THE NEXT BEST THING TO A P-POINT

ANDREAS BLASS, NATASHA DOBRINEN, AND DILIP RAGHAVAN

ABSTRACT. We study ultrafilters on ω^2 produced by forcing with the quotient of $\mathcal{P}(\omega^2)$ by the Fubini square of the Fréchet filter on ω. We show that such an ultrafilter is a weak P-point but not a P-point and that the only non-principal ultrafilters strictly below it in the Rudin-Keisler order are a single isomorphism class of selective ultrafilters. We further show that it enjoys the strongest square-bracket partition relations that are possible for a non-P-point. We show that it is not basically generated but that it shares with basically generated ultrafilters the property of not being at the top of the Tukey ordering. In fact, it is not Tukey-above $[\omega_1]^{<\omega}$, and it has only continuum many ultrafilters Tukey-below it. A tool in our proofs is the analysis of similar (but not the same) properties for ultrafilters obtained as the sum, over a selective ultrafilter, of non-isomorphic selective ultrafilters.

1. Introduction

Quotients of the form $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an analytic ideal on ω are referred to as analytic quotients. Analytic quotients have been well studied in the literature (see [11], [17], and [8]). These studies have usually focused on the structure of gaps in such quotients or on lifting isomorphisms between $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$, topics that are closely related. $\mathcal{P}(\omega)/\mathcal{I}$ is a Boolean algebra, and hence is a notion of forcing.[1] If forcing with an analytic quotient $\mathcal{P}(\omega)/\mathcal{I}$ does not add any new subset of ω, then the generic filter it adds is in fact an ultrafilter on ω that is disjoint from $\mathcal{I}$. Ultrafilters added by analytic quotients have not been as extensively investigated, except for the most familiar

Blass was partially supported by NSF grant DMS-0653696.

Dobrinen was partially supported by the Fields Institute and the National Science Foundation during the Thematic Program in Forcing and Its Applications 2012, and by a Simons Foundation Collaborative Grant.

Raghavan was partially supported by a National University of Singapore research grant and by the Fields Institute during its Thematic Program on Forcing and Its Applications, 2012.

[1] Strictly speaking, the notion of forcing is the Boolean algebra minus its zero element; such deletions or additions of zero elements will be tacitly assumed wherever necessary.

analytic quotient $\mathcal{P}(\omega)/\mathrm{Fin}$, which adds a selective ultrafilter. Recall the following definitions.

Definition 1. An ultrafilter $\mathcal{U}$ on ω is *selective* if, for every function $f : \omega \to \omega$, there is a set $A \in \mathcal{U}$ on which f is either one-to-one or constant. It is a *P-point* if, for every $f : \omega \to \omega$, there is $A \in \mathcal{U}$ on which f is finite-to-one or constant.

Indeed, selective ultrafilters can be completely characterized in terms of genericity over $\mathcal{P}(\omega)/\mathrm{Fin}$ – a well known theorem of Todorcevic ([3]) states that in the presence of large cardinals an ultrafilter on ω is selective if and only if it is $(\mathbf{L}(\mathbb{R}), \mathcal{P}(\omega)/\mathrm{Fin})$-generic. Similar characterizations were recently shown for a large class of ultrafilters forming a precise hierarchy above selective ultrafilters (see [6], [7], and [12]).

The generic ultrafilter added by $\mathcal{P}(\omega)/\mathrm{Fin}$ has a simple Rudin-Keisler type as well as a simple Tukey type. Let $\mathcal{F}$ be a filter on a set X and $\mathcal{G}$ a filter on a set Y. Recall that we say that $\mathcal{F}$ is *Rudin-Keisler(RK) reducible to* $\mathcal{G}$ or *Rudin-Keisler(RK) below* $\mathcal{G}$, and we write $\mathcal{F} \leq_{RK} \mathcal{G}$, if there is a map $f : Y \to X$ such that for each $a \subseteq X$, $a \in \mathcal{F}$ iff $f^{-1}(a) \in \mathcal{G}$. $\mathcal{F}$ and $\mathcal{G}$ are *RK equivalent*, written $\mathcal{F} \equiv_{RK} \mathcal{G}$, if $\mathcal{F} \leq_{RK} \mathcal{G}$ and $\mathcal{G} \leq_{RK} \mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, then $\mathcal{F} \equiv_{RK} \mathcal{G}$ if and only if there is a permutation $f : \omega \to \omega$ such that $\mathcal{F} = \{a \subseteq \omega : f^{-1}(a) \in \mathcal{G}\}$. For this reason, ultrafilters that are RK equivalent are sometimes said to be *(RK) isomorphic*. It is well-known that selective ultrafilters are minimal in the Rudin-Keisler ordering, meaning that any ultrafilter that is RK below a selective ultrafilter is RK equivalent to that selective ultrafilter.

The Tukey types of selective ultrafilters have also been completely characterized. We say that a poset $\langle D, \leq \rangle$ is *directed* if any two members of D have an upper bound in D. A set $X \subseteq D$ is *unbounded in* D if it doesn't have an upper bound in D. A set $X \subseteq D$ is said to be *cofinal in* D if $\forall y \in D \, \exists x \in X \, [y \leq x]$. Given directed sets D and E, a map $f : D \to E$ is called a *Tukey map* if the image of every unbounded subset of D is unbounded in E. A map $g : E \to D$ is called a *convergent map* if the image of every cofinal subset of E is cofinal in D. It is not difficult to show that there is a Tukey map $f : D \to E$ if and only if there is a convergent $g : E \to D$. When this situation obtains, we say that D is *Tukey reducible* to E, and we write $D \leq_T E$. The relation $\leq_T$ is a quasi order, so it induces an equivalence relation in the usual way: $D \equiv_T E$ if and only if both $D \leq_T E$ and $E \leq_T D$ hold. If $D \equiv_T E$, we say that D and E are *Tukey equivalent* or have the same *cofinal type*. It is worth noting that if κ is an infinite cardinal and if D is a directed set of size κ, then $D \leq_T \langle [\kappa]^{<\omega}, \subseteq\rangle$. If $\mathcal{U}$ is any ultrafilter, then $\langle \mathcal{U}, \supseteq\rangle$ is a directed set. When ultrafilters are viewed as directed sets in this way, Tukey reducibility is a coarser quasi order than RK reducibility. It is also worth noting that there is always an ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} \equiv_T \langle [\mathfrak{c}]^{<\omega}, \subseteq\rangle$; every ultrafilter on ω is Tukey below this $\mathcal{U}$. In [14], it is shown that selective ultrafilters are Tukey minimal. Moreover, if $\mathcal{U}$ is selective and $\mathcal{V}$ is any ultrafilter that is Tukey below $\mathcal{U}$, then $\mathcal{V} \equiv_{RK} \mathcal{U}^\alpha$, for some $\alpha < \omega_1$ (see [14] for the definition of $\mathcal{U}^\alpha$). Similar results have been proved in [6] and [7] for a large class of rapid p-points forming a hierarchy of ultrafilters Tukey above selective ultrafilters.

On the other hand, the situation is not as clear for ultrafilters added by other analytic quotients; neither their Rudin-Keisler type nor their Tukey type has been well-studied. In this paper we investigate the generic ultrafilter added by a specific Borel quotient. In order to describe it, we introduce some preliminary notation.

Definition 2. For $p \subseteq \omega^2$ and $n \in \omega$, let $p(n) = \{m \in \omega : \langle n, m\rangle \in p\}$. For $x \in {}^\omega(\mathcal{P}(\omega))$ and $a \subseteq \omega$, $x{\restriction}a = \{\langle n, m\rangle \in \omega^2 : n \in a \text{ and } m \in x(n)\}$. According to these conventions, we notationally identify a set $p \subseteq \omega^2$ with the sequence of its sections $p(n)$; conversely, we shall sometimes notationally identify a sequence $x \in {}^\omega(\mathcal{P}(\omega))$ and the corresponding set $x \restriction \omega$. π_1 is the projection of ω^2 to the first co-ordinate and π_2 is projection to the second. More formally, for $\langle n, m\rangle \in \omega^2$, $\pi_1(\langle n, m\rangle) = n$ and $\pi_2(\langle n, m\rangle) = m$.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}_n$ for $n \in \omega$ be filters on ω. We define the *$\mathcal{U}$-indexed sum of the $\mathcal{V}_n$'s* as

$$\mathcal{U}\text{-}\sum_n \mathcal{V}_n = \{X \subseteq \omega^2 : \{n \in \omega : X(n) \in \mathcal{V}_n\} \in \mathcal{U}\}.$$

In the case when all the ultrafilters $\mathcal{V}_n$ are the same $\mathcal{V}$, we write $\mathcal{U} \otimes \mathcal{V}$ for $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$. If, furthermore, $\mathcal{V} = \mathcal{U}$, then we abbreviate $\mathcal{U} \otimes \mathcal{U}$ as $\mathcal{U}^{\otimes 2}$.

For a set A, the *Fréchet ideal on A* is $\{B \subseteq A : B \text{ is finite}\}$ and the *Fréchet filter on A* is $\{B \subseteq A : A - B \text{ is finite}\}$. $\mathcal{F}$ denotes the *Fréchet filter* on ω.

For a filter $\mathcal{G}$ on a set X, $\mathcal{G}^*$ is the *dual ideal to $\mathcal{G}$* – that is $\mathcal{G}^* = \{X - a : a \in \mathcal{G}\}$. $\mathcal{G}^+ = \mathcal{P}(X) - \mathcal{G}^*$.

The ideal $\mathcal{I} = (\mathcal{F}^{\otimes 2})^*$ is a $F_{\sigma\delta\sigma}$ ideal, and it is easy to show that it is not $G_{\delta\sigma\delta}$. We will abuse notation and write the quotient $\mathcal{P}(\omega^2)/\mathcal{I}$ as $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$. Forcing with the Boolean algebra $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$ is equivalent to forcing with $(\mathcal{F}^{\otimes 2})^+$ ordered by inclusion. This quotient does not add any new reals (see Lemma 26), and hence adds an ultrafilter on

ω^2. It is not hard to see that this ultrafilter is not a P-point; indeed, the first projection π_1 is neither finite-to-one nor constant on any set in this ultrafilter (or even on any set in $(\mathcal{F}^{\otimes 2})^+$).

In Section 3, we study some combinatorial properties of the generic ultrafilter added by $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$. We determine the extent to which the generic ultrafilter possesses a certain partition property which has been investigated by Galvin and Blass ([1]) before. An ultrafilter $\mathcal{W}$ on a countable set S is said to be (n, h)-*weakly Ramsey*, where n and h are natural numbers, if for every partition of $[S]^n$ into finitely many pieces, there is $W \in \mathcal{W}$ such that $[W]^n$ intersects at most h pieces of the partition. For $n \geq 2$, the property of being $(n, 1)$-weakly Ramsey is equivalent to being selective. Moreover, for each $n \geq 2$, there is a largest natural number $T(n)$ such that, provably in ZFC, if an ultrafilter is $(n, T(n) - 1)$-weakly Ramsey, then it is a P-point. Theorem 31 shows that the generic ultrafilter added by $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$ is $(n, T(n))$-weakly Ramsey. Thus the generic ultrafilter satisfies the strongest partition property which a non-P-point is able to satisfy. Furthermore, the location of the generic ultrafilter in the Rudin-Keisler ordering is fully determined. It is shown that all non-principal ultrafilters that are strictly RK below the generic ultrafilter $\mathcal{G}$ are RK equivalent to $\pi_1(\mathcal{G})$. We also show in Section 3 that the generic ultrafilter is a weak P-point.

In Sections 4 and 5 we prove canonization theorems for monotone maps from the generic ultrafilter to $\mathcal{P}(\omega)$. Let D and E be directed sets. A map $f : E \to D$ is called *monotone* if

$$\forall e_0, e_1 \in E\ [e_0 \leq e_1 \implies f(e_0) \leq f(e_1)]\,.$$

f is said to be *cofinal in* D if $\forall d \in D \exists e \in E\,[d \leq f(e)]$. It is clear that if f is monotone and cofinal in D, then f is convergent. It can be checked that if $\mathcal{U}$ is an ultrafilter and D is any directed set such that $\mathcal{U} \leq_T D$, then there is a map from D to $\mathcal{U}$ which is monotone and cofinal in $\mathcal{U}$. Therefore, understanding monotone maps from the generic ultrafilter to $\mathcal{P}(\omega)$ is necessary to analyze its Tukey type.

