of observables

\[
\{\lambda [\lambda_iM_i + (1 - \lambda_i)S_i] + (1 - \lambda) [\mu_iM_i + (1 - \mu_i)T_i]\}
\]

is compatible. Since

\[
\lambda(1 - \lambda_i) + (1 - \lambda)(1 - \mu_i) = 1 - \lambda\lambda_i - \mu_i + \lambda\mu_i
\]

letting \(\alpha_i = \lambda\lambda_i + (1 - \lambda)\mu_i\) we have that

\[
U_i = \frac{1}{1 - \alpha_i} [\lambda(1 - \lambda_i)S_i + (1 - \lambda)(1 - \mu_i)T_i] \in T(\mathcal{K})
\]

Since \(\{\alpha_iM_i + (1 - \alpha_i)U_i\}_{i=1}^n\) forms a compatible set, we conclude that \((\alpha_1, \ldots, \alpha_n) \in J(M_1, \ldots, M_n)\).

\(\square\)

Let \(\Delta_n = \{(\lambda_1, \ldots, \lambda_n) \in [0, 1]^n : \sum \lambda_i \leq 1\}\). To show that \(\Delta_n\) forms a convex subset of \([0, 1]^n \subseteq \mathbb{R}^n\), let \((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in \Delta_n\) and \(\lambda \in [0, 1]\). Then \(\lambda(\lambda_1, \ldots, \lambda_n) + (1 - \lambda)(\mu_1, \ldots, \mu_n) \in [0, 1]^n\) and

\[
\sum_{i=1}^n [\lambda\lambda_i + (1 - \lambda)\mu_i] = \lambda \sum \lambda_i + (1 - \lambda) \sum \mu_i \leq \lambda + (1 - \lambda) = 1
\]

**Theorem 2.4.** If \(\{M_1, \ldots, M_n\} \subseteq \mathcal{O}(\mathcal{K})\), then \(\Delta_n \subseteq J(M_1, \ldots, M_n)\).

**Proof.** Let \(\delta_0 = (0, 0, \ldots, 0) \in \mathbb{R}^n\), \(\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n\), \(i = 1, \ldots, n\) where 1 is in the \(i\)th coordinate. It is clear that

\[
\delta_i \in J(M_1, \ldots, M_n) \cap \Delta_n, \quad i = 0, 1, \ldots, n
\]
If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$, letting $\mu = \sum \lambda_i$ we have that $0 \leq \mu \leq 1$, 
$$
\lambda = \sum_{i=1}^{n} \lambda_i \delta_i + (1 - \mu) \delta_0
$$

It follows that $\Delta_n$ is the convex hull of $\{\delta_0, \delta_1, \ldots, \delta_n\}$. Since 
$$
\{\delta_0, \delta_1, \ldots, \delta_n\} \subseteq J(M_1, \ldots, M_n)
$$

and $J(M_1, \ldots, M_n)$ is convex, it follows that $\Delta_n \in J(M_1, \ldots, M_n)$. \hfill \Box

The $n$-dimensional compatibility region for PT $\mathcal{K}$ is defined by 
$$
J_n(\mathcal{K}) = \cap \{J(M_1, \ldots, M_n) : M_i \in \mathcal{O}(\mathcal{K}), i = 1, \ldots, n\}
$$

We have that $\Delta_n \subseteq J_n(\mathcal{K}) \subseteq [0,1]^n$ and $J_n(\mathcal{K})$ is a convex set that gives a measure of the incompatibility of observables on $\mathcal{K}$. As $J_n(\mathcal{K})$ gets smaller, $\mathcal{K}$ gets more incompatible and the maximal incompatibility is when $J_n(\mathcal{K}) = \Delta_n$. For the case of quantum states $\mathcal{K}$, the set $J_2(\mathcal{K})$ has been considered in [1].

