COMPATIBILITY FOR PROBABILISTIC THEORIES

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@du.edu

Abstract

We define an index of compatibility for a probabilistic theory (PT). Quantum mechanics with index 0 and classical probability theory with index 1 are at the two extremes. In this way, quantum mechanics is at least as incompatible as any PT. We consider a PT called a concrete quantum logic that may have compatibility index strictly between 0 and 1, but we have not been able to show this yet. Finally, we show that observables in a PT can be represented by positive, vector-valued measures.

1 Observables in Probabilistic Theories

This paper is based on the stimulating article [1] by Busch, Heinosaari and Schultz. The authors should be congratulated for introducing a useful new tool for measuring the compatibility of a probabilistic theory (PT). In this paper, we present a simpler, but coarser, measure of compatibility that we believe will also be useful.

A *probabilistic theory* is a σ-convex subset $\mathcal{K}$ of a real Banach space $\mathcal{V}$. That is, if $0 \le \lambda_i \le 1$ with $\sum \lambda_i = 1$ and $v_i \in \mathcal{K}$, $i = 1, 2, \ldots$, then $\sum \lambda_i v_i$ converges in norm to an element of $\mathcal{K}$. We call the elements of $\mathcal{K}$ *states*. There is no loss of generality in assuming that $\mathcal{K}$ generates $\mathcal{V}$ in the sense that the closed linear hull of $\mathcal{K}$ equals $\mathcal{V}$. Denote the collection of Borel subsets of $\mathbb{R}^n$ by $\mathcal{B}(\mathbb{R}^n)$ and the set of probability measures on $\mathcal{B}(\mathbb{R}^n)$ by $\mathcal{M}(\mathbb{R}^n)$. If $\mathcal{K}$ is a PT, an *n-dimensional observable* on $\mathcal{K}$ is a σ-affine map $M\colon \mathcal{K} \to \mathcal{M}(\mathbb{R}^n)$. We denote the set of n-dimensional observables by $\mathcal{O}_n(\mathcal{K})$ and write $\mathcal{O}(\mathcal{K}) = \mathcal{O}_1(\mathcal{K})$. We call the elements of $\mathcal{O}(\mathcal{K})$ *observables*. For $M \in \mathcal{O}(\mathcal{K})$, $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$, we interpret $M(s)(A)$ as the probability that M has a value in A when the system is in state s.

A set of observables $\{M_1, \ldots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$ is *compatible* or *jointly measurable* if there exists an $M \in \mathcal{O}_n(\mathcal{K})$ such that for every $A \in \mathcal{B}(\mathbb{R})$ and every $s \in \mathcal{K}$ we have

$$
\begin{aligned}
M(s)(A \times \mathbb{R} \times \cdots \times \mathbb{R}) &= M_1(s)(A) \\
M(s)(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R}) &= M_2(s)(A) \\
&\vdots \\
M(s)(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) &= M_n(s)(A)
\end{aligned}
$$

In this case, we call M a *joint observable* for $\{M_1, \ldots, M_n\}$ and we call $\{M_1, \ldots, M_n\}$ the *marginals* for M. It is clear that if $\{M_1, \ldots, M_n\}$ is compatible, then any proper subset is compatible. However, we suspect that the converse is not true. If a set of observables is not compatible we say it is *incompatible*.

It is clear that convex combinations of observables give an observable so $\mathcal{O}(\mathcal{K})$ forms a convex set. In the same way, $\mathcal{O}_n(\mathcal{K})$ is a convex set. Another way of forming new observables is by taking functions of an observable. If $f\colon \mathbb{R} \to \mathbb{R}$ is a Borel function and $M \in \mathcal{O}(\mathcal{K})$, the observable $f(M)\colon \mathcal{K} \to \mathcal{M}(\mathbb{R})$ is defined by $f(M)(s)(A) = M(s)\left(f^{-1}(A)\right)$ for all $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$.

Theorem 1.1. *If $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ are functions of a single observable M, then M_1, M_2 are compatible.*

Proof. Suppose $M_1 = f(M)$, $M_2 = g(M)$ where f and g are Borel functions. For $A, B \in \mathcal{B}(\mathbb{R})$, $s \in \mathcal{K}$ define $\widetilde{M}(s)$ on $A \times B$ by

$$
\widetilde{M}(s)(A \times B) = M(s)\left[f^{-1}(A) \cap g^{-1}(B)\right]
$$

By the Hahn extension theorem, $\widetilde{M}(s)$ extends to a measure in $\mathcal{M}(\mathbb{R}^2)$. Hence, $\widetilde{M} \in \mathcal{O}_2(\mathcal{K})$ and the marginals of $\widetilde{M}$ are $f(M)$ and $g(M)$. We conclude that $M_1 = f(M)$ and $M_2 = g(M)$ are compatible $\square$

It follows from Theorem 1.1 that an observable is compatible with any Borel function of itself and in particular with itself. In a similar way we obtain the next result.