The first canonization of monotone maps from an ultrafilter to $\mathcal{P}(\omega)$ was done for the basic ultrafilters by Dobrinen and Todorcevic in [5]. The crucial notion of a basic poset was identified by Solecki and Todorcevic in [15], who showed the importance of this concept for the Tukey theory of definable ideals. Dobrinen and Todorcevic [5] proved that within the category of ultrafilters on ω, the basic posets are precisely the P-points. Their canonization result, showing that monotone maps on P-points are continuous on some filter base, allowed them to prove that there are only continuum many ultrafilters that are Tukey below any fixed P-point. They also introduced a weakened version of the property of being basic, which they called being basically generated.

Definition 4. Let $\mathcal{U}$ be an ultrafilter. A set $\mathcal{B} \subseteq \mathcal{U}$ is said to be a *base* for $\mathcal{U}$ if $\forall a \in \mathcal{U}\, \exists b \in \mathcal{B}\, [b \subseteq a]$.

Definition 5. Let $\mathcal{U}$ be an ultrafilter on a countable set X. We say that $\mathcal{U}$ is *basically generated* if there is a base $\mathcal{B} \subseteq \mathcal{U}$ with the property that for every $\langle b_n : n \in \omega\rangle \subseteq \mathcal{B}$ and $b \in \mathcal{B}$, if $\langle b_n : n \in \omega\rangle$ converges to b, then there exists $X \in [\omega]^{\omega}$ such that $\bigcap_{n\in X} b_n \in \mathcal{U}$. Here convergence is with respect to the natural topology on $\mathcal{P}(X)$ induced by identifying $\mathcal{P}(X)$ with the product space 2^X, with $2 = \{0, 1\}$ having the discrete topology.

It is not hard to see that if $\mathcal{U}$ is basically generated, then $[\omega_1]^{<\omega} \not\leq_T \mathcal{U}$. Dobrinen and Todorcevic [5] pointed out that all the ultrafilters obtained by closing the class of P-points under the operation of taking sums as in Definition 3 are basically generated by a base that is also closed under finite intersections.

In [14] a canonization theorem was proved for monotone maps to $\mathcal{P}(\omega)$ from any ultrafilter that is basically generated by a base that is closed under finite intersections. This canonization result was used in [14] to prove that there are only continuum many ultrafilters that are Tukey below any fixed ultrafilter that is basically generated by a base that is closed under finite intersections. The question of whether there are any ultrafilters that are not above $[\omega_1]^{<\omega}$ and also not basically generated was left open in both [5] and [14].

The canonization results in Sections 4 and 5 imply that the generic ultrafilter added by $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$ has only continuum many ultrafilters Tukey below it. In particular it is not of the maximal possible Tukey type, $[\mathfrak{c}]^{<\omega}$. This is strengthened in Section 4 to prove that the generic ultrafilter is not Tukey above $[\omega_1]^{<\omega}$. The canonization obtained in Theorem 57 is used in Theorem 59 to show that any Tukey reduction from the generic ultrafilter $\mathcal{G}$ to an arbitrary ultrafilter $\mathcal{V}$ can be replaced with a *Rudin-Keisler* reduction from an associated filter $\mathcal{G}(P)$ to $\mathcal{V}$. This is an exact analogue of Theorem 17 of [14], which was established there for all ultrafilters that are basically generated by a base that is closed under finite intersections. These results show that the generic ultrafilter added by $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$ has much in common with basically generated ultrafilters. However, we show in Section 6 that it fails to be basically generated. Thus this generic ultrafilter provides the first known example of an ultrafilter that is not basically generated and is not of the maximal possible Tukey type.

Problem 6. Investigate the Rudin-Keisler and Tukey types of ultrafilters added by forcing with other analytic quotients.

2. Sums of Selective Ultrafilters

Although the primary topic of this paper is the study of generic ultrafilters obtained by forcing with $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$, this study will make much use of certain other ultrafilters on ω^2, namely sums of non-isomorphic selective ultrafilters indexed by another selective ultrafilter. The present section is devoted to the study of such sums.

The results in Sections 2 and 3 are due to Blass. Most of them were presented in a lecture at the Fields Institute in September, 2012, but Theorem 36 was obtained later in 2012.

2.1. Definition and Basic Facts.

Throughout this section, $\mathcal{U}$ and $\mathcal{V}_n$ for $n \in \omega$ are pairwise non-isomorphic selective ultrafilters on ω. Our primary subject here will be the $\mathcal{U}$-indexed sum of the $\mathcal{V}_n$'s, defined above as

$$\mathcal{U}\text{-}\sum_n \mathcal{V}_n = \{X \subseteq \omega^2 : \{n \in \omega : X(n) \in \mathcal{V}_n\} \in \mathcal{U}\}.$$

Recall that $X(n)$ denotes the n^{th} vertical section of X, $\{r : \langle n, r\rangle \in X\}$. So a set X is in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ if and only if its vertical sections $X(n)$ are in the corresponding ultrafilters $\mathcal{V}_n$ for $\mathcal{U}$-almost all n.

Remark 7. The sum $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ would be unchanged if we replaced the ultrafilters $\mathcal{V}_n$ by arbitrary other ultrafilters $\mathcal{V}'_n$ for a set of n's that is not in $\mathcal{U}$. Thus, what we say in this section about pairwise non-isomorphic selective ultrafilters would remain true under the weaker assumption that $\mathcal{U}$ is selective and that $\mathcal{U}$-almost all of the $\mathcal{V}_n$ are selective and not isomorphic to each other (or to $\mathcal{U}$).

The following lemma collects some elementary facts about sums of ultrafilters. The proofs are omitted because they amount to just inspection of the definition.

Lemma 8.
(1) *$\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is an ultrafilter on ω^2.*
(2) *$\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ contains the sets*

$$\{\langle x, y\rangle \in \omega^2 : n < x \text{ and } f(x) < y\}$$

for all $n \in \omega$ and all $f : \omega \to \omega$.
(3) *The projection $\pi_1 : \omega^2 \to \omega$ to the first factor sends $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ to $\mathcal{U}$.*
(4) *The projection π_1 is neither finite-to-one nor constant on any set in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$. Thus $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is not a P-point.*

Part (2) of the lemma implies in particular that $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ contains the above-diagonal set $\{\langle x, y\rangle : x < y\}$, which we can identify with the set $[\omega]^2$ of two-element subsets of ω.

The preceding lemma used only that $\mathcal{U}$ and the $\mathcal{V}_n$'s are non-principal ultrafilters. The next lemma, which describes how π_2 acts, uses our assumptions of selectivity and non-isomorphism. It requires somewhat more work, so we give the proof even though it has long been known.

Lemma 9. *There is a set in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ on which the projection $\pi_2 : \omega^2 \to \omega$ to the second factor is one-to-one. Thus, $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is isomorphic to its π_2-image, which is the limit, with respect to $\mathcal{U}$, of the sequence $\langle \mathcal{V}_n : n \in \omega\rangle$.*

Proof. We prove only the first statement, because the second, identifying the image under π_2 with the limit in $\beta\omega$, is a straightforward verification.

The first assertion will follow if we find pairwise disjoint sets $A_n \subseteq \omega$ with each $A_n \in \mathcal{V}_n$, for then $\{\langle n, y\rangle : n \in \omega \text{ and } y \in A_n\}$ is as required. We obtain such sets A_n in a sequence of three steps.

For each $m \neq n$ in ω, the ultrafilters $\mathcal{V}_m$ and $\mathcal{V}_n$ are not isomorphic, so in particular they are distinct. Thus, there is a set $C_{n,m} \in \mathcal{V}_n$ that is not in $\mathcal{V}_m$.

Because $\mathcal{V}_n$ is a P-point, it contains a set B_n that is almost included in $C_{n,m}$ for all $m \in \omega - \{n\}$. Choose such a B_n for each n. Thus, each $\mathcal{V}_n$ contains the B_n with the same subscript but the complements $\omega - B_m$ of all the other B_m's.

Let
$$A_n = B_n \cap \bigcap_{m<n} (\omega - B_m).$$
Then A_n is the intersection of finitely many sets from $\mathcal{V}_n$, so it is in $\mathcal{V}_n$. For any $m < n$, A_m and A_n are clearly disjoint, so the sets A_n are as required. $\square$

An amplification of this argument gives the following complete classification, modulo $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$, of all functions $f : \omega^2 \to \omega$.

Proposition 10. *If $f : \omega^2 \to \omega$, then there is a set $X \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ such that $f \upharpoonright X$ is one of the following.*

- *a constant function*
- *π_1 followed by a one-to-one function*
- *a one-to-one function*

Proof. Consider first, for each $n \in \omega$, the n^{th} section of f, that is, the function $f_n : \omega \to \omega : y \to f(n, y)$. Since $\mathcal{V}_n$ is selective, f_n is constant or one-to-one on some set $A_n \in \mathcal{V}_n$. Furthermore, as $\mathcal{U}$ is an ultrafilter, it contains a set B such that either f_n is constant on A_n for all $n \in B$ or f_n is one-to-one on A_n for all $n \in B$.

We treat first the case where f_n is constant on A_n for all $n \in B$. Then, on the set

$$Y = \{\langle n, y\rangle : n \in B \text{ and } y \in A_n\},$$

which is in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$, our function f factors through π_1; $f(x, y) = g(x) = g(\pi_1(x, y))$, where $g(n)$ is defined as the constant value taken by f_n on A_n.

As $\mathcal{U}$ is selective, g is either constant or one-to-one on a set $C \in \mathcal{U}$. If g is constant on C then f is constant on $X = Y \cap {\pi_1}^{-1}(C)$, and we have the first alternative in the proposition. If, on the other hand, g is one-to-one on C, then the same X gives us the second alternative in the proposition.

It remains to treat the case where f_n is one-to-one on A_n for all $n \in B$. In what follows, n is intended to range only over B. Then $f_n(\mathcal{V}_n)$ is an ultrafilter isomorphic to $\mathcal{V}_n$, so, as n varies over B, these are pairwise non-isomorphic selective ultrafilters.

Arguing as in the proof of Lemma 9, we find pairwise disjoint sets $Z_n \in f_n(\mathcal{V}_n)$ for all $n \in B$. We claim that f is one-to-one on

$$X = \{\langle n, y\rangle : n \in B,\ y \in A_n, \text{ and } f_n(y) \in Z_n\}.$$

Indeed, if we had $\langle n, y\rangle, \langle n', y'\rangle \in X$ and $f(n, y) = f(n', y')$, i.e., $f_n(y) = f_{n'}(y')$, then this would be an element of $Z_n \cap Z_{n'}$, so disjointness requires $n = n'$. Furthermore, as f_n is one-to-one on A_n, we would have $y = y'$. This completes the proof of the claim that f is one-to-one on X. Also, since $\mathcal{U}$ contains B and since $\mathcal{V}_n$ contains both A_n and ${f_n}^{-1}(Z_n)$ for all $n \in B$, we have $X \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$, and so we have the third alternative in the proposition. □

Corollary 11. *$\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is a Q-point.*

Proof. If $f : \omega^2 \to \omega$ is finite-to-one, then the first two alternatives in the proposition are impossible (in view of part (4) of Lemma 8), and the only remaining alternative is that f is one-to-one on a set in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$. □

Corollary 12. *The only non-principal ultrafilters strictly below $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ in the Rudin-Keisler ordering are the isomorphic copies of $\mathcal{U}$.*

2.2. **Weak Partition Properties and P-points.** The goal of this subsection and the next is to show that $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ satisfies the strongest "square-bracket" partition properties that are possible for a non-P-point. In the present subsection, we explain square-bracket partition relations for ultrafilters, and we show that some of these relations require the ultrafilter to be a P-point. In the next subsection, we shall show that the strongest relations not covered by this result are satisfied by $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$.

Definition 13. Let n and h be natural numbers, let S be a countable set (for our purposes S is usually ω or ω^2), and let $\mathcal{W}$ be an ultrafilter on S. Then $\mathcal{W}$ is (n, h)-*weakly Ramsey* if, for every partition of $[S]^n$ into finitely many pieces, there is a set $W \in \mathcal{W}$ such that $[W]^n$ meets at most h of the pieces.

Remark 14. It would make no difference in this definition if, instead of partitions into an arbitrary finite number of pieces, we referred only to partitions into $h + 1$ pieces, the first non-trivial case. The remaining cases would then follow by a routine induction on the number of pieces.