We now introduce a measure of compatibility that we believe is simpler and easier to investigate than $J_2(M, N)$ For $M, N \in \mathcal{O}(\mathcal{K})$, the compatibility interval $I(M, N)$ is the set of $\lambda \in [0,1]$ for which there exists a $T \in T(\mathcal{K})$ such that $M$ is compatible with $\lambda N + (1 - \lambda)T$. Of course, $0 \in T(M, N)$ and $M$ and $N$ are compatible if and only if $1 \in I(M, N)$. We do not know whether $I(M, N) = I(N, M)$. It follows from Lemma 2.2(b) that if $\lambda \in T(M, N)$ and $0 \leq \mu \leq \lambda$, then $\mu \in I(M, N)$. Thus, $I(M, N)$ is an interval with left endpoint 0. The index of compatibility of $M$ and $N$ is $\lambda(M, N) = \sup \{\lambda : \lambda \in I(M, N)\}$. We do not know whether $\lambda(M, N) \in I(M, N)$ but in any case $I(M, N) = [0, \lambda(M, N)]$ or $I(M, N) = [0, \lambda(M, N))$. For a PT $\mathcal{K}$, we define the interval of compatibility for $\mathcal{K}$ to be 
$$
I(\mathcal{K}) = \cap \{I(M, N) : M, N \in \mathcal{O}(\mathcal{K})\}
$$

The index of compatibility of $\mathcal{K}$ is 
$$
\lambda(\mathcal{K}) = \inf \{\lambda(M, N) : M, N \in \mathcal{O}(\mathcal{K})\}
$$

and $I(\mathcal{K}) = [0, \lambda(\mathcal{K})]$ or $I(\mathcal{K}) = [0, \lambda(\mathcal{K}))$. Again, $\lambda(\mathcal{K}) = 0$ gives a measure of incompatibility of the observables in $\mathcal{O}(\mathcal{K})$. 7
Example 1. (Classical Probability Theory) Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{V}$ be the Banach space of real-valued measures on $\mathcal{A}$ with the total variation norm. If $\mathcal{K}$ is the $\sigma$-convex set of probability measures on $\mathcal{A}$, then $\mathcal{K}$ generates $\mathcal{V}$. There are two types of observables on $\mathcal{K}$, the sharp and fuzzy observables. The sharp observables have the form $M_f$ where $f$ is a measurable function $f: \Omega \to \mathbb{R}$ and $M_f(s)(A) = s[f^{-1}(A)]$. If $M_f, M_g$ are sharp observables, form the unique 2-dimensional observable $\tilde{M}$ satisfying

$$\tilde{M}(s)(A \times B) = s[f^{-1}(A) \cap g^{-1}(B)]$$

Then $\tilde{M}$ is a joint observable for $M_f, M_g$ so $M_f$ and $M_g$ are compatible. The unsharp observables are obtained as follows. Let $\mathcal{F}(\Omega)$ be the set of measurable functions $f: \Omega \to [0, 1]$. Let $\tilde{M}: \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\Omega)$ satisfy $\tilde{M}(\mathbb{R}) = 1$, $\tilde{M}(\bigcup A_i) = \sum \tilde{M}(A_i)$. An unsharp observable has the form

$$M(s)(A) = \int \tilde{M}(A)ds$$

Two unsharp observables $M, N$ are also compatible because we can form the joint observable $\tilde{M}$ given by

$$\tilde{M}(S)(A \times B) = \int \tilde{M}(A)\tilde{N}(B)ds$$

We conclude that $J(\mathcal{K}) = [0, 1] \times [0, 1]$ and $I(\mathcal{K}) = [0, 1]$ so $\mathcal{K}$ has the maximal amount of compatibility.

Example 2. (Quantum Theory) Let $H$ be a separable complex Hilbert space and let $\mathcal{K}$ be the $\sigma$-convex set of all trace 1 positive operators on $H$. Then $\mathcal{K}$ generates the Banach space of self-adjoint trace-class operators with the trace norm. It is well known that $M \in \mathcal{O}(\mathcal{K})$ if and only if there exists a positive operator-valued measure (POVM) $P$ such that $M(s)(A) = \text{tr}[sP(A)]$ for every $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$. It is shown in [1] that if $\dim H = \infty$, then there exist $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ such that $J_2(M_1, M_2) = \Delta_2$ and hence $J(\mathcal{K}) = \Delta_2$. If $\dim H < \infty$, then $J(\mathcal{K})$ is not known, although partial results have been obtained and it is known that $J(\mathcal{K}) \to \Delta_2$ as $\dim H \to \infty$.