Theorem 1.2. *If $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ are compatible and f, g are Borel functions, then $f(M_1)$ and $g(M_2)$ are compatible.*

Proof. Since M_1, M_2 are compatible, they have a joint observable $M \in \mathcal{O}_2(\mathcal{K})$. For $A, B \in \mathcal{B}(\mathbb{R})$, $s \in \mathcal{K}$ define $\widetilde{M}(s)$ on $A \times B$ by

$$\widetilde{M}(s)(A \times B) = M(s)\left[f^{-1}(A) \times g^{-1}(B)\right]$$

As in the proof of Theorem 1.1, $\widetilde{M}(s)$ extends to a measure in $\mathcal{M}(\mathbb{R}^2)$. Hence, $\widetilde{M} \in \mathcal{O}(\mathcal{K})$ and the marginals of $\widetilde{M}$ are

$$\widetilde{M}(s)(A \times \mathbb{R}) = M(s)\left[f^{-1}(A) \times \mathbb{R}\right] = M_1(s)\left[f^{-1}(A)\right] = f(M_1)(s)(A)$$
$$\widetilde{M}(s)(\mathbb{R} \times A) = M(s)\left[\mathbb{R} \times g^{-1}(A)\right] = M_2(s)\left[g^{-1}(A)\right] = g(M_2)(s)(A)$$

We conclude that $f(M_1)$ and $g(M_2)$ are compatible. $\square$

The next result is quite useful and somewhat surprising.

Theorem 1.3. *Let $M_i^j \in \mathcal{O}(\mathcal{K})$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and suppose $\{M_i^1, \ldots, M_i^m\}$ is compatible, $i = 1, \ldots, n$. If $\lambda_i \in [0,1]$ with $\sum \lambda_i = 1$, $i = 1, \ldots, n$, then*

$$\left\{\sum_{i=1}^n \lambda_i M_i^1, \sum_{i=1}^n \lambda_i M_i^2, \ldots, \sum_{i=1}^n \lambda_i M_i^m\right\}$$

is compatible.

Proof. Let $\widetilde{M}_i \in \mathcal{O}_m(\mathcal{K})$ be the joint observable for $\{M_i^1, \ldots, M_i^m\}$, $i = 1, \ldots, n$. Then $\widetilde{M} = \sum_{i=1}^n \lambda_i \widetilde{M}_i$ is an m-dimensional observable with marginals

$$\widetilde{M}(s)(A\times\mathbb{R}\times\cdots\times\mathbb{R})=\sum_{i=1}^{n}\lambda_i\widetilde{M}_i(s)(A\times\mathbb{R}\times\cdots\times\mathbb{R})=\sum_{i=1}^{n}\lambda_iM_i^1(s)(A)$$

$$\begin{aligned}\widetilde{M}(s)(\mathbb{R}\times A\times\mathbb{R}\times\cdots\times\mathbb{R})&=\sum_{i=1}^{n}\lambda_i\widetilde{M}_i(s)(\mathbb{R}\times A\times\mathbb{R}\times\cdots\times\mathbb{R})\\&=\sum_{i=1}^{n}\lambda_iM_i^2(s)(A)\\&\vdots\\\widetilde{M}(s)(\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\times A)&=\sum_{i=1}^{n}\lambda_i\widetilde{M}_i(s)(\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\times A)\\&=\sum_{I=1}^{n}\lambda_iM_i^m(s)(A)\end{aligned}$$

The result now follows □

Corollary 1.4. *Let $M, N, P \in \mathcal{O}(\mathcal{K})$ and $\lambda\in[0,1]$. If M is compatible with N and P, then M is compatible with $\lambda N+(1-\lambda)P$.*

Proof. Since $\{M,N\}$ and $\{M,P\}$ are compatible sets, by Theorem 1.3, we have that $M=\lambda M+(1-\lambda)M$ is compatible with $\lambda N+(1-\lambda)P$. □

2 Noisy Observables

If $p\in\mathcal{M}(\mathbb{R})$, we define the *trivial observable* $T_p\in\mathcal{O}(\mathcal{K})$ by $T_p(s)=p$ for every $s\in\mathcal{K}$. A trivial observable represents noise in the system. We denote the set of trivial observables on $\mathcal{K}$ by $\mathcal{T}(\mathcal{K})$. The set $\mathcal{T}(\mathcal{K})$ is convex with

$$\lambda T_p+(1-\lambda)T_q=T_{\lambda p+(1-\lambda)q}$$

for every $\lambda\in[0,1]$ and $p,q\in\mathcal{M}(\mathbb{R})$. An observable $M\in\mathcal{O}(\mathcal{K})$ is compatible with any $T_p\in\mathcal{T}(\mathcal{K})$ and a joint observable $\widetilde{M}\in\mathcal{O}_2(\mathcal{K})$ is given by

$$\widetilde{M}(s)(A\times B)=p(A)M(s)(B)$$

If $M\in\mathcal{O}(\mathcal{K})$, $T\in\mathcal{T}(\mathcal{K})$ and $\lambda\in[0,1]$ we consider $\lambda M+(1-\lambda)T$ as the observable M together with noise. Stated differently, we consider $\lambda M+$

$(1-\lambda)T$ to be a noisy version of M. The parameter $1-\lambda$ gives a measure of the proportion of noise and is called the *noise index*. Smaller λ gives a larger proportion of noise. As we shall see, incompatible observables may have compatible noisy versions.