What we have defined as (n, h)-weakly Ramsey is often expressed in the partition calculus notation as

$$S \to [\mathcal{W}]^n_{h+1},$$

meaning that, if $[S]^n$ is partitioned into $h + 1$ pieces then $\mathcal{W}$ contains a set W such that $[W]^n$ misses a piece. The square brackets around $\mathcal{W}$ in this notation are used to indicate that we have only weak homogeneity, missing one piece. Round brackets conventionally denote full homogeneity, namely meeting only one piece.

When considering (n, h)-weak Ramseyness, we always assume $n \geq 2$ and $h \geq 1$ to avoid trivialities.

Clearly, if we increase h while keeping n fixed, the property of (n, h)-weak Ramseyness becomes weaker.

As mentioned in Section 1, for each $n \geq 2$, the property of being $(n, 1)$-weakly Ramsey is equivalent to selectivity. We intend to show next that, as we increase h to values greater than 2, (n, h)-weak Ramseyness continues to imply that $\mathcal{W}$ is a P-point, until h reaches a certain critical value $T(n)$, which we shall compute.

To show that, for certain n and h, all (n, h)-weakly Ramsey ultrafilters are P-points, we shall consider ultrafilters $\mathcal{W}$ that are not P-points, and we shall describe $h + 1$ different types of n-element sets that must occur inside each $W \in \mathcal{W}$. It will be convenient to describe these types first in an abstract, formal way, and only afterward to connect them with n-element sets.

The following definition describes the abstract types.

Definition 15. An n-type is a list of $2n$ variables, namely x_i and y_i for $1 \leq i \leq n$, with either $<$ or $=$ between each consecutive pair in the list, such that

- the y_i's occur in the list in increasing order of their subscripts,
- equality signs can occur only between two x_i's, and
- each x_i precedes the y_i that has the same subscript.

Two such lists are considered the same if they differ only by permuting x_i's that are connected by equality signs.

Example 16. A fairly typical 7-type is

$$x_3 = x_1 < x_5 < y_1 < x_2 = x_7 < y_2 < y_3 < x_4 < y_4 < x_6 < y_5 < y_6 < y_7.$$

Interchanging x_3 with x_1 or interchanging x_2 with x_7 or both would result in a new representation of the same 7-type. Any other rearrangements would result in a different 7-type (or in a list that fails to be a 7-type).

Remark 17. An equivalent definition of n-type is as a linear pre-ordering of the set of $2n$ x_i's and y_i's subject to the requirements that its restriction to the y_i's is the strict ordering according to subscripts, that any equivalence class with more than one element must consist entirely of x_i's, and that each x_i strictly precedes the corresponding y_i.

The number of n-types is the critical number $T(n)$ mentioned above, the border between those (n, h)-weak Ramsey properties that imply P-point and those that do not. We shall later describe a more effective means to compute $T(n)$, but first we develop the connection between types and Ramsey properties of ultrafilters.

Definition 18. Let $n \geq 2$, let τ be an n-type, let $f : \omega \to \omega$, and let $\mathbf{a} = \{a_1 < a_2 < \cdots < a_n\}$ be an n-element subset of ω with its elements listed in increasing order. We say that $\mathbf{a}$ *realizes* τ with respect to f and that τ is the *f-type of* $\mathbf{a}$ if the equations and inequalities in τ become true when every y_i is interpreted as a_i and every x_i is interpreted as $f(a_i)$.

Remark 19. Because of the restrictions in the definition of n-types, there can be n-element sets $\mathbf{a}$ that have no f-type. This could happen if $a_i = f(a_j)$ for some i and j, because n-types cannot say that $y_i = x_j$. It could also happen if $f(a_i) \geq a_i$ for some i, because x_i must strictly precede y_i in any n-type.

The following proposition is, as far as we know, due to Galvin but never published. A brief indication of the proof is given, along with an explicit statement of the result for $n = 24$ attributed to Galvin, on page 85 of [1]. Since that brief indication used ultrapowers, we take this opportunity to give a complete and purely combinatorial proof.

Proposition 20. *Let $\mathcal{W}$ be a non-principal ultrafilter on ω that is not a P-point, and let $f : \omega \to \omega$ be a function that is neither finite-to-one nor constant on any set in $\mathcal{W}$. For any natural number n, every set $W \in \mathcal{W}$ includes n-element subsets realizing with respect to f all n-types.*

Proof. Fix $\mathcal{W}$, W, f, and n as in the proposition, and fix an n-type τ. Let

$$B = \{v \in \omega : W \cap f^{-1}(\{v\}) \text{ is infinite}\}$$

and note that $f^{-1}(B) \in \mathcal{W}$; indeed, f is finite-to-one on the intersection of W with the complement of $f^{-1}(B)$, so this complement cannot be in $\mathcal{W}$. Note also that B is infinite, for otherwise f would take only finitely many values on $W \cap f^{-1}(B)$ and would therefore be constant on a set in $\mathcal{W}$.

Now we shall produce a set $\mathbf{a} \subseteq W$ realizing the given n-type τ. We go through the list τ of variables x_i and y_i in the order given by τ, assigning a value to each variable in turn. At each step, we proceed as follows, assuming we have just reached a certain variable and have assigned values to the earlier variables in τ.

- If we have reached x_i and there is an equality symbol immediately before it in τ, say $x_j = x_i$, then we assign to x_i the same value that was already assigned to x_j.
- If we have reached x_i and there is no equality symbol immediately before it (either because there is $<$ there or because x_i is first in the list τ), then we assign to x_i a value in B that is larger than all of the (finitely many) values already assigned to other variables.
- If we have reached y_i, then x_i has already been assigned a value v; we assign to y_i a value a_i in $W \cap f^{-1}(\{v\})$ that is larger than all of the values already assigned to other variables.

In the second and third cases, where a new, large value must be chosen, we use the fact that B and $W \cap f^{-1}(\{v\})$ (for $v \in B$) are infinite, so sufficiently large values are available. The values a_i assigned to the y_i's are all in W, and they are in increasing order. They were chosen so that their images $f(a_i)$ are the values assigned to the corresponding x_i's. Using these observations and the fact that, whenever a new value was chosen, it was larger than all previously chosen values, it is easy to see that $\mathbf{a} = \{a_1 < a_2 < \cdots < a_n\}$ realizes τ, as required. □

Definition 21. Let $T(n)$ denote the number of n-types.

Corollary 22. *Every $(n, T(n) - 1)$-Ramsey ultrafilter is a P-point.*

Proof. Suppose $\mathcal{W}$ were a counterexample. Let $f : \omega \to \omega$ witness that $\mathcal{W}$ is not a P-point. Partition the n-element subsets of ω into $T(n)$ pieces C_τ, one for each n-type τ, by putting into C_τ all those n-element sets that realize τ with respect to f. Sets that realize no type with respect to f can be thrown into any of the pieces. By the proposition, every set in $\mathcal{W}$ has n-element subsets in all of the pieces. So this partition is a counterexample to $(n, T(n)-1)$-weak Ramseyness. □

In the next subsection, we shall show that this corollary is optimal; a non-P-point can be $(n, T(n))$-weakly Ramsey. It is therefore of some interest to understand $T(n)$, and we devote the rest of this subsection to this finite combinatorial topic.

There is a simple recurrence relation, not for $T(n)$ but for a closely related and more informative two-variable function $T(n,k)$ defined as the number of n-types τ that have exactly k equivalence classes of x_i's. Here two x_i's are called equivalent[2] if, between them in τ, there are only $=$ signs, not $<$ signs. (It follows, by the definition of types, that the only variables occurring between these two x's are other x's, all in the same equivalence class.) In general, k can be as small as 1 (in just one n-type, namely $x_1 = \cdots = x_n < y_1 < \cdots < y_n$) and as large as n (if there are no $=$ signs in τ). Clearly,

$$T(n) = \sum_{k=1}^{n} T(n,k).$$

We have the recursion relation

$$T(n,k) = kT(n-1,k) + (n+k-1)T(n-1,k-1)$$

for $n \geq k \geq 1$. To see this, consider an arbitrary n-type τ with k equivalence classes of x's. Let σ be the induced $n-1$-type. That is, obtain σ by deleting x_n and y_n from τ and combining the $=$ and $<$ signs around x_n in the obvious way. (If both are $=$, use $=$, and otherwise use $<$. Note that no combining is needed around y_n since it is at the end of the list τ. Also, x_n could be at the beginning of τ, in which case it too would need no combining.) The number of equivalence classes in σ is either k or $k-1$, depending on whether x_n was, in τ, equivalent to another x_i or not. Each of the $T(n-1,k)$ possible σ's of the first sort arises from k possible τ's, because x_n could have been in any of the k equivalence classes. This accounts for the first term on the right side of our recurrence. Each of the $T(n-1,k-1)$ possible σ's of the second sort arises from $n+k-1$ possible τ's, because x_n can be put into

[2]This is the same as the equivalence relation arising from the pre-order in Remark 17.

any of the intervals determined by the $n-1$ y's and $k-1$ equivalence classes of x's in σ. Here the possible intervals include the degenerate "intervals" at the left and right ends of σ, so the number of intervals is $n+k-1$. This accounts for the second term on the right side and thus completes the proof of our recurrence relation.

There are other combinatorial interpretations of $T(n)$. For example, it is the number of rooted trees with $n+1$ labeled leaves, subject to the requirement that every non-leaf vertex must have at least two children, i.e., there is genuine branching at each internal node. The number of internal nodes can be as small as 1 (if all the leaves are children of the root) and as large as n (if all branching is binary). $T(n,k)$ counts these trees according to the number k of internal nodes.

For more information, tables of values, and references, see [18], where $T(n)$ is (up to a shift of the indexing) sequence A000311 and $T(n,k)$ is (rearranged into a single sequence) A134991.

2.3. **Weak Partition Properties and Sums of Selective Ultrafilters.** The goal of this subsection is to prove that ultrafilters of the form $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ (still subject to this section's convention that $\mathcal{U}$ and the $\mathcal{V}_n$ are pairwise non-isomorphic selective ultrafilters) have the strongest weak-Ramsey properties that Corollary 22 permits for a non-P-point.

Theorem 23. *$\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is $(n, T(n))$-weakly Ramsey.*

Proof. We shall need to refer to the types of n-element subsets of the set ω^2 underlying our ultrafilter $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$. Types were defined above for n-element subsets of ω, not ω^2. In principle, this is no problem, because we know, from Lemma 9, that π_2 is an isomorphism from $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ to an ultrafilter $\mathcal{W}$ on ω, so we can just transfer, via this isomorphism, any concepts and facts that we need. In practice, though, the need to repeatedly apply this isomorphism makes statements unpleasantly complicated, so we begin by reformulating the relevant facts about types in a way that lets us use $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ directly.

Our discussion of types above was relative to a fixed function f witnessing that the ultrafilter is not a P-point. We know, from Lemma 8, that π_1 is such a function for $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$; therefore $f = \pi_1 \circ {\pi_2}^{-1}$ is such a function for $\mathcal{W} = \pi_2(\mathcal{U}\text{-}\sum_n \mathcal{V}_n)$. Notice, in this connection, that although π_2 is not globally one-to-one, it has a one-to-one restriction to a suitable set in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ (Lemma 9); enlarging this set slightly if necessary, we get a set $G \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ on which π_2 is bijective. By ${\pi_2}^{-1} : \omega \to G$ we mean the inverse of this bijection.

Now any n-element subset $\mathbf{a}$ of G corresponds to an n-element subset $\mathbf{b} = \pi_2(\mathbf{a})$ of ω. With $f = \pi_1 \circ {\pi_2}^{-1}$ as above, the f-type of $\mathbf{b}$ will be called simply the *type* of $\mathbf{a}$. Unraveling the definitions, we find that this type can be described as follows, directly in terms of $\mathbf{a}$. Enumerate the elements of $\mathbf{a}$ as

$$\langle p_1, q_1\rangle, \langle p_2, q_2\rangle, \dots, \langle p_n, q_n\rangle$$

in order of increasing second components, so $q_1 < q_2 < \cdots < q_n$. Then $\mathbf{a}$ realizes the n-type τ if and only if all the equations and inequalities in τ become true when the variables x_i are interpreted as p_i and the y_i are interpreted as q_i.

Let us use, for points in ω^2, the familiar terminology "x-coordinate" and "y-coordinate" for the first and second components. Then, the x and y variables in a type represent the x and y coordinates of the points of a set realizing the type.