Now let $H$ be an arbitrary complex Hilbert space with $\dim H \geq 2$. Although the Pauli matrices $\sigma_x, \sigma_y$ are 2-dimensional, we can extend them from a 2-dimensional subspace $H_0$ of $H$ to all of $H$ by defining $\sigma_x \psi = 0$ for
all \( \psi \in H_{0+}^1 \). Define the POVMs \( M_x, M_y \) on \( H \) by \( M_x(\pm 1) = \frac{1}{2}(I \pm \sigma_x) \), 
\( M_y(\pm 1) = \frac{1}{2}(I \pm \sigma_y) \). It is shown in [1] that 
\[
J(M_x, M_y) = \{ (\lambda, \mu) \in [0, 1] \times [0, 1] : \lambda^2 + \mu^2 \leq 1 \}
\]
Thus, \( J(M_x, M_y) \) is a quadrant of the unit disk. We conclude that \( M_x \) is compatible with \( \mu M_y + (1 - \mu)T \) for \( T \in T(\mathcal{K}) \) if and only if \( 1 + \mu^2 \leq 1 \). Therefore, \( \mu = 0 \), so \( I(M_x, M_y) = \{ 0 \} \) and \( \lambda(M_x, M_y) = 0 \). Thus, \( I(\mathcal{K}) = \{ 0 \} \) and \( \lambda(\mathcal{K}) = 0 \). We conclude that quantum mechanics has the smallest index of compatibility possible for a PT. The index of compatibility for a classical system is 1, so we have the two extremes. It would be interesting to find \( \lambda(\mathcal{K}) \) for other PTs.

3 Concrete Quantum Logics

We now consider a PT that seems to be between the classical and quantum PTs of Examples 1 and 2. A collection of subsets \( \mathcal{A} \) of a set \( \Omega \) is a \( \sigma \)-class if \( \emptyset \in \mathcal{A}, A^c \in \mathcal{A} \) whenever \( A \in \mathcal{A} \) and if \( A_i \) are mutually disjoint, \( i = 1, 2, \ldots, \), then \( \bigcup A_i \in \mathcal{A} \). If \( \mathcal{A} \) is a \( \sigma \)-class on \( \Omega \), we call \((\Omega, \mathcal{A})\) a concrete quantum logic. A \( \sigma \)-state on \( \mathcal{A} \) is a map \( s : \mathcal{A} \to [0, 1] \) such that \( s(\Omega) = 1 \) and if \( A_i \in \mathcal{A} \) are mutually disjoint, then \( s(\bigcup A_i) = \sum s(A_i) \). If \( \mathcal{K} \) is the set of \( \sigma \)-states on \((\Omega, \mathcal{A})\), we call \( \mathcal{K} \) a concrete quantum logic PT. Let \( \mathcal{A}_\sigma \) be the \( \sigma \)-algebra generated by \( \mathcal{A} \). A \( \sigma \)-state \( s \) is classical if there exists a probability measure \( \mu \) on \( \mathcal{A}_\sigma \) such that \( s = \mu | \mathcal{A} \). As in the classical case, an observable is sharp if it has the form \( M_f(s)(A) = s \cdot [f^{-1}(A)] \) for an \( \mathcal{A} \)-measurable function \( f : \Omega \to \mathbb{R} \). If \( f \) and \( g \) are \( \mathcal{A} \)-measurable functions satisfying \( f^{-1}(A) \cap g^{-1}(B) \in \mathcal{A} \) for all \( A, B \in \mathcal{B}(\mathbb{R}) \), then \( M_f \) and \( M_g \) are compatible because they have a joint observable \( M \) satisfying \( M(s)(A \times B) = s \cdot [f^{-1}(A) \cap g^{-1}(B)] \) for all \( s \in \mathcal{K}, A, B \in \mathcal{B}(\mathbb{R}) \). We do not know whether \( M_f \) and \( M_g \) compatible implies that \( f^{-1}(A) \cap g^{-1}(B) \in \mathcal{A} \) holds for every \( A, B \in \mathcal{B}(\mathbb{R}) \), although we suspect it does not.