The next lemma follows directly from Corollary 1.4. It shows that if M is compatible with N, then M is compatible with any noisy version of N.

Lemma 2.1. *If $M \in \mathcal{O}(\mathcal{K})$ is compatible with $N \in \mathcal{O}(\mathcal{K})$, then M is compatible with $\lambda N + (1-\lambda)T$ for any $\lambda \in [0,1]$ and $T \in \mathcal{T}(\mathcal{K})$.*

The following lemma shows that for any $M, N \in \mathcal{O}(\mathcal{K})$ a noisy version of N with noise index λ is compatible with any noisy version of M with noise index $1-\lambda$. The lemma also shows that if M is compatible with a noisy version of N, then M is compatible with a still noisier version of N.

Lemma 2.2. *Let $M, N \in \mathcal{O}(\mathcal{K})$ and $S, T \in \mathcal{T}(\mathcal{K})$.* (a) *If $\lambda \in [0,1]$, then $\lambda M + (1-\lambda)T$ and $(1-\lambda)N + \lambda S$ are compatible.* (b) *If M is compatible with $\lambda N + (1-\lambda)T$, then M is compatible with $\mu N + (1-\mu)T$ where $0 \le \mu \le \lambda \le 1$.*

Proof. (a) Since $\{M, S\}$ and $\{T, N\}$ are compatible sets, by Theorem 1.3 $\lambda M + (1-\lambda)T$ is compatible with $\lambda S + (1-\lambda)N$. (b) We can assume that $\lambda > 0$ and we let $\alpha = \mu/\lambda$ so $0 \le \alpha \le 1$. Since $\{M, \lambda N + (1-\lambda)T\}$ and $\{M, T\}$ are compatible sets, by Theorem 1.3, $M = \alpha M + (1-\alpha)M$ is compatible with

$$\begin{aligned}\alpha\left[\lambda N + (1-\lambda)T\right] + (1-\alpha)T &= \alpha\lambda N + \left[\alpha(1-\lambda) + (1-\alpha)\right]T \\ &= \mu N + (1-\mu)T\end{aligned}$$

□

The *compatibility region* $J(M_1, M_2, \ldots, M_n)$ of observables $M_i \in \mathcal{O}(\mathcal{K})$, $i = 1, \ldots, n$, is the set of points $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in [0,1]^n$ for which there exist $T_i \in \mathcal{T}(\mathcal{K})$, $i = 1, 2, \ldots, n$, such that

$$\{\lambda_i M_i + (1-\lambda_i)T_i\}_{i=1}^n$$

form a compatible set. Thus, $J(M_1, M_2, \ldots, M_n)$ gives parameters for which there exist compatible noisy versions of $M_1, M_2, \ldots, M_n$. It is clear that $0 = (0, \ldots, 0) \in J(M_1, M_2, \ldots, M_n)$ and we shall show that $J(M_1, M_2, \ldots, M_n)$ contains many points. We do not know whether $J(M_1, M_2, \ldots, M_n)$ is symmetric under permutations of the M_i. For example, is $J(M_1, M_2) = J(M_2, M_1)$?

Theorem 2.3. *$J(M_1, M_2, \ldots M_n)$ is a convex subset of $[0,1]^n$.*

Proof. Suppose $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in J(M_1, \ldots, M_n)$. We must show that

$$\begin{aligned}\lambda(\lambda_1, \ldots, \lambda_n) + &(1-\lambda)(\mu_1, \ldots, \mu_n) \\ &= (\lambda\lambda_1 + (1-\lambda)\mu_1, \ldots, \lambda\lambda_n + (1-\lambda)\mu_n) \in J(M_1, \ldots, M_n)\end{aligned}$$

for all $\lambda \in [0,1]$. Now there exist $S_1, \ldots, S_n, T_1, \ldots, T_n \in \mathcal{T}(\mathcal{K})$ such that $\{\lambda_i M_i + (1-\lambda_i)S_i\}_{i=1}^n$ and $\{\mu_i M_i + (1-\mu_i)T_i\}_{i=1}^n$ are compatible. By Theorem 1.3 the set of observables