Transferring Proposition 20 and its corollaries via the isomorphism π_2, we learn that every set in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ contains n-element subsets realizing all $T(n)$ of the n-types. The proof of the theorem will consist of showing that this is all one can say in the direction of existence of many different kinds of n-element subsets in all sets from $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$.

Consider an arbitrary partition Π of the set $[\omega^2]^n$ of n-element subsets of ω^2 into a finite number z of pieces. We shall show that there is a set $W \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ such that

- every n-element subset of W realizes an n-type, and
- any two subsets realizing the same n-type are in the same piece of the partition Π.

This will clearly suffice to prove the theorem.

Furthermore, we can treat the various types separately. That is, it suffices to find

- a set $X \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ such that all n-element subsets of X realize n-types and
- for each n-type τ, a set $Y_\tau \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ such that all n-element subsets of Y_τ that realize τ lie in the same piece of the partition Π.

Indeed, the intersection of X and all $T(n)$ of the sets Y_τ's will then be a set W as required above.

Let us first produce the required X all of whose n-element subsets realize types. Inspecting the descriptions of types and realization, we find that X needs to have the following properties, in addition to being in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$.

(1) No two distinct elements of X have the same y-coordinate (so that the y_i can all be properly ordered in the type).

(2) Each element of X has its x-coordinate smaller than its y-coordinate (so that each x_i can precede y_i in the type).
(3) No x-coordinate of a point in X equals the y-coordinate of another point in X (so that the type has no equalities between any x_i and y_j).

Furthermore, these three requirements can be treated independently; if we find a different $X \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ for each one, we can just intersect those three X's to complete the job. For the first two requirements, we already have the necessary X's. The first says that π_2 is one-to-one on X, so Lemma 9 gives what we need. The second is satisfied by $\{\langle x, y\rangle \in \omega^2 : x < y\}$ which we noted, right after Lemma 8, is in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$. We therefore concentrate on the third requirement.

For this requirement, it suffices to find a set $B \in \mathcal{U}$ that is in none of the $\mathcal{V}_n$, for then $B \times (\omega - B)$ serves as the required X. Since $\mathcal{U}$ is distinct from all the $\mathcal{V}_n$, we can find, for each n, a set $B_n \in \mathcal{U} - \mathcal{V}_n$. Then, since $\mathcal{U}$ is a P-point, we can find a single $B \in \mathcal{U}$ almost included in each B_n. Since each $\mathcal{V}_n$ is non-principal and doesn't contain B_n, it cannot contain B either.

This completes the proof that $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ contains a set X all of whose n-element subsets represent types. It remains to consider an arbitrary type τ and find a set $Y_\tau \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ all of whose n-element subsets of type τ are in the same piece of our partition Π.

Fix, therefore, a particular τ for the remainder of the proof. It suffices to find an appropriate set Y_τ in the case that Π is a partition into only two pieces. The general case where Π has any finite number z of pieces then follows. Just consider all the 2-piece partitions coarser than Π, find an appropriate Y for each of these, and intersect these finitely many Y's. So from now on, in addition to working with a fixed τ, we work with a fixed partition $\Pi = \{R, [\omega^2]^n - R\}$ of $[\omega^2]^n$ into two pieces. Our goal is to find a set $Y \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ such that either all n-element subsets of Y realizing τ are in R or none of them are.

Fortunately, a stronger result than this was already proved as Theorem 7 in [2, page 236]. We shall quote that result and then indicate how it implies what we need here.

Proposition 24 ([2], Theorem 7)**.** *Let there be given selective ultrafilters $\mathcal{W}_s$ on ω for all finite subsets s of ω. Assume that, for all $s, t \in [\omega]^{<\omega}$, the ultrafilters $\mathcal{W}_s$ and $\mathcal{W}_t$ are either equal or not isomorphic. Let there also be given an analytic subset $\mathcal{X}$ of the set $[\omega]^\omega$ of infinite subsets of ω. Then there is a function Z assigning, to each ultrafilter $\mathcal{W}$ that occurs among the $\mathcal{W}_s$'s, some element $Z(\mathcal{W}) \in \mathcal{W}$ such* *that $\mathcal{X}$ contains all or none of the infinite subsets* $\{z_0 < z_1 < z_2 < \dots\}$ *that satisfy* $z_n \in Z(\mathcal{W}_{\{z_0,\dots,z_{n-1}\}})$ *for all* $n \in \omega$.

It is important here that Z assigns sets $Z(\mathcal{W})$ to ultrafilters $\mathcal{W}$, not to their occurrences in the system $\langle \mathcal{W}_s : s \in [\omega]^{<\omega}\rangle$. That is, if the same ultrafilter $\mathcal{W}$ occurs as $\mathcal{W}_s$ for several sets s, then the same $Z(\mathcal{W})$ is used for all these occurrences of $\mathcal{W}$.

There is an essentially unique reasonable way to regard our partition Π of $[\omega^2]^n$, restricted to sets of type τ, as a clopen (and therefore analytic) partition of $[\omega]^\omega$ and to choose ultrafilters $\mathcal{W}_s$ among our $\mathcal{U}$ and $\mathcal{V}_n$'s, so that Proposition 24 completes the proof of Theorem 23. We spell this out explicitly in what follows.

For our fixed type τ, we say that a variable x_i or y_i is in *position* α if it is preceded in τ by exactly α occurrences of $<$. The values of α that can occur here range from 0 to $n+k-1$, where k is the number of equivalence classes of x_i's as in our discussion of the recurrence for $T(n,k)$ in subsection 2.2. Notice that the variables in any particular position are either an equivalence class of x's or a single y.

Given any finite or infinite increasing sequence $\vec{z}$ of natural numbers, say $z_0 < z_1 < \cdots < z_l\,(< \dots)$, we associate to it an assignment of values to the variables in some initial segment of τ by giving the value z_α to the variable(s) in position α. If the sequence $\vec{z}$ is longer than $n+k$ then the terms from z_{n+k} on have no effect on this assignment. If $\vec{z}$ is shorter than $n+k$ then not all of the variables in τ receive values.

For an infinite increasing sequence $\vec{z}$ (and indeed also for finite $\vec{z}$ of length $\geq n+k$), we obtain in this way values, say p_i, for all the x_i's and values, say q_i, for all the y_i's ($1 \leq i \leq n$), and we combine these values to form an n-element subset of ω^2:

$$\mathbf{a}(\vec{z}) = \{\langle p_1, q_1\rangle, \langle p_2, q_2\rangle, \dots, \langle p_n, q_n\rangle\}.$$

Note that the pairs $\langle p_i, q_i\rangle$ occur in this list in order of increasing y-coordinates (because τ is a type and $\vec{z}$ is increasing) and that $\mathbf{a}(\vec{z})$ has type τ. We define $\mathcal{X}$ to consist of those infinite subsets of ω whose increasing enumeration $\vec{z}$ has $\mathbf{a}(\vec{z})$ in the piece R of our partition Π. Thus, membership of a set A in $\mathcal{X}$ depends only on the first $n+k$ elements of A. In particular, $\mathcal{X}$ is clopen, and therefore certainly analytic, in $[\omega]^\omega$.

To finish preparing an application of Proposition 24, we define ultrafilters $\mathcal{W}_s$ for all finite $s \subseteq \omega$ as follows. Let $\vec{z}$ be the increasing enumeration of s, and use it as above to assign values to the variables in the positions 0 through $\min\{|s|, n+k\} - 1$ in τ.

(1) If $|s| < n+k$ and the variables in position $|s|$ (the first variables not assigned values) are an equivalence class of x_i's, then let $\mathcal{W}_s = \mathcal{U}$.
(2) If $|s| < n + k$ and the variable in position $|s|$ is y_i, then x_i, occurring earlier in τ, has been assigned a value v. Let $\mathcal{W}_s = \mathcal{V}_v$.
(3) If $|s| \geq n + k$ then set $\mathcal{W}_s = \mathcal{U}$. (This case is unimportant; we could use any selective ultrafilters here as long as we satisfy the "equal or not isomorphic" requirement in Proposition 24.)

Since our $\mathcal{X}$ is analytic and our $\mathcal{W}_s$ are selective ultrafilters every two of which are either equal or not isomorphic, Proposition 24 applies and provides us with sets $Z(\mathcal{U})$ and $Z(\mathcal{V}_r)$ for all $r \in \omega$ such that $\mathcal{X}$ contains all or none of the infinite sets whose increasing enumerations $\vec{z} = \langle z_0 < z_1 < \dots \rangle$ have $z_\alpha \in Z(\mathcal{W}_{\{z_0,\dots,z_{\alpha-1}\}})$ for all α. It remains to untangle the definitions and see what this homogeneity property actually means. Since membership of an infinite set in $\mathcal{X}$ depends only on that set's first $n + k$ members, the homogeneity property is really not about infinite sets but about $(n + k)$-element sets and their increasing enumerations $\vec{z}$. We look separately at the assumption and the conclusion in the homogeneity statement of Proposition 24.

The assumption is that $z_\alpha \in Z(\mathcal{W}_{\{z_0,\dots,z_{\alpha-1}\}})$, which means that, when we use $\vec{z}$ to give values to the variables in τ, all the values given to the x_i's are in $Z(\mathcal{U})$ and the value given to any y_i is in $Z(\mathcal{V}_v)$, where v is the value given to the corresponding x_i. This means exactly that $\mathbf{a}(\vec{z})$ has all its elements in the set

$$Y = \{\langle p, q\rangle : p \in Z(\mathcal{U}) \text{ and } q \in Z(\mathcal{V}_p)\}.$$

Conversely, any n-element subset of type τ in this Y, listed in increasing order of y-coordinates, is $\mathbf{a}(\vec{z})$ for an enumeration $\vec{z}$ satisfying the assumption in Proposition 24.

Because each $Z(\mathcal{W})$ is in the corresponding ultrafilter $\mathcal{W}$, we have $Y \in \mathcal{U}\text{-}\sum_r \mathcal{V}_r$.

The conclusion in the homogeneity statement is that all or none of the sets satisfying the hypothesis are in $\mathcal{X}$. This means, in view of our choice of $\mathcal{X}$, that all or none of the associated $\mathbf{a}(\vec{z})$ are in R.

Collecting all this information, we have that R contains all or none of the n-element subsets of type τ in Y. Thus, Y is as required to complete the proof of Theorem 23. □

Corollary 25. *Let $X \in (\mathcal{F}^{\otimes 2})^+$, and let $[X]^n$ be partitioned into finitely many pieces. Then there is $Y \subseteq X$ such that $Y \in (\mathcal{F}^{\otimes 2})^+$ and*

- *every n-element subset of Y realizes an n-type and*

- *for each n-type τ, all the n-element subsets of Y that realize it are in the same piece of the given partition.*

In particular, $[Y]^n$ meets at most $T(n)$ pieces of the given partition.

Proof. Suppose for a moment that the continuum hypothesis holds, so that there are $2^{\mathfrak{c}}$ non-isomorphic selective ultrafilters containing any given infinite subset of ω. Let $B = \{n \in \omega : X(n) \text{ is infinite}\}$. The assumption that $X \in (\mathcal{F}^{\otimes 2})^+$ means that B is infinite. Choose non-isomorphic selective ultrafilters $\mathcal{U}$ and $\mathcal{V}_n$ for all $n \in \omega$ so that $B \in \mathcal{U}$ and $X(n) \in \mathcal{V}_n$ for all $n \in B$. Then $X \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$. Extend the given partition arbitrarily to all of $[\omega^2]^n$ and invoke the proof of Theorem 23 to get a set $Y \in \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ in which every n-element subset realizes a type and different subsets realizing the same type are in the same piece of our partition. Intersecting Y with X, we get a set, still in $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ and thus in $(\mathcal{F}^{\otimes 2})^+$, with the required homogeneity property for the given partition of $[X]^n$. This completes the proof under the assumption of the continuum hypothesis.

There are (at least) two ways to show that the result remains true in the absence of the continuum hypothesis. One way is to pass to a forcing extension that has no new reals but satisfies the continuum hypothesis. The corollary holds in the extension, but it is only about reals, so it must have held in the ground model. The other way is note that the corollary is a Π^1_2 statement, so, by Shoenfield's absoluteness theorem, it is absolute between the whole universe V and the constructible sub-universe L. Since L satisfies the continuum hypothesis, the corollary holds there and thus also holds in V. □

3. Ultrafilters Generically Extending $\mathcal{F} \otimes \mathcal{F}$

We turn now to the main subject of this paper, ultrafilters on ω^2 produced by forcing with $\mathbb{P} = (\mathcal{F}^{\otimes 2})^+$ partially ordered by inclusion. Recall that we write $\mathcal{F}$ for the Fréchet filter, the set of cofinite subsets of ω. Its Fubini square $\mathcal{F}^{\otimes 2}$ is the filter on ω^2 consisting of sets X such that the sections $X(n)$ are cofinite for cofinitely many n. A convenient basis for $\mathcal{F}^{\otimes 2}$ is the family of "wedges"

$$\{\langle x, y\rangle \in \omega^2 : x > n \text{ and } y > f(x)\}.$$

where n ranges over natural numbers and f ranges over functions $\omega \to \omega$.