Example 3. This is a simple example of a concrete quantum logic. Let \( \Omega = \{1, 2, 3, 4\} \) and let \( \mathcal{A} \) be the collection of subsets of \( \Omega \) with even cardinality. Then 
\[
\mathcal{A} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}
\]
Let $K$ be the sets of all states on $\mathcal{A}$. Letting $a = \{1, 2\}$, $a' = \{3, 4\}$, $b = \{1, 3\}$, $b' = \{3, 4\}$, $c = \{1, 4\}$, $c' = \{2, 3\}$ we can represent an $s \in K$ by
\[
\hat{s} = (s(a), s(a'), s(b), s(b'), s(c), s(c')) = (s(a), 1 - s(a), s(b), 1 - s(b), s(c), 1 - s(c))
\]
Thus, every $s \in K$ has the form
\[
s = (\lambda_1, 1 - \lambda_1, \lambda_2, 1 - \lambda_2, \lambda_3, 1 - \lambda_3)
\]
for $0 \leq \lambda_i \leq 1$, $i = 1, 2, 3$. The pure (extremal) classical states are the 0-1 states: $\delta_1 = (1, 0, 1, 0, 1, 0)$, $\delta_3 = (1, 0, 0, 1, 0, 1)$, $\delta_3 = (0, 1, 1, 0, 0, 1)$, $\delta_4 = (0, 1, 0, 1, 1, 0)$. The pure nonclassical states are the 0-1 states: $\gamma_1 = 1 - \delta_1$, $\gamma_2 = 1 - \delta_2$, $\gamma_3 = 1 - \delta_3$, $\gamma_4 = 1 - \delta_4$ where $1 = (1, 1, 1, 1, 1, 1)$. For example, to see that $\gamma_1$ is not classical, we have that $\gamma_1 = (0, 1, 0, 1, 0, 1)$. Hence, $\gamma_1(\{3, 4\}) = \gamma_1(\{2, 4\}) = \gamma_1(\{2, 3\}) = 1$. If there exists a probability measure $\mu$ such that $\gamma_1 = \mu | \mathcal{A}$ we would have $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 0$ which is a contradiction. The collection of sharp observable is very limited because a measurable function $f: \Omega \rightarrow \mathbb{R}$ can have at most two values. Thus, if $M_f$ is a sharp observable there exists $a, b \in \mathbb{R}$ such that $M_f(s)(\{a, b\}) = 1$ for every $s \in K$. There are many observables with more than two values (non-binary observables) and these are not sharp. Even for this simple example, it appears to be challenging to investigate the region and interval of compatibility.

\section{Vector-Valued Measures}

Let $K$ be a PT with generated Banach space $\mathcal{V}$ and $\mathcal{V}^*$ be the Banach space dual of $\mathcal{V}$. A \textit{normalized vector-valued measure} (NVM) for $K$ is a map $\Gamma: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{V}^*$ such that $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R})$ for every $s \in K$. Thus, $\Gamma$ satisfies the conditions:

1. $\Gamma(\mathbb{R})(s) = 1$ for every $s \in K$,
2. $0 \leq \Gamma(A)(s) \leq 1$ for every $s \in K$, $A \in \mathcal{B}(\mathbb{R})$,
3. If $A_i \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, $i = 1, 2, \ldots$, then
\[
\Gamma(\bigcup A_i)(s) = \sum \Gamma(A_i)(s)
\]
for every $s \in K$.
This section shows that there is a close connection between observables on $\mathcal{K}$ and NVMs for $\mathcal{K}$.