$$\begin{aligned}&\{\lambda\left[\lambda_i M_i + (1-\lambda_i)S_i\right] + (1-\lambda)\left[\mu_i M_i + (1-\mu_i)T_i\right]\} \\ &\quad = \{(\lambda\lambda_i + (1-\lambda)\mu_i)M_i + \lambda(1-\lambda_i)S_i + (1-\lambda)(1-\mu_i)T_i\}\end{aligned}$$

is compatible. Since

$$\begin{aligned}\lambda(1-\lambda_i) + (1-\lambda)(1-\mu_i) &= 1 - \lambda\lambda_i - \mu_i + \lambda\mu_i \\ &= 1 - [\lambda\lambda_i + (1-\lambda)\mu_i]\end{aligned}$$

letting $\alpha_i = \lambda\lambda_i + (1-\lambda)\mu_i$ we have that

$$U_i = \frac{1}{1-\alpha_i}\left[\lambda(1-\lambda_i)S_i + (1-\lambda)(1-\mu_i)T_i\right] \in \mathcal{T}(\mathcal{K})$$

Since $\{\alpha_i M_i + (1-\alpha_i)U_i\}_{i=1}^n$ forms a compatible set, we conclude that $(\alpha_1, \ldots, \alpha_n) \in J(M_1, \ldots, M_n)$. $\square$

Let $\Delta_n = \{(\lambda_1, \ldots, \lambda_n) \in [0,1]^n : \sum \lambda_i \leq 1\}$. To show that Δ_n forms a convex subset of $[0,1]^n \subseteq \mathbb{R}^n$, let $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in \Delta_n$ and $\lambda \in [0,1]$. Then $\lambda(\lambda_1, \ldots, \lambda_n + (1-\lambda)(\mu_1, \ldots, \mu_n) \in [0,1]^n$ and

$$\sum_{i=1}^n \left[\lambda\lambda_i + (1-\lambda)\mu_i\right] = \lambda\sum\lambda_i + (1-\lambda)\sum\mu_i \leq \lambda + (1-\lambda) = 1$$

Theorem 2.4. *If $\{M_1, \ldots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$, then $\Delta_n \subseteq J(M_1, \ldots, M_n)$.*

Proof. Let $\delta_0 = (0, 0, \ldots, 0) \in \mathbb{R}^n$, $\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$, $i = 1, \ldots, n$ where 1 is in the ith coordinate. It is clear that

$$\delta_i \in J(M_1, \ldots, M_n) \cap \Delta_n, \quad i = 0, 1, \ldots, n$$

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$, letting $\mu = \sum \lambda_i$ we have that $0 \leq \mu \leq 1$, $\sum \lambda_i + (1 - \mu) = 1$ and

$$\lambda = \sum_{i=1}^{n} \lambda_i \delta_i + (1 - \mu)\delta_0$$

It follows that Δ_n is the convex hull of $\{\delta_0, \delta_1, \ldots, \delta_n\}$. Since

$$\{\delta_0, \delta_1, \ldots, \delta_n\} \subseteq J(M_1, \ldots, M_n)$$

and $J(M_1, \ldots, M_n)$ is convex, it follows that $\Delta_n \in J(M_1, \ldots, M_n)$. $\square$

The *n-dimensional compatibility region* for PT $\mathcal{K}$ is defined by

$$J_n(\mathcal{K}) = \cap \{J(M_1, \ldots, M_n) \colon M_i \in \mathcal{O}(\mathcal{K}), i = 1, \ldots, n\}$$

We have that $\Delta_n \subseteq J_n(\mathcal{K}) \subseteq [0,1]^n$ and $J_n(\mathcal{K})$ is a convex set that gives a measure of the incompatibility of observables on $\mathcal{K}$. As $J_n(\mathcal{K})$ gets smaller, $\mathcal{K}$ gets more incompatible and the maximal incompatibility is when $J_n(\mathcal{K}) = \Delta_n$. For the case of quantum states $\mathcal{K}$, the set $J_2(\mathcal{K})$ has been considered in detail in [1].

We now introduce a measure of compatibility that we believe is simpler and easier to investigate than $J_2(M,N)$ For $M, N \in \mathcal{O}(\mathcal{K})$, the *compatibility interval* $I(M,N)$ is the set of $\lambda \in [0,1]$ for which there exists a $T \in \mathcal{T}(\mathcal{K})$ such that M is compatible with $\lambda N + (1-\lambda)T$. Of course, $0 \in T(M,N)$ and M and N are compatible if and only if $1 \in I(M,N)$. We do not know whether $I(M,N) = I(N,M)$. It follows from Lemma 2.2(b) that if $\lambda \in T(M,N)$ and $0 \leq \mu \leq \lambda$, then $\mu \in I(M,N)$. Thus, $I(M,N)$ is an interval with left endpoint 0. The *index of compatibility* of M and N is $\lambda(M,N) = \sup\{\lambda \colon \lambda \in I(M,N)\}$. We do not know whether $\lambda(M,N) \in I(M,N)$ but in any case $I(M,N) = [0, \lambda(M,N)]$ or $I(M,N) = [0, \lambda(M,N))$. For a PT $\mathcal{K}$, we define the *interval of compatibility* for $\mathcal{K}$ to be