Our forcing conditions in $\mathbb{P}$ are the sets of positive measure with respect to $\mathcal{F}^{\otimes 2}$; these are the sets X whose sections $X(n)$ are infinite for infinitely many n.

The partially ordered set $\mathbb{P}$ is not separative. Its separative quotient is obtained by identifying any two conditions that have the same intersection with a set in $\mathcal{F}^{\otimes 2}$. So the separative quotient is the Boolean algebra $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$.

We call a forcing condition $X \in \mathbb{P}$ *standard* if every nonempty section $X(n)$ is infinite. Every condition can be shrunk to a standard one by simply deleting its finite sections; the shrunk condition is equivalent, in the separative quotient, to the original condition. We can therefore assume, without loss of generality, that we always work with standard conditions.

Lemma 26. *The separative quotient of* $\mathbb{P}$ *is countably closed. In particular, forcing with* $\mathbb{P}$ *adds no new reals.*

Proof. The second assertion of the lemma is a well-known consequence of the first, so we just verify the first. Given a decreasing ω-sequence in $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2} - \{0\}$, we can choose representatives in $\mathbb{P}$, say $A_0 \supseteq A_1 \supseteq \dots$, where we have arranged actual inclusion (rather than inclusion modulo $\mathcal{F}^{\otimes 2}$) by intersecting each set in the sequence with all its predecessors. Each A_n, being in $\mathbb{P}$, has infinitely many infinite sections, so we can choose numbers $x_0 < x_1 < \dots$ such that $A_n(x_n)$ is infinite for every n. Let $B = \{\langle x_n, y\rangle : n \in \omega \text{ and } y \in A_n\}$. Then $B \in \mathbb{P}$ and B is included, modulo $\mathcal{F}^{\otimes 2}$, in each A_n. (In fact, it's included modulo $\mathcal{F} \otimes \{\omega\}$.) □

In what follows, we work with a ground model V and its forcing extensions. Depending on the reader's preferences, V can be the whole universe of sets, in which case its forcing extensions are Boolean valued models; or V can be a countable transitive model of ZFC, in which case forcing extensions are also countable transitive models; or V can be an arbitrary model of ZFC, in which case its forcing extensions are again merely models of ZFC. What we do below is valid in any of these contexts. Unless the contrary is specified, "generic" means generic over V.

Corollary 27. *Any generic filter* $\mathcal{G} \subseteq \mathbb{P}$ *is, in the forcing extension* $V[\mathcal{G}]$*, an ultrafilter on* ω^2*.*

Proof. Since every subset X of ω in $V[\mathcal{G}]$ is in V, it suffices to notice that, for each such X, the set $\{A \in \mathbb{P} : A \subseteq X \text{ or } A \subseteq \omega - X\}$ is in V and dense in $\mathbb{P}$, so it meets $\mathcal{G}$. □

In view of the corollary, we have written $\mathcal{G}$ for the generic filter, rather than the more customary G, since we have been using script letters for ultrafilters.

The rest of this section is devoted to establishing the basic combinatorial properties of $\mathbb{P}$-generic ultrafilters $\mathcal{G}$. Many of these properties are the same as what we established in the preceding section for selective-indexed sums of non-isomorphic selective ultrafilters. In particular, $\mathcal{G}$ is not a P-point, and it satisfies the strongest weak-Ramsey property compatible with not being a P-point. But $\mathcal{G}$ also differs in an essential way from the sums considered earlier, in that it is not a limit of countably many other non-principal ultrafilters.

3.1. **Basic Properties.** In this subsection, we collect some of the basic facts about $\mathbb{P}$-generic ultrafilters $\mathcal{G}$ on ω^2. From now on, $\mathcal{G}$ will always denote such an ultrafilter and $\dot{\mathcal{G}}$ its canonical name in the forcing language.

Lemma 28. *For $A \in \mathbb{P}$ and $X \subseteq \omega^2$, we have $A \Vdash X \in \dot{\mathcal{G}}$ if and only if $A \subseteq X$ modulo $\mathcal{F}^{\otimes 2}$. The ultrafilter $\mathcal{G}$ is an extension of $\mathcal{F}^{\otimes 2}$.*

Proof. The first assertion of the lemma is a consequence of the well-known facts that forcing is unchanged when a notion of forcing is replaced by its separative quotient and that, for a separative notion of forcing, p forces q to belong to the canonical generic filter if and only if $p \leq q$.

It follows from the first assertion that $\mathcal{G} \subseteq (\mathcal{F}^{\otimes 2})^+$. Since $\mathcal{G}$ is an ultrafilter, this is equivalent to $\mathcal{G} \supseteq \mathcal{F}^{\otimes 2}$. □

Corollary 29. *The first projection $\pi_1 : \omega \to \omega$ is not finite-to-one or constant on any set in $\mathcal{G}$. Thus, $\mathcal{G}$ is not a P-point.*

Proof. π_1 is not finite-to-one or constant on any set in $(\mathcal{F}^{\otimes 2})^+$. □

Proposition 30. *The image $\mathcal{U} = \pi_1(\mathcal{G})$ of $\mathcal{G}$ under the first projection is V-generic for $\mathcal{P}(\omega)/\mathcal{F}$. In particular, it is a selective ultrafilter on ω.*

Proof. Consider the function $c : [\omega]^\omega \to \mathbb{P}$ that sends each infinite $X \subseteq \omega$ to the cylinder over it in ω^2, i.e., $c(X) = {\pi_1}^{-1}(X)$. We claim that this function is a complete embedding of the forcing notion $([\omega]^\omega, \subseteq)$ into $\mathbb{P}$. In the first place, c clearly preserves the ordering relation $\subseteq$. It also preserves incompatibility; if X and Y are incompatible in $[\omega]^\omega$, i.e., if their intersection is finite, then the intersection of $c(X)$ and $c(Y)$ is not in $(\mathcal{F}^{\otimes 2})^+$, so $c(X)$ and $c(Y)$ are also incompatible. Finally, if $\mathcal{A}$ is a maximal antichain in $[\omega]^\omega$, then the antichain $\{c(A) : A \in \mathcal{A}\}$ is maximal in $\mathbb{P}$. To see this, consider an arbitrary $X \in \mathbb{P}$. We intend to show that it is compatible with $c(A)$ for some $A \in \mathcal{A}$. Let $B = \{n \in \omega : X(n) \text{ is infinite}\}$. Then B is infinite. By maximality of $\mathcal{A}$, find

$A \in \mathcal{A}$ such that $A \cap B$ is infinite. Then $X \cap c(A \cap B)$ is in $\mathbb{P}$ and is an extension of both X and $c(A)$, as required. This completes the proof that c is a complete embedding.

By a well-known general fact about complete embeddings, it follows that $c^{-1}(\mathcal{G})$ is a V-generic subset of $[\omega]^\omega$. But $c^{-1}(\mathcal{G})$ is exactly $\pi_1(\mathcal{G})$, so the first assertion of the lemma is proved. The second follows, as it it well-known that the generic object adjoined by forcing with $[\omega]^\omega$ is a selective ultrafilter on ω. □

3.2. **Partition Property and Consequences.**

Theorem 31. *The generic ultrafilter $\mathcal{G}$ is $(n, T(n))$-weakly Ramsey for every n. In more detail, for every partition Π of $[\omega^2]^n$ into finitely many pieces, there is a set $H \in \mathcal{G}$ such that*

- *every n-element subset of H realizes an n-type, and*
- *for each n-type τ, all the n-element subsets of H that realize τ are in the same piece of Π.*

Proof. The first assertion of the the theorem follows from the second because there are only $T(n)$ n-types. The second assertion, in turn, follows immediately from Corollary 25, which says that conditions H with the desired properties are dense in $\mathbb{P}$, so the generic $\mathcal{G}$ must contain such an H. □

Note that the $T(n)$ in the theorem is optimal, because of Corollary 22 and the fact that $\mathcal{G}$ is not a P-point.

The next two corollaries could be proved by direct density arguments, but it seems worthwhile to point out how they follow from the partition properties in Theorem 31.

Corollary 32. *The second projection π_2 is one-to-one on a set in $\mathcal{G}$.*

Proof. By Theorem 31, let $H \in \mathcal{G}$ be a set all of whose 2-element subsets realize types. Then π_2 is one-to-one on H because no 2-type can have $y_1 = y_2$. □

Corollary 33. *If $f : \omega^2 \to \omega$, then there is a set $X \in \mathcal{G}$ such that $f \restriction X$ is one of the following.*

- *a constant function*
- *π_1 followed by a one-to-one function*
- *a one-to-one function*

Proof. Partition the set $[\omega^2]^2$ into two pieces by putting $\{\langle p_1, q_1\rangle, \langle p_2, q_2\rangle\}$ into the first piece if $f(p_1, q_1) = f(p_2, q_2)$ and into the second piece if $f(p_1, q_1) \neq f(p_2, q_2)$. Let $H \in \mathcal{G}$ be as in Theorem 31 for $n = 2$ and this partition.

Consider first those 2-element subsets of H that realize the type $x_1 = x_2 < y_1 < y_2$, i.e., 2-element subsets of columns. If these all lie in the first piece of our partition, then the restriction $f \restriction H$ is constant in each column, so it is $g \circ \pi_1$ for some $g : \omega \to \omega$. Furthermore, as $\mathcal{U} = \pi_1(\mathcal{G})$ is selective, by Proposition 30, g is constant or one-to-one on a set $A \in \mathcal{U}$. Then the restriction of f to $H \cap {\pi_1}^{-1}(A)$ satisfies the first or second conclusion of the corollary.

So we may assume from now on that the 2-element subsets of H realizing $x_1 = x_2 < y_1 < y_2$ are in the second piece of our partition. That is, $f \restriction H$ is one-to-one in each column. We shall complete the proof by showing that f is one-to-one on all of H. Suppose, toward a contradiction, that f took the same value at two elements of H, necessarily in different columns. Then the set of those two elements realizes one of the three 2-types

$$x_2 < x_1 < y_1 < y_2$$
$$x_1 < x_2 < y_1 < y_2$$
$$x_1 < y_1 < x_2 < y_2,$$

because these are the only 2-types that don't have $x_1 = x_2$. By the homogeneity of H, all 2-element sets $\{\langle p_1, q_1\rangle, \langle p_2, q_2\rangle\}$ realizing that 2-type have $f(p_1, q_1) = f(p_2, q_2)$. We can associate to each of the three relevant 2-types a 3-type in which $x_2 = x_3$, namely

$$x_2 = x_3 < x_1 < y_1 < y_2 < y_3$$
$$x_1 < x_2 = x_3 < y_1 < y_2 < y_3$$
$$x_1 < y_1 < x_2 = x_3 < y_2 < y_3,$$

respectively. This 3-type (like all types) is realized in H, say by $\{\langle p_1, q_1\rangle, \langle p_2, q_2\rangle, \langle p_3, q_3\rangle\}$. Then both of the pairs $\{\langle p_1, q_1\rangle, \langle p_2, q_2\rangle\}$ and $\{\langle p_1, q_1\rangle, \langle p_3, q_3\rangle\}$ realize the 2-type that guarantees $f(p_1, q_1) = f(p_2, q_2)$ and $f(p_1, q_1) = f(p_3, q_3)$. Therefore $f(p_2, q_2) = f(p_3, q_3)$. But $p_2 = p_3$, so this contradicts the fact that f is one-to-one on columns in H. □

Corollary 34. *The generic ultrafilter $\mathcal{G}$ is a Q-point. The only non-principal ultrafilters strictly below it in the Rudin-Keisler order are the isomorphic copies of $\mathcal{U} = \pi_1(\mathcal{G})$.*

Proof. This follows from the preceding corollary by the same proofs as for Corollaries 11 and 12 above. □

3.3. **Weak P-point.** The results proved so far about the $\mathbb{P}$-generic ultrafilter $\mathcal{G}$ mirror the properties of selective-indexed sums of selective ultrafilters proved in Section 2. Nevertheless, there is an important difference between $\mathcal{G}$ and these sums; in particular, $\mathcal{G}$ is not such a sum.