**Theorem 4.1.** If $\Gamma$ is a NVM for $\mathcal{K}$, then $M: \mathcal{K} \to \mathcal{M}({\mathbb{R}})$ given by $M(s)(A) = \Gamma(A)(s)$, $s \in \mathcal{K}$, $A \in \mathcal{B}({\mathbb{R}})$, is an observable on $\mathcal{K}$.

**Proof.** Since $A \mapsto \Gamma(A)(s) \in \mathcal{M}({\mathbb{R}})$ we have that $A \mapsto M(s)(A) \in \mathcal{M}({\mathbb{R}})$. Let $\lambda_i \in [0, 1]$ with $\sum \lambda_i = 1$, $s_i \in \mathcal{K}$, $i = 1, 2, \ldots$, and suppose that $s = \sum \lambda_i s_i$. Then $\lim n \sum_i = 1 \lambda_i s_i = s$ in norm and since $s \mapsto \Gamma(A)(s) \in \mathcal{V}^*$, for every $A \in \mathcal{B}({\mathbb{R}})$ we have

$$M(s)(A) = M \left( \sum \lambda_i s_i \right)(A) = \Gamma(A) \left( \sum \lambda_i s_i \right) = \Gamma(A) \left( \lim n \sum_{i=1}^{n} \lambda_i s_i \right)$$

$$= \lim_{n \to \infty} \Gamma(A) \left( \sum_{i=1}^{n} \lambda_i s_i \right) = \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_i \Gamma(A)(s_i)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_i M(s_i)(A) = \sum_{i=1}^{\infty} \lambda_i M(s_i)(A)$$

It follows that $M \left( \sum \lambda_i s_i \right) = \sum \lambda_i M(s_i)$ so $M \in \mathcal{O}(\mathcal{K})$. \qed

The converse of Theorem 4.1 holds if some mild conditions are satisfied. To avoid some topological and measure-theoretic technicalities, we consider the special case where $\mathcal{V}$ is finite-dimensional. Assuming that $\mathcal{K}$ is the base of a generating positive cone $\mathcal{V}^+$, we have that every element $v \in \mathcal{V}^+$ has a unique form $v = \alpha s$, $\alpha \geq 0$, $s \in \mathcal{K}$ and that $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ where $\mathcal{V}^- = -\mathcal{V}^+$ and $\mathcal{V}^+ \cap \mathcal{V}^- = \{0\}$. If $M \in \mathcal{O}(\mathcal{K})$, then for every $A \in \mathcal{B}({\mathbb{R}})$, $s \mapsto M(s)(A)$ is a convex, real-valued function on $\mathcal{K}$. A standard argument shows that this function has a unique linear extension $\widehat{M}(A) = \mathcal{V}^*$ for every $A \in \mathcal{B}({\mathbb{R}})$. Hence

$$\widehat{M}(A)(s) = M(s)(A) \quad (4.1)$$

for every $s \in \mathcal{K}$, $A \in \mathcal{B}({\mathbb{R}})$. Since $A \mapsto \widehat{M}(A)(s) = M(s)(A) \in \mathcal{M}({\mathbb{R}})$ we conclude that $A \mapsto \widehat{M}(A)$ is a NVM and $\widehat{M}$ is the unique NVM satisfying (4.1). It follows that the converse of Theorem 4.1 holds in this case.

**Example 1’. (Classical Probability Theory)** In this example $\mathcal{V}^*$ is the Banach space of bounded measurable functions $f: \Omega \to {\mathbb{R}}$ with norm $\|f\| = \ldots$
\[ \sup |f(\omega)| < \infty \] and duality given by
\[ \langle \mu, f \rangle = f(\mu) = \int f d\mu \]