$$I(\mathcal{K}) = \cap \{I(M,N) \colon M, N \in \mathcal{O}(\mathcal{K})\}$$

The *index of compatibility* of $\mathcal{K}$ is

$$\lambda(\mathcal{K}) = \inf \{\lambda(M,N) \colon M, N \in \mathcal{O}(\mathcal{K})\}$$

and $I(\mathcal{K}) = [0, \lambda(\mathcal{K})]$ or $I(\mathcal{K}) = [0, \lambda(\mathcal{K}))$. Again, $\lambda(\mathcal{K}) = 0$ gives a measure of incompatibility of the observables in $\mathcal{O}(\mathcal{K})$.

Example 1. (Classical Probability Theory) Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{V}$ be the Banach space of real-valued measures on $\mathcal{A}$ with the total variation norm. If $\mathcal{K}$ is the σ-convex set of probability measures on $\mathcal{A}$, then $\mathcal{K}$ generates $\mathcal{V}$. There are two types of observables on $\mathcal{K}$, the *sharp* and *fuzzy* observables. The sharp observables have the form M_f where f is a measurable function $f\colon \Omega \to \mathbb{R}$ and $M_f(s)(A) = s\left[f^{-1}(A)\right]$. If M_f, M_g are sharp observables, form the unique 2-dimensional observable $\widetilde{M}$ satisfying

$$\widetilde{M}(s)(A \times B) = s\left[f^{-1}(A) \cap g^{-1}(B)\right]$$

Then $\widetilde{M}$ is a joint observable for M_f, M_g so M_f and M_g are compatible. The unsharp observables are obtained as follows. Let $\mathcal{F}(\Omega)$ be the set of measurable functions $f\colon \Omega \to [0,1]$. Let $\widehat{M}\colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\Omega)$ satisfy $\widehat{M}(\mathbb{R}) = 1$, $\widehat{M}(\dot{\cup} A_i) = \sum \widehat{M}(A_i)$. An unsharp observable has the form

$$M(s)(A) = \int \widehat{M}(A)ds$$

Two unsharp observables M, N are also compatible because we can form the joint observable $\widetilde{M}$ given by

$$\widehat{M}(S)(A \times B) = \int \widehat{M}(A)\widehat{N}(B)ds$$

We conclude that $J(\mathcal{K}) = [0,1] \times [0,1]$ and $I(\mathcal{K}) = [0,1]$ so $\mathcal{K}$ has the maximal amount of compatibility.

Example 2. (Quantum Theory) Let H be a separable complex Hilbert space and let $\mathcal{K}$ be the σ-convex set of all trace 1 positive operators on H. Then $\mathcal{K}$ generates the Banach space of self-adjoint trace-class operators with the trace norm. It is well known that $M \in \mathcal{O}(\mathcal{K})$ if and only if there exists a positive operator-valued measure (POVM) P such that $M(s)(A) = \operatorname{tr}\left[sP(A)\right]$ for every $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$. It is shown in [1] that if $\dim H = \infty$, then there exist $M_1, M_2 \in \mathcal{O}(\mathcal{K})$ such that $J_2(M_1, M_2) = \Delta_2$ and hence $J(\mathcal{K}) = \Delta_2$. If $\dim H < \infty$, then $J(\mathcal{K})$ is not known, although partial results have been obtained and it is known that $J(\mathcal{K}) \to \Delta_2$ as $\dim H \to \infty$

Now let H be an arbitrary complex Hilbert space with $\dim H \geq 2$. Although the Pauli matrices σ_x, σ_y are 2-dimensional, we can extend them from a 2-dimensional subspace H_0 of H to all of H by defining $\sigma_x \psi = 0$ for all $\psi \in H_0^\perp$. Define the POVMs M_x, M_y on H by $M_x(\pm 1) = \frac{1}{2}(I \pm \sigma_x)$, $M_y(\pm 1) = \frac{1}{2}(I \pm \sigma_y)$. It is shown in [1] that

$$J(M_x, M_y) = \left\{(\lambda, \mu) \in [0,1] \times [0,1] : \lambda^2 + \mu^2 \leq 1\right\}$$

Thus, $J(M_x, M_y)$ is a quadrant of the unit disk. We conclude that M_x is compatible with $\mu M_y + (1-\mu)T$ for $T \in \mathcal{T}(\mathcal{K})$ if and only if $1 + \mu^2 \leq 1$. Therefore, $\mu = 0$, so $I(M_x, M_y) = \{0\}$ and $\lambda(M_x, M_y) = 0$. Thus, $I(\mathcal{K}) = \{0\}$ and $\lambda(\mathcal{K}) = 0$. We conclude that quantum mechanics has the smallest index of compatibility possible for a PT. The index of compatibility for a classical system is 1, so we have the two extremes. It would be interesting to find $\lambda(\mathcal{K})$ for other PTs.