Notice that any sum $\mathcal{U}$-$\sum_n \mathcal{V}_n$ is, in the Stone-Čech compactification $\beta(\omega^2)$, the limit with respect to $\mathcal{U}$ of copies in columns of ω^2 of the ultrafilters $\mathcal{V}_n$, namely the images $i_n(\mathcal{V}_n)$ where $i_n : \omega \to \omega^2 : y \mapsto \langle n, y\rangle$. In contrast, $\mathcal{G}$ is not such a limit; indeed, we shall show in this subsection that it is not a limit point of any countable set of other, non-principal ultrafilters.

We begin by recalling some standard terminology and results.

Definition 35. A non-principal ultrafilter $\mathcal{W}$ on a countable set S is a *weak P-point* if, for any countably many non-principal ultrafilters $\mathcal{X}_n \neq \mathcal{W}$ on S, there is a set $A \in \mathcal{W}$ such that $A \notin \mathcal{X}_n$ for all n.

In topological terms, this means that $\mathcal{W}$ is not in the closure in $\beta(S)$ of a countable set of other non-principal ultrafilters.

The terminology "weak P-point" is justified by the observation that any P-point $\mathcal{W}$ is also a weak P-point. Indeed, given countably many $\mathcal{X}_n$ as in the definition, we have for each n, since $\mathcal{W} \neq \mathcal{X}_n$, some $A_n \in \mathcal{W} - \mathcal{X}_n$. As $\mathcal{W}$ is a P-point, it contains a set A almost included in each A_n, and this A is clearly not in any of the $\mathcal{X}_n$'s. Unlike P-points, weak P-points can be proved to exist in ZFC; see [10].

Theorem 36. *The generic ultrafilter $\mathcal{G}$ is a weak P-point.*

Proof. In accordance with the definition of "weak P-point", we shall need to consider non-principal ultrafilters $\mathcal{X} \neq \mathcal{G}$ on ω^2. It will be useful to distinguish four sorts of such ultrafilters $\mathcal{X}$:

(1) ultrafilters $\mathcal{X}$ such that $\pi_1(\mathcal{X})$ is a principal ultrafilter,
(2) ultrafilters $\mathcal{X}$ such that $\pi_1(\mathcal{X})$ is non-principal and distinct from $\mathcal{U} = \pi_1(\mathcal{G})$,
(3) ultrafilters $\mathcal{X}$ such that $\pi_1(\mathcal{X}) = \mathcal{U}$ and, for some $f : \omega \to \omega$, the set $\{\langle x, y\rangle : y \leq f(x)\}$ is in $\mathcal{X}$,
(4) ultrafilters $\mathcal{X} \neq \mathcal{G}$ such that $\pi_1(\mathcal{X}) = \mathcal{U}$ and, for every $f : \omega \to \omega$, the set $\{\langle x, y\rangle : y > f(x)\}$ is in $\mathcal{X}$.

We shall show that, for any countably many ultrafilters $\mathcal{X}_n$ of any one of these four sorts, $\mathcal{G}$ contains a set A that is in none of these $\mathcal{X}_n$'s. This will suffice, because then, if we are given a countable set of $\mathcal{X}_n$'s of possibly different sorts, we can partition it into four subsets, one for each sort, find suitable sets A for each of the four subsets, and then intersect those four A's to get a single A that is in $\mathcal{G}$ but in none of the original $\mathcal{X}_n$'s. In other words, it suffices to treat each sort of $\mathcal{X}$ separately. This we now proceed to do, starting with the easier cases.

Suppose we are given countably many ultrafilters $\mathcal{X}_n$ of sort (2). The countably many ultrafilters $\pi_1(\mathcal{X}_n)$ on ω are non-principal and distinct from $\mathcal{U}$, which is (by Proposition 30) selective, hence a P-point, and hence a weak P-point. So there is a set $B \in \mathcal{U}$ that is in none of the ultrafilters $\pi_1(\mathcal{X}_n)$. Then ${\pi_1}^{-1}(B)$ is in $\mathcal{G}$ but in none of the $\mathcal{X}_n$, as required. This completes the proof for sort (2).

Suppose next that we are given countably many ultrafilters $\mathcal{X}_n$ of sort (3). For each n, fix $f_n : \omega \to \omega$ such that $\{\langle x, y\rangle : y \leq f_n(x)\} \in \mathcal{X}_n$. By diagonalization, let $g : \omega \to \omega$ eventually majorize each of the countably many functions f_n. Thus, for each n, the set $\{\langle x, y\rangle : y \leq f_n(x)\}$ is covered by the union of $\{\langle x, y\rangle : y \leq g(x)\}$ and ${\pi_1}^{-1}(F_n)$ for a finite set F_n. So $\mathcal{X}_n$ must contain $\{\langle x, y\rangle : y \leq g(x)\}$ or ${\pi_1}^{-1}(F_n)$. It cannot contain the latter, because $\pi_1(\mathcal{X}_n)$ is the non-principal ultrafilter $\mathcal{U}$. So each $\mathcal{X}_n$ must contain $\{\langle x, y\rangle : y \leq g(x)\}$. But the complement of this set is in $\mathcal{F}^{\otimes 2}$ and therefore in $\mathcal{G}$; it therefore serves as the required A. This completes the proof for sort (3).

Suppose next that we are given countably many ultrafilters $\mathcal{X}_n$ of sort (4). In contrast to the previous cases, we shall now need to use the fact that our ultrafilters are in the forcing extension $V[\mathcal{G}]$. Fix names $\dot{\mathcal{X}}_n$ for the ultrafilters $\mathcal{X}_n$, and recall that we already fixed the canonical name $\dot{\mathcal{G}}$ for $\mathcal{G}$. We shall complete the proof for this case by showing that any condition $A \in \mathbb{P}$ that forces "The $\dot{\mathcal{X}}_n$ are ultrafilters of sort (4)" can be extended to one forcing "$\dot{\mathcal{G}}$ contains a set that is in none of the $\dot{\mathcal{X}}_n$."

Let such a condition A be given. It forces that, for each n, the difference $\dot{\mathcal{G}} - \dot{\mathcal{X}}_n$ is nonempty. Since the forcing (in its separative form) is countably closed, we can extend A to a condition B that forces, for each $n \in \omega$, a specific set C_n in the ground model to be in $\dot{\mathcal{G}} - \dot{\mathcal{X}}_n$. By Lemma 28, we have that $B \subseteq C_n$ modulo $\mathcal{F}^{\otimes 2}$ for each n. But B, as an extension of A, forces $\dot{\mathcal{X}}_n$ to be of sort (4) and therefore to be a superset of the filter $\mathcal{F}^{\otimes 2}$. So from $B \Vdash C_n \notin \dot{\mathcal{X}}_n$ it follows that $B \Vdash B \notin \dot{\mathcal{X}}_n$. Of course B forces itself to be in the generic $\mathcal{G}$. Summarizing, we have an extension B of A forcing some set, namely B itself, to be in $\mathcal{G}$ but in none of the $\mathcal{X}_n$. This completes the proof for sort (4).

It remains to treat the case where we are given ultrafilters $\mathcal{X}_n$ of sort (1). So each $\pi_1(\mathcal{X}_n)$ is a principal ultrafilter, say generated by $\{f(n)\}$. Note that, although the sequence $\langle \mathcal{X}_n \rangle$ and the individual ultrafilters $\mathcal{X}_n$ need not be in the ground model, f, being a real, is in the ground model.

We shall complete the proof by showing that any condition A forcing "The $\dot{\mathcal{X}}_n$ are ultrafilters of sort (1)" can be extended to a condition B

forcing "$\dot{\mathcal{G}}$ contains a set that is in none of the $\dot{\mathcal{X}}_n$." We begin with a few simplifying steps, shrinking A to normalize it in certain ways.

By extending (i.e., shrinking) the given condition A, we can arrange that it is standard; recall that this means that all its nonempty sections are infinite. We can also arrange that it forces the f introduced above to be a specific function in the ground model. If the set $Z = \pi_1(A) - \mathrm{ran}(f)$ is infinite, then $B = A \cap {\pi_1}^{-1}(Z)$ is a condition forcing that ${\pi_1}^{-1}(Z)$ is in $\mathcal{G}$ but in none of the $\mathcal{X}_n$, so the proof is complete in this case. We therefore assume that Z is finite. Removing ${\pi_1}^{-1}(Z)$ from A, we can arrange that $\pi_1(A) \subseteq \mathrm{ran}(f)$. From now on, we assume that all these arrangements have been made.

The rest of the proof will consist of an ω-sequence of successive extensions of A, approaching the desired condition B. Each step will involve a simple construction, which we isolate in the following lemma.

Lemma 37. *Any countably infinite set S admits a sequence $\langle \Pi_n : n \in \omega \rangle$ of partitions Π_n of S into two pieces each, such that, whenever pieces $C_n \in \Pi_n$ are chosen for each n, there is an infinite set $P \subseteq S$ almost disjoint from all the chosen C_n's.*

Proof. It suffices to choose the partitions to be sufficiently independent. For example, suppose, without loss of generality, that S is the set of finite sequences of zeros and ones and let Π_n partition the sequences according to their n^{th} term (where sequences shorter than n are considered to have n^{th} term zero). For any chosen pieces $C_n \in \Pi_n$, their complements (the unchosen pieces) have the finite intersection property, so there is an infinite P almost included in all these complements. □

We shall construct a sequence of standard conditions

$$A_0 \geq A_1 \geq \cdots \geq A_k \geq A_{k+1} \geq \dots,$$

starting with $A_0 = A$, along with an increasing sequence $x_0 < x_1 < \dots$ of natural numbers and a sequence of infinite subsets P_k of ω, with the following properties.

(1) If $i < j$ then $\{x_i\} \times P_i \subseteq A_j$.

(2) If $f(n) = x_i$ then $A_{i+1} \Vdash \{x_i\} \times (\omega - P_i) \in \dot{\mathcal{X}}_n$.

Note that we require the A_k's to be a decreasing sequence in $\mathbb{P}$, not just in the separative quotient, so they are genuine subsets of each other, not just modulo $\mathcal{F}^{\otimes 2}$. In particular, we shall have $\pi_1(A_k) \subseteq \mathrm{ran}(f)$ for all k.

We proceed by induction, assuming that at the beginning of stage k we already have A_i for $i \leq k$ and x_i and P_i for $i < k$, satisfying all our requirements insofar as they involve only these initial segments of our construction. At the beginning of stage 0, we have the situation already described: $A_0 = A$.

In stage k, we must define A_{k+1}, x_k, and P_k so as to maintain our requirements. First choose x_k to be any natural number (say the first, for definiteness) larger than x_{k-1} and in $\pi_1(A_k)$. (Ignore "larger than x_{k-1} if $k = 0$.) Of course, such an x_k exists because A_k is a condition. Let $S = A_k(x_k)$ be the corresponding section of A_k. Note that S is infinite because A_k is standard. Apply Lemma 37 to obtain sequence of partitions Π_n as there.

Consider those natural numbers n such that $f(n) = x_k$. For each of these, $\mathcal{X}_n$ is forced (by A and a fortiori by A_k) to concentrate on $\{x_k\} \times \omega$ and therefore on a set of the form $\{x_k\} \times Z$, where Z is one of the two pieces of Π_n or $\omega - S$. Furthermore, since our forcing adds no new reals, the condition A_k has an extension B that decides these options for each relevant n. If B forces $\mathcal{X}_n$ to concentrate on $\{x_k\} \times Z$ where $Z \in \Pi_n$, let C_n be that Z. For all other n's (those for which $f(n) \neq x_k$ and those for which B forces $\mathcal{X}_n$ to concentrate on $\{x_k\} \times (\omega - S)$), choose $C_n \in \Pi_n$ arbitrarily. By Lemma 37, we can find an infinite subset of S almost disjoint from all the chosen C_n's; let P_k be such a set. Note that, for each n with $f(n) = x_k$, the condition B forces $\mathcal{X}_n$ to concentrate on $\{x_k\} \times (\omega - P_k)$.