The function \( 1(\omega) = 1 \) for every \( \omega \in \Omega \) is the natural unit satisfying \( 1(\mu) = 1 \) for every \( \mu \in \mathcal{K} \). In this case, \( \mathcal{K} \) is a base for the generating positive cone \( \mathcal{V}^+ \) of bounded measures and the converse of Theorem 4.1 holds. Then a NVM \( \Gamma \) has the form \( 0 \leq \Gamma(A)(\omega) \leq 1 \) for every \( A \in \mathcal{B}(\mathbb{R}) \), \( \omega \in \Omega \) and \( \Gamma(\mathbb{R}) = 1 \). Thus \( \Gamma(A) \in \mathcal{F}(\Omega) \) and if \( M \) is the corresponding observable, then
\[ M(\mu)(A) = \Gamma(A)(\mu) = \int \Gamma(A) d\mu \]

In particular, if \( T_p \in \mathcal{T}(\mathcal{K}) \) then the corresponding NVM \( \Gamma_p \) has the form
\[ \Gamma_p(A)(\mu) = T_p(\mu)(A) = p(A) \]
so \( \Gamma_p(A) \) is the constant function \( p(A) \). Moreover, if \( M_p \in \mathcal{O}(\mathcal{K}) \) is sharp, then the corresponding NVM \( \Gamma_f \) satisfies
\[ \int \Gamma_f(A) d\mu = \Gamma_f(A)(\mu) = M_f(\mu)(A) = \mu [f^{-1}(A)] = \int \chi_{f^{-1}(A)} d\mu \]
Hence, \( \Gamma_f(A) = \chi_{f^{-1}(A)} \) for every \( A \in \mathcal{B}(\mathbb{R}) \).

**Example 2’. (Quantum Theory)** In this example \( \mathcal{V}^* \) is the Banach space \( \mathcal{B}(H) \) of bounded linear operators on \( H \) with norm
\[ \|L\| = \sup \{ \|L\psi\| : \|\psi\| = 1 \} \]
and duality given by
\[ \langle s, L \rangle = L(a) = \text{tr}(sL) \]
The identity operator \( I \) is the natural unit satisfying \( I(s) = 1 \) for all \( s \in \mathcal{K} \). In this case, \( \mathcal{K} \) is a base for the generating cone \( \mathcal{V}^+ \) of positive trace class operators and the converse of Theorem 4.1 holds, If \( \Gamma \) is a NVM, then \( \Gamma(A) \) is a positive operator satisfying \( 0 \leq \Gamma(A) \leq I \) called an effect and \( \Gamma(\mathbb{R}) = I \). According to the converse of Theorem 4.1, if \( M \) is an observable, then there exists a POVM \( \Gamma \) such that
\[ M(s)(A) = \text{tr}[s\Gamma(A)] \]
for every $s \in \mathcal{K}$ and $A \in \mathcal{B}(\mathbb{R})$. In particular, if $T_p \in \mathcal{T}(\mathcal{K})$, then the corresponding NVM $\Gamma_p$ has the form

$$\text{tr}[s \Gamma_p(A)] = \Gamma_p(A)(s) = T_p(s)(A) = p(A) = \text{tr}[sp(A)I]$$

so $\Gamma_p(A) = p(A)I$ for all $A \in \mathcal{B}(\mathbb{R})$.

Similar to a NVM, we define an $n$-dimensional NVM to be a map $\Gamma: \mathcal{B}(\mathbb{R}^n) \to \mathcal{V}^*$ such that $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R}^b)$ for every $s \in \mathcal{K}$. Moreover, a set $\{\Gamma_1, \ldots, \Gamma_n\}$ of NVMs for $\mathcal{K}$ is compatible if there exists an $n$-dimensional NVM $\Gamma$ such that

$$\Gamma(A \times \mathbb{R} \times \cdots \times \mathbb{R}) = \Gamma_1(A)$$

$$\vdots$$

$$\Gamma(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) = \Gamma_n(A)$$

for every $A \in \mathcal{B}(\mathbb{R})$. The proof of the following theorem is straightforward.

**Theorem 4.2.** If $\{M_1, \ldots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$ and $\{\Gamma_1, \ldots, \Gamma_n\}$ are the corresponding NVM for $\mathcal{K}$, then $\{M_1, \ldots, M_n\}$ are compatible if and only if $\{\Gamma_1, \ldots, \Gamma_n\}$ are compatible.

**References**