3 Concrete Quantum Logics

We now consider a PT that seems to be between the classical and quantum PTs of Examples 1 and 2. A collection of subsets $\mathcal{A}$ of a set Ω is a *σ-class* if $\emptyset \in \mathcal{A}$, $A^c \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and if A_i are mutually disjoint, $i = 1, 2, \ldots$, then $\cup A_i \in \mathcal{A}$. If $\mathcal{A}$ is a σ-class on Ω, we call $(\Omega, \mathcal{A})$ a *concrete quantum logic*. A *σ-state* on $\mathcal{A}$ is a map $s \colon \mathcal{A} \to [0,1]$ such that $s(\Omega) = 1$ and if $A_i \in \mathcal{A}$ are mutually disjoint, then $s(\cup A_i) = \sum s(A_i)$. If $\mathcal{K}$ is the set of σ-states on $(\Omega, \mathcal{A})$, we call $\mathcal{K}$ a *concrete quantum logic* PT. Let $\mathcal{A}_\sigma$ be the σ-algebra generated by $\mathcal{A}$. A σ-state s is *classical* if there exists a probability measure μ on $\mathcal{A}_\sigma$ such that $s = \mu \mid \mathcal{A}$. As in the classical case, an observable is *sharp* if it has the form $M_f(s)(A) = s\left[f^{-1}(A)\right]$ for an $\mathcal{A}$-measurable function $f \colon \Omega \to \mathbb{R}$. If f and g are $\mathcal{A}$-measurable functions satisfying $f^{-1}(A) \cap g^{-1}(B) \in \mathcal{A}$ for all $A, B \in \mathcal{B}(\mathbb{R})$, then M_f and M_g are compatible because they have a joint observable M satisfying $M(s)(A \times B) = s\left[f^{-1}(A) \cap g^{-1}(B)\right]$ for all $s \in \mathcal{K}$, $A, B \in \mathcal{B}(\mathbb{R})$. We do not know whether M_f and M_g compatible implies that $f^{-1}(A) \cap g^{-1}(B) \in \mathcal{A}$ holds for every $A, B \in \mathcal{B}(\mathbb{R})$, although we suspect it does not.

Example 3. This is a simple example of a concrete quantum logic. Let $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{A}$ be the collection of subsets of Ω with even cardinality. Then

$$\mathcal{A} = \{\emptyset, \Omega, \{1,2\}, \{3,4\}, \{1,3\}, \{2,4\}, \{1,4\}, \{2,3\}\}$$

Let $\mathcal{K}$ be the sets of all states on $\mathcal{A}$. Letting $a = \{1, 2\}$, $a' = \{3, 4\}$, $b = \{1, 3\}$, $b' = \{3, 4\}$, $c = \{1, 4\}$, $c' = \{2, 3\}$ we can represent an $s \in \mathcal{K}$ by

$$\begin{aligned}\widehat{s} &= (s(a), s(a'), s(b), s(b'), s(c), s(c')) \\ &= (s(a), 1 - s(a), s(b), 1 - s(b), s(c), 1 - s(c))\end{aligned}$$

Thus, every $s \in \mathcal{K}$ has the form

$$s = (\lambda_1, 1 - \lambda_1, \lambda_2, 1 - \lambda_2, \lambda_3, 1 - \lambda_3)$$

for $0 \leq \lambda_i \leq 1$, $i = 1, 2, 3$. The pure (extremal) classical states are the 0-1 states: $\delta_1 = (1, 0, 1, 0, 1, 0)$, $\delta_3 = (1, 0, 0, 1, 0, 1)$, $\delta_3 = (0, 1, 1, 0, 0, 1)$, $\delta_4 = (0, 1, 0, 1, 1, 0)$. The pure nonclassical states are the 0-1 states: $\gamma_1 = 1 - \delta_1$, $\gamma_2 = 1 - \delta_2$, $\gamma_3 = 1 - \delta_3$, $\gamma_4 = 1 - \delta_4$ where $1 = (1, 1, 1, 1, 1, 1)$. For example, to see that γ_1 is not classical, we have that $\gamma_1 = (0, 1, 0, 1, 0, 1)$. Hence, $\gamma_1(\{3, 4\}) = \gamma_1(\{2, 4\}) = \gamma_1(\{2, 3\}) = 1$. If there exists a probability measure μ such that $\gamma_1 = \mu \mid \mathcal{A}$ we would have $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 0$ which is a contradiction. The collection of sharp observable is very limited because a measurable function $f \colon \Omega \to \mathbb{R}$ can have at most two values. Thus, if M_f is a sharp observable there exists $a, b \in \mathbb{R}$ such that $M_f(s)(\{a, b\}) = 1$ for every $s \in \mathcal{K}$. There are many observables with more than two values (non-binary observables) and these are not sharp. Even for this simple example, it appears to be challenging to investigate the region and interval of compatibility.