Obtain A_{k+1} from B by standardizing (i.e., removing all finite sections) and adjoining all the sets $\{x_i\} \times P_i$ for all $i \leq k$. None of this affects B in the separative quotient, since the union of the finite columns removed and the finitely many columns added is in the ideal dual to $\mathcal{F}^{\otimes 2}$. Furthermore, $A_{k+1} \subseteq A_k$. Indeed, we had $B \subseteq A_k$, and standardization only shrinks B. As for the additional columns $\{x_i\} \times P_i$, the ones for $i < k$ were subsets of A_k by induction hypothesis, and the one for $i = k$ is also included in A_k because $P_k \subseteq S = A_k(x_k)$.

This completes the inductive construction of the sequences of A_k's, x_k's, and P_k's. Finally, let

$$B = \{\langle x_k, y\rangle : k \in \omega \text{ and } y \in P_k\}.$$

This B is a condition, as there are infinitely many x_k's and each P_k is infinite. It is an extension of all the A_k's because each of its elements $\langle x_k, y\rangle$ is in A_j for all $j > k$ by our inductive construction, and therefore also for $j \leq k$ because the A_j's form a decreasing sequence. We claim that B forces all the $\mathcal{X}_n$ to concentrate on $\omega^2 - B$. Since it forces $\mathcal{G}$ to concentrate on B, this will complete the proof of the theorem.

So consider an arbitrary $n \in \omega$. If $f(n)$ is not of the form x_k, then $\mathcal{X}_n$ concentrates on $\{f(n)\} \times \omega$, which is disjoint from B, as required for the claim. So suppose that $f(n) = x_k$. Then A_{k+1} was constructed to force $\mathcal{X}_n$ to concentrate on $\{x_k\} \times (\omega - P_k)$, which is disjoint from B. As an extension of A_{k+1}, B forces the same, and the proof is therefore complete. □

4. $\mathcal{G}$ does not have maximal Tukey type

We prove a canonization theorem showing that every monotone map with domain $\mathcal{G}$ is almost continuous (represented by a finitary function) on a cofinal subset of $\mathcal{G}$. It follows that every ultrafilter Tukey reducible to $\mathcal{G}$ has Tukey type of cardinality continuum. In particular, $\mathcal{G}$ is strictly below the top Tukey degree, thus answering a question of Blass stated during a talk at the Fields Institute, September 2012. Moreover, we show that $\mathcal{G} \not\geq_T [\omega_1]^{<\omega}$, answering a question of Raghavan in [13].

The results in this section were obtained by Dobrinen and completed on the following dates. Theorems 42 and 47 were completed on October 8, 2012. Theorem 49 was completed on October 31, 2012. Proposition 39 was completed on November 1, 2012.

We begin by recalling some useful facts. For ultrafilters $\mathcal{U}, \mathcal{V}$, a map $f : \mathcal{U} \to \mathcal{V}$ is called *monotone* if whenever $u \supseteq u'$ are in $\mathcal{U}$, then $f(u) \supseteq f(u')$. By [Fact 6, in [5]], whenever $\mathcal{U} \geq_T \mathcal{V}$, there is a monotone convergent map $f : \mathcal{U} \to \mathcal{V}$ witnessing this reduction. Recall the following [Theorem 20, in [5]] canonizing Tukey reductions from P-points as continuous maps. Here, $\mathcal{P}(\omega)$ is endowed with the Cantor topology on 2^ω by associating subsets of ω with their characteristic functions.

Theorem 38 (Dobrinen/Todorcevic, [5])**.** *Suppose $\mathcal{U}$ is a P-point on ω and $\mathcal{V}$ is an arbitrary ultrafilter on a countable base set such that $\mathcal{U} \geq_T \mathcal{V}$. For each monotone convergent map $f : \mathcal{U} \to \mathcal{V}$, there is an $\tilde{x} \in \mathcal{U}$ such that $f \restriction (\mathcal{U} \restriction \tilde{x})$ is continuous. Moreover, there is a continuous monotone map $f^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f^* \restriction (\mathcal{U} \restriction \tilde{x}) = f \restriction (\mathcal{U} \restriction \tilde{x})$. Hence, there is a continuous monotone convergent map $f^* \restriction \mathcal{U}$ from $\mathcal{U}$ into $\mathcal{V}$ which extends $f \restriction (\mathcal{U} \restriction \tilde{x})$.*

The proof of Theorem 38 holds whenever $\mathcal{V}$ is an ultrafilter on any countable base set B, the topology given by enumerating B in order type ω and considering the Cantor topology on 2^B. In the next proposition, we shall apply this theorem with $\mathcal{G}$ in place of $\mathcal{V}$.

Recall Proposition 30 which implies that $\pi_1(\mathcal{G})$ is selective (hence a P-point) and that $\pi_1(\mathcal{G}) <_{RK} \mathcal{G}$. It follows that $\pi_1(\mathcal{G}) \leq_T \mathcal{G}$. We begin this section by showing the stronger fact that the inequality is also strict for the Tukey reduction.

Proposition 39. $\mathcal{G} >_T \pi_1(\mathcal{G})$.

Proof. We shall show that for each monotone map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega^2)$ in V, $f \restriction \pi_1(\mathcal{G})$ is not a convergent map from $\pi_1(\mathcal{G})$ into $\mathcal{G}$ in $V[\mathcal{G}]$. This shows, in particular, there are no monotone continuous maps in V witnessing a Tukey reduction from $\pi_1(\mathcal{G})$ into $\mathcal{G}$. Since $\pi_1(\mathcal{G})$ is a P-point, by Theorem 38, every Tukey reduction from $\pi_1(\mathcal{G})$ to any other ultrafilter is witnessed by a monotone continuous map. Since $\mathbb{P}$ is σ-closed, every continuous map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega^2)$ in $V[\mathcal{G}]$ is actually in V. Thus, $\pi_1(\mathcal{G}) \not\geq_T \mathcal{G}$.

Let $x \in \mathbb{P}$ be given and let $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega^2)$ be a monotone map in V. If there is a $y \in \mathbb{P}$ such that $y \subseteq x$ and $f(\pi_1[y]) \cap y \notin \mathbb{P}$, then $y \Vdash f(\pi_1(\dot{\mathcal{G}})) \not\subseteq \dot{\mathcal{G}}$. Otherwise, for all $y \in \mathbb{P}$ with $y \subseteq x$, $f(\pi_1[y]) \cap y \in \mathbb{P}$. If there is a $y \in \mathbb{P}$ such that $y \subseteq x$ and for all $z \in \mathbb{P}$ with $z \subseteq y$, $f(\pi_1[z]) \not\subseteq y$, then $y \Vdash$ "$f \restriction \pi_1(\dot{\mathcal{G}}) : \pi_1(\dot{\mathcal{G}}) \to \dot{\mathcal{G}}$ is not a convergent map."

If none of the above cases holds, then for each $y \in \mathbb{P}$ with $y \subseteq x$, (a) $f(\pi_1[y]) \cap y \in \mathbb{P}$, and (b) there is a $z \in \mathbb{P}$ with $z \subseteq y$ such that $f(\pi_1[z]) \subseteq y$ Fix some $y, w \in \mathbb{P}$ with $y, w \subseteq x$ such that $\pi_1[y] = \pi_1[w] = \pi_1[x]$ but $y \cap w = 0$. Take a $y' \in \mathbb{P}$ with $y' \subseteq y$ such that $f(\pi_1[y']) \subseteq y$. Next, take a $w' \in \mathbb{P}$ with $w' \subseteq w$ such that $\pi_1[w'] \subseteq \pi_1[y']$; then take a $w'' \in \mathbb{P}$ with $w'' \subseteq w'$ such that $f(\pi_1[w'']) \subseteq w'$. Then $\pi_1[w''] \subseteq \pi_1[y']$, so $f(\pi_1[w'']) \subseteq f(\pi_1[y'])$, since f is monotone. Hence, $f(\pi_1[w'']) \subseteq y$. At the same time, $f(\pi_1[w'']) \subseteq w' \subseteq w$. Thus, $f(\pi_1[w'']) \subseteq w \cap y = 0$. Hence, it is dense to force that $\dot{f}$ is constantly zero on some cofinal subset of $\pi_1(\dot{\mathcal{G}})$.

Therefore, for each monotone map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega^2)$ in V, it is dense to force that "$f \restriction \pi_1(\dot{\mathcal{G}})$ is not a convergent map from $\pi_1(\dot{\mathcal{G}})$ into $\dot{\mathcal{G}}$." □

Notation 40. Throughout this section, we shall let $\mathbb{P}_s$ denote the collection of the standard forcing conditions; that is, those conditions such that each non-empty fiber is infinite. We shall let $\mathcal{G}_s$ denote the members of $\mathcal{G}$ which are standard; that is, $\mathcal{G}_s = \mathcal{G} \cap \mathbb{P}$. For $u \in \mathbb{P}_s$ and $n < \omega$, we shall let $u \cap n^2$ denote $u \cap (n \times n)$. Recall that for any $p \in \mathbb{P}$ and $i \in \omega$, $p(i)$ denotes $\{j \in \omega : (i, j) \in p\}$, the i-th fiber of p.

The next theorem provides a canonization for all monotone maps with domain $\mathcal{G}_s$ in terms of monotone finitary maps on the base $[\omega^2]^{<\omega}$. The argument of Theorem 42 combines some of the key traits of the proof of Theorem 20 in [5] for canonizing monotone convergent maps on P-points and the proof of Theorem 13 in [4] canonizing monotone convergent maps on iterated Fubini products of P-points, making new adjustments for this setting.

Definition 41. We say that a monotone map f on $\mathcal{G}_s \restriction x$ is *represented by a monotone finitary map* φ if there is a map $\varphi : [\omega^2]^{<\omega} \to [\omega]^{<\omega}$ such that for all $s, t \in [\omega^2]^{<\omega}$,

(1) (Monotonicity) $s \subseteq t \to \varphi(s) \subseteq \varphi(t)$;
(2) (φ represents f) For each $u \in \mathcal{G}_s \restriction x$, $f(u) = \bigcup_{n<\omega} \varphi(u \cap n^2)$.

Theorem 42 (Canonization of monotone maps as almost continuous)**.** *In $V[\mathcal{G}]$, for each monotone function $f : \mathcal{G} \to \mathcal{P}(\omega)$, there is an $x \in \mathcal{G}_s$ such that $f \restriction (\mathcal{G}_s \restriction x)$ is represented by a monotone finitary map. Hence, every monotone cofinal map $f : \mathcal{G} \to \mathcal{V}$ is represented by a monotone finitary map on the filter base consisting of members of $\mathcal{G}_s$ below some $x \in \mathcal{G}_s$, where $\mathcal{V}$ is any ultrafilter on a countable base.*

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name such that $\Vdash$ "$\dot{f} : \dot{\mathcal{G}} \to \mathcal{P}(\omega)$ is monotone." Unless stated otherwise, all conditions are assumed to be in $\mathbb{P}_s$. Fix a $p \in \mathbb{P}_s$, and let $p_{-1} = p$. In the first step of the proof, we shall construct a sequence $p \geq x_0 \geq p_0 \geq x_1 \geq p_1 \geq \dots$ of members of $\mathbb{P}_s$ such that the following (a)-(c) hold for each $k < \omega$:

(a) $x_{k+1} \subseteq x_k$.
(b) There is a sequence $i_0 < i_1 < \dots$ such that $i_0 = \min(\pi_1[p])$, and in general,

$$i_{k+1} = \min(\pi_1[x_k] \setminus \{i_0, \dots, i_k\}) = \min(\pi_1[x_k] \setminus (i_k + 1));$$

and for each $n < \omega$,

$$\pi_1[x_{k+n}] \cap (i_{k+1} + 1) = \pi_1[x_k] \cap (i_{k+1} + 1) = \{i_0, \dots, i_{k+1}\}.$$

(c) Let S_k denote the set of all triples $(s, t, j) \in \mathcal{P}(\{i_0, \dots, i_k\} \times (i_k + 1)) \times \mathcal{P}(\{i_0, \dots, i_k\}) \times (i_k + 1)$ such that $\pi_1[s] \subseteq t$. Fix an enumeration of S_k as (s_k^l, t_k^l, j_k^l), $l \leq l_k := |S_k| - 1$. For all $k < \omega$ and $l \leq l_k$, x_k and p_k satisfy the following:

If there are $v, q \in \mathbb{P}_s$ with $q \leq v \leq p_{k-1}$ such that
(i) $v \cap (i_k + 1)^2 = s_k^l$;
(ii) $\pi_1[v] \cap (i_k + 1) = t_k^l$;
(iii) For all $i \in t_k^l$, $v(i) \setminus (i_k + 1) \subseteq x_k(i)$;
(iv) $v \cap ((i_k, \omega) \times \omega) \subseteq x_k$;
(v) $q \Vdash j_k^l \notin \dot{f}(v)$;

Then for each $u \in \mathbb{P}_s$ such that
(vi) $u \cap (i_k + 1)^2 = s_k^l$;
(vii) $\pi_1(u) \cap (i_k + 1) = t_k^l$;
(viii) For all $i \in t_k^l$, $u(i) \setminus (i_k + 1) \subseteq x_k(i)$;

(ix) $u \cap ((i_k, \omega) \times \omega) \subseteq x_k$;
we have that $p_k \Vdash j_k^l \notin \dot{f}(u)$.