4 Vector-Valued Measures

Let $\mathcal{K}$ be a PT with generated Banach space $\mathcal{V}$ and $\mathcal{V}^*$ be the Banach space dual of $\mathcal{V}$. A *normalized vector-valued measure* (NVM) for $\mathcal{K}$ is a map $\Gamma \colon \mathcal{B}(\mathbb{R}) \to \mathcal{V}^*$ such that $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R})$ for every $s \in \mathcal{K}$. Thus, Γ satisfies the conditions:

(1) $\Gamma(\mathbb{R})(s) = 1$ for every $s \in \mathcal{K}$,

(2) $0 \leq \Gamma(A)(s) \leq 1$ for every $s \in \mathcal{K}$, $A \in \mathcal{B}(\mathbb{R})$,

(3) If $A_i \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, $i = 1, 2, \ldots$, then

$$\Gamma(\cup A_i)(s) = \sum \Gamma(A_i)(s)$$

for every $s \in \mathcal{K}$.

This section shows that there is a close connection between observables on $\mathcal{K}$ and NVMs for $\mathcal{K}$.

Theorem 4.1. *If Γ is a NVM for $\mathcal{K}$, then $M\colon \mathcal{K}\to\mathcal{M}(\mathbb{R})$ given by $M(s)(A)=\Gamma(A)(s)$, $s\in\mathcal{K}$, $A\in\mathcal{B}(\mathbb{R})$, is an observable on $\mathcal{K}$.*

Proof. Since $A\mapsto\Gamma(A)(s)\in\mathcal{M}(\mathbb{R})$ we have that $A\mapsto M(s)(A)\in\mathcal{M}(\mathbb{R})$. Let $\lambda_i\in[0,1]$ with $\sum\lambda_i=1$, $s_i\in\mathcal{K}$, $i=1,2,\ldots$, and suppose that $s=\sum\lambda_i s_i$. Then $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n\lambda_i s_i=s$ in norm and since $s\mapsto\Gamma(A)(s)\in\mathcal{V}^*$, for every $A\in\mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned}M(s)(A)&=M\left(\sum\lambda_i s_i\right)(A)=\Gamma(A)\left(\sum\lambda_i s_i\right)=\Gamma(A)\left(\lim_{n\to\infty}\sum_{i=1}^n\lambda_i s_i\right)\\&=\lim_{n\to\infty}\Gamma(A)\left(\sum_{i=1}^n\lambda_i s_i\right)=\lim_{n\to\infty}\sum_{i=1}^n\lambda_i\Gamma(A)(s_i)\\&=\lim_{n\to\infty}\sum_{i=1}^n\lambda_i M(s_i)(A)=\sum_{i=1}^\infty\lambda_i M(s_i)(A)\end{aligned}$$

It follows that $M\left(\sum\lambda_i s_i\right)=\sum\lambda_i M(s_i)$ so $M\in\mathcal{O}(\mathcal{K})$. □

The converse of Theorem 4.1 holds if some mild conditions are satisfied. To avoid some topological and measure-theoretic technicalities, we consider the special case where $\mathcal{V}$ is finite-dimensional. Assuming that $\mathcal{K}$ is the base of a generating positive cone $\mathcal{V}^+$, we have that every element $v\in\mathcal{V}^+$ has a unique form $v=\alpha s$, $\alpha\geq 0$, $s\in\mathcal{K}$ and that $\mathcal{V}=\mathcal{V}^+\oplus\mathcal{V}^-$ where $\mathcal{V}^-=-\mathcal{V}^+$ and $\mathcal{V}^+\cap\mathcal{V}^-=\{0\}$. If $M\in\mathcal{O}(\mathcal{K})$, then for every $A\in\mathcal{B}(\mathbb{R})$, $s\mapsto M(s)(A)$ is a convex, real-valued function on $\mathcal{K}$. A standard argument shows that this function has a unique linear extension $\widehat{M}(A)=\mathcal{V}^*$ for every $A\in\mathcal{B}(\mathbb{R})$. Hence

$$\widehat{M}(A)(s)=M(s)(A)\tag{4.1}$$

for every $s\in\mathcal{K}$, $A\in\mathcal{B}(\mathbb{R})$. Since $A\mapsto\widehat{M}(A)(s)=M(s)(A)\in\mathcal{M}(\mathbb{R})$ we conclude that $A\mapsto\widehat{M}(A)$ is a NVM and $\widehat{M}$ is the unique NVM satisfying (4.1). It follows that the converse of Theorem 4.1 holds in this case.

Example $1'$**.** (Classical Probability Theory) In this example $\mathcal{V}^*$ is the Banach space of bounded measurable functions $f\colon\Omega\to\mathbb{R}$ with norm $\|f\|=$

$\sup |f(\omega)| < \infty$ and duality given by

$$\langle \mu, f \rangle = f(\mu) = \int f d\mu$$

The function $1(\omega) = 1$ for every $\omega \in \Omega$ is the natural unit satisfying $1(\mu) = 1$ for every $\mu \in \mathcal{K}$. In this case, $\mathcal{K}$ is a base for the generating positive cone $\mathcal{V}^+$ of bounded measures and the converse of Theorem 4.1 holds. Then a NVM Γ has the form $0 \leq \Gamma(A)(\omega) \leq 1$ for every $A \in \mathcal{B}(\mathbb{R})$, $\omega \in \Omega$ and $\Gamma(\mathbb{R}) = 1$. Thus $\Gamma(A) \in \mathcal{F}(\Omega)$ and if M is the corresponding observable, then

$$M(\mu)(A) = \Gamma(A)(\mu) = \int \Gamma(A) d\mu$$

In particular, if $T_p \in \mathcal{T}(\mathcal{K})$ then the corresponding NVM Γ_p has the form

$$\Gamma_p(A)(\mu) = T_p(\mu)(A) = p(A)$$

so $\Gamma_p(A)$ is the constant function $p(A)$. Moreover, if $M_p \in \mathcal{O}(\mathcal{K})$ is sharp, then the corresponding NVM Γ_f satisfies

$$\int \Gamma_f(A) d\mu = \Gamma_f(A)(\mu) = M_f(\mu)(A) = \mu\left[f^{-1}(A)\right] = \int \chi_{f^{-1}(A)} d\mu$$

Hence, $\Gamma_f(A) = \chi_{f^{-1}(A)}$ for every $A \in \mathcal{B}(\mathbb{R})$.

Example $2'$**.** (Quantum Theory) In this example $\mathcal{V}^*$ is the Banach space $\mathcal{B}(H)$ of bounded linear operators on H with norm

$$\|L\| = \sup\{\|L\psi\| : \|\psi\| = 1\}$$

and duality given by

$$\langle s, L \rangle = L(a) = \operatorname{tr}(sL)$$

The identity operator I is the natural unit satisfying $I(s) = 1$ for all $s \in \mathcal{K}$. In this case, $\mathcal{K}$ is a base for the generating cone $\mathcal{V}^+$ of positive trace class operators and the converse of Theorem 4.1 holds, If Γ is a NVM, then $\Gamma(A)$ is a positive operator satisfying $0 \leq \Gamma(A) \leq I$ called an *effect* and $\Gamma(\mathbb{R}) = I$. According to the converse of Theorem 4.1, if M is an observable, then there exists a POVM Γ such that

$$M(s)(A) = \operatorname{tr}\left[s\Gamma(A)\right]$$

for every $s \in \mathcal{K}$ and $A \in \mathcal{B}(\mathbb{R})$. In particular, if $T_p \in \mathcal{T}(\mathcal{K})$, then the corresponding NVM Γ_p has the form

$$\operatorname{tr}[s\Gamma_p(A)] = \Gamma_p(A)(s) = T_p(s)(A) = p(A) = \operatorname{tr}[sp(A)I]$$

so $\Gamma_p(A) = p(A)I$ for all $A \in \mathcal{B}(\mathbb{R})$.

Similar to a NVM, we define an *n-dimensional* NVM to be a map $\Gamma\colon \mathcal{B}(\mathbb{R}^n) \to \mathcal{V}^*$ such that $A \mapsto \Gamma(A)(s) \in \mathcal{M}(\mathbb{R}^b)$ for every $s \in \mathcal{K}$. Moreover, a set $\{\Gamma_1, \ldots, \Gamma_n\}$ of NVMs for $\mathcal{K}$ is compatible if there exists an n-dimensional NVM Γ such that

$$\begin{aligned}\Gamma(A \times \mathbb{R} \times \cdots \times \mathbb{R}) &= \Gamma_1(A)\\ &\vdots\\ \Gamma(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A) &= \Gamma_n(A)\end{aligned}$$

for every $A \in \mathcal{B}(\mathbb{R})$. The proof of the following theorem is straightforward.

Theorem 4.2. *If $\{M_1, \ldots, M_n\} \subseteq \mathcal{O}(\mathcal{K})$ and $\{\Gamma_1, \ldots \Gamma_n\}$ are the corresponding NVM for $\mathcal{K}$, then $\{M_1, \ldots, M_n\}$ are compatible if and only if $\{\Gamma_1, \ldots, \Gamma_n\}$ are compatible.*

References

[1] P. Busch, T. Heinosaari and J. Schultz, Quantum theory contains maximally incompatible observables, arXiv: 1210.4142 v1 [quant-ph], Oct. 15, 2012.