Note that it follows from (b) that $\pi_1[x_k] \to \{i_0, i_1, \dots\}$ as $k \to \infty$.

We now begin the recursive construction of the sequences $(x_n)_{n<\omega}$ and $(p_n)_{n<\omega}$. Let $p_{-1} = x_{-1} = p$. Let $k \in \omega$ be given, and suppose we have chosen x_{k-1}, p_{k-1}. If $k = 0$, let $i_0 = \min(\pi_1[p])$; if $k \geq 1$, let $i_k = \min(\pi_1[x_{k-1}] \setminus (i_{k-1} + 1))$. Recall that S_k denotes the set of all triples (s, t, j) such that $s \in \mathcal{P}(\{i_0, \dots, i_k\} \times (i_k + 1))$, $t \in \mathcal{P}(\{i_0, \dots, i_k\})$, and $j \in (i_k + 1)$ such that $\pi_1[s] \subseteq t$. Let $x_k^{-1} = x_{k-1}$ and $p_k^{-1} = p_{k-1}$. Suppose we have chosen x_k^{l-1} and p_k^{l-1} for $l \leq l_k = |S_k| - 1$.

If there are $v, q \in \mathbb{P}_s$ and $q \leq v \leq p_k^{l-1}$ such that

(i) $v \cap (i_k + 1)^2 = s_k^l$;
(ii) $\pi_1[v] \cap (i_k + 1) = t_k^l$;
(iii) For all $i \in t_k^l$, $v(i) \setminus (i_k + 1) \subseteq x_k^{l-1}(i)$;
(iv) $v \cap ((i_k, \omega) \times \omega) \subseteq x_k^{l-1}$;
(v) $q \Vdash j_k^l \notin \dot{f}(v)$;

then take p_k^l and v_k^l be some such q and v. Note that $p_k^l \Vdash j_k^l \notin \dot{f}(v_k^l)$. Hence, by monotonicity, $p_k^l \Vdash j_k^l \notin \dot{f}(v)$, for every $v \subseteq v_k^l$. In this case, let

$$\begin{aligned} x_k^l = & \bigcup\{\{i\} \times (v_k^l(i) \setminus (i_k + 1)) : i \in t_k^l\} \\ & \cup \bigcup\{\{i\} \times x_k^{l-1}(i) : i \in \{i_0, \dots, i_k\} \setminus t_k^l\} \\ & \cup v_k^l \cap ((i_k, \omega) \times \omega). \end{aligned} \tag{1}$$

Thus, x_k^l is empty on the square $(i_k + 1)^2$; on indices $i \in t_k^l$, the i-th fiber of x_k^l equals the i-th fiber of v_k^l above i_k; on indices $i \leq i_k$ which are not in t_k^l, the i-th fiber of x_k^l equals the i-th fiber of x_k^{l-1}; and for all $i > i_k$, the i-th fiber of x_k^l is exactly the same as v_k^l.

Otherwise, for all $v \leq p_k^{l-1}$ satisfying (i) - (iv), there is no $q \leq v$ which forces j_k^l to not be in $\dot{f}(v)$. Thus, for all $v \leq p_k^{l-1}$ satisfying (i) - (iv), $v \Vdash j_k^l \in \dot{f}(v)$. In particular, $p_k^l \Vdash j_k^l \in \dot{f}(v_k^l)$. In this case, let $p_k^l = p_k^{l-1}$ and $x_k^l = x_k^{l-1}$, and define

$$v_k^l = s_k^l \cup (x_k^l \cap ((t_k^l \cup (i_k, \omega)) \times \omega)). \tag{2}$$

In the 'If' case, $p_k^l \Vdash j_k^l \notin \dot{f}(v_k^l)$. In the 'Otherwise' case, $p_k^l \Vdash j_k^l \in \dot{f}(v_k^l)$. Thus,

($*$) For all $k < \omega$ and $l \leq l_k$, p_k^l decides the statement "$j_k^l \in \dot{f}(v_k^l)$".

After the l_k steps, let $p_k = p_k^{l_k}$ and $x_k = x_k^{l_k}$. Note that $x_k \cap (i_k+1)^2 = \emptyset$, and $x_k \subseteq x_{k-1}$ by (iii) and (iv). This ends the recursive construction of the p_k and x_k. It is not hard to see that the sequence of x_k, p_k satisfies (a) - (c).

In the second step, we diagonalize through the x_n to obtain x as follows. Let $k_0 = 0$. Take an $a_{0,0}$ in the fiber $x_0(i_0)$. Choose k_1 so that $i_{k_1} > a_{0,0}$. In general, given k_n, for each $j \le i_{k_n}$, take an $a_{j,n} \in x_{k_n}(i_j) \setminus (i_{k_n}+1)$. Then choose k_{n+1} so that $i_{k_1} > \max\{a_{j,n} : j \le k_n\}$. Let

$$x = \{(i_j, a_{j,n}) : n < \omega, j \le k_n\}.$$

Note that $x \in \mathbb{P}$, $x \subseteq p$, $\pi_1[x] = \{i_{k_0}, i_{k_1}, i_{k_2}, \dots\}$, and in fact, $x \subseteq^* x_k$ for all $k < \omega$. Thus, also $x \le p_k$, for all $k < \omega$. Moreover, for each n, $x \setminus (i_{k_n}+1)^2 \subseteq x_{k_n}$. From now on, we work below x.

Let us establish some useful notation. Given any $(s,t) \in [\omega^2]^{<\omega} \times [\omega]^{<\omega}$, $k \ge \max(\pi_1[s] \cup t)$, and $v \subseteq x$, define

$$z(k,s,t,v) = s \cup (\bigcup_{i \in t}\{i\} \times (v(i) \setminus (k+1))) \cup (v \cap ((k,\omega) \times \omega)), \tag{3}$$

and

$$\dot{f}(k,s,t,v) = \dot{f}(z(k,s,t,v)). \tag{4}$$

Thus, $z(k,s,t,v)$ is the following condition $z \in \mathbb{P}_s$: Inside the square $(k+1) \times (k+1)$, z is exactly s. For indices $i \le k$, z has nonempty fibers if and only if $i \in t$; and if $i \in t$, then the tail of the i-th fiber of z above k is exactly the tail of the i-th fiber of v above k. All fibers of z with index greater than k are exactly the fibers of v with index greater than k.

Our construction was geared toward establishing the following.

Claim 43. Let $v \subseteq x$, $n < \omega$, and $j \le i_{k_n}$ be given; and let $l \le l_{k_n}$ be the integer satisfying $s^l_{k_n} = v \cap (i_{k_n}+1)^2$, $t^l_{k_n} = \pi_1[v] \cap (i_{k_n}+1)$, and $j^l_{k_n} = j$. Then

$$v \Vdash j \in \dot{f}(v) \iff x \Vdash j \in \dot{f}(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x) \iff x \Vdash j \in \dot{f}(v^l_{k_n}).$$

Proof. Assume $v \subseteq x$, $n < \omega$, $j \le i_{k_n}$, and $l \le l_{k_n}$ satisfy the hypothesis. By $(*)$, $p^l_{k_n}$ decides whether or not $j \in \dot{f}(v^l_{k_n})$. Since $x \le p^l_{k_n}$, x also decides whether or not $j \in \dot{f}(v^l_{k_n})$.

Suppose $x \Vdash j \notin \dot{f}(v^l_{k_n})$. Our choice of l implies that

$$v = z(i_{k_n}, s^l_{k_n}, t^l_{k_n}, v) \subseteq z(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x) \subseteq z(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x_{k_n}) \tag{5}$$

$$\subseteq z(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x^l_{k_n}) \subseteq z(i_{k_n}, s^l_{k_n}, t^l_{k_n}, v^l_{k_n}) = v^l_{k_n}. \tag{6}$$

By monotonicity of $\dot{f}$, we have $x \Vdash j \notin \dot{f}(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x)$, and $x \Vdash j \notin \dot{f}(v)$. Hence, also $v \Vdash j \notin \dot{f}(v)$.

Suppose now that $x \Vdash j \in \dot{f}(v^l_{k_n})$. Then in the construction of $x^l_{k_n}$, we were in the 'Otherwise' case. Thus, for all pairs $q \leq v' \leq p^{l-1}_{k_n}$ satisying (i) - (iv) for the pair k_n, l, we have that $q \Vdash j \in \dot{f}(v')$. In particular, $v \Vdash j \in \dot{f}(v)$. Further, since $x \leq z(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x)$, we have $x \Vdash j \in \dot{f}(i_{k_n}, s^l_{k_n}, t^l_{k_n}, x)$. □

It follows immediately from Claim 43 that

(†) For each $y \subseteq x$ and $j < \omega$, taking n so that $j < i_{k_n}$, the pair of finite sets $(y \cap (i_{k_n}+1)^2, \pi_1[y] \cap (i_{k_n}+1))$ completely determines whether or not $y \Vdash j \in \dot{f}(y)$.

Now we define a finitary monotone function ψ which x forces to represent $\dot{f}$ on a cofinal subset of $\dot{\mathcal{G}}$. For all pairs $(s,t) \in [x]^{<\omega} \times [\pi_1[x]]^{<\omega}$, take n least such that $i_{k_n} \geq \max(\pi_1[s] \cup t)$ and define

$$\psi(s,t) = \{j \leq i_{k_n} : x \Vdash j \in \dot{f}(i_{k_n}, s, t, x)\}. \tag{7}$$

Note that ψ is monotone: Suppose $s \subseteq s'$ and $t \subseteq t'$. Let n be least such that $\max(\pi_1[s] \cup t) \leq i_{k_n}$, and let n' be least such that $\max(\pi_1[s'] \cup t') \leq i_{k_{n'}}$. Let $s'' = s' \cap (i_{k_n}+1)^2$ and $t'' = t' \cap (i_{k_n}+1)$. Let $j \leq i_{k_n}$. By Claim 43, (7) and monotonicity of $\dot{f}$,

$$\begin{aligned} j \in \psi(s,t) &\Longleftrightarrow x \Vdash j \in \dot{f}(k_n, s, t, x) \\ &\Longrightarrow x \Vdash j \in \dot{f}(k_n, s'', t'', x) \\ &\Longrightarrow x \Vdash j \in \dot{f}(k_{n'}, s', t', x) \\ &\Longleftrightarrow j \in \psi(s', t'). \end{aligned} \tag{8}$$

Claim 44. If x is in $\mathcal{G}$, then ψ represents f on $\mathcal{G}_s \restriction x$. That is, for each $y \in \mathbb{P}_s$ with $y \subseteq x$, $y \Vdash \dot{f}(y) = \bigcup\{\psi(y \cap (i_{k_n}+1)^2, \pi[y] \cap (i_{k_n}+1)) : n < \omega\}$.

Proof. Let $y \in \mathbb{P}_s$ such that $y \subseteq x$. By the definition of ψ, for each j, if $x \Vdash j \in \dot{f}(y)$, then $j \in \psi(y \cap (i_{k_n}+1)^2, \pi_1[y] \cap (i_{k_n}+1))$ for every $i_{k_n} \geq j$. On the other hand, if $j \in \psi(y \cap (i_{k_n}+1)^2, \pi_1[y] \cap (i_{k_n}+1))$, then $x \Vdash j \in \dot{f}(k_n, y \cap (i_{k_n}+1)^2, \pi_1[y] \cap (i_{k_n}+1), x)$. Thus, $y \Vdash j \in \dot{f}(y)$, by Claim 43. □

We simplify φ a bit more. Define the map $\varphi : [x]^{<\omega} \to [\omega]^{<\omega}$ by letting $\varphi(s) = \psi(s, \pi_1[s])$, for each $s \in [x]^{<\omega}$.

Claim 45. φ is monotone and represents $\dot{f} \restriction \mathbb{P}_s \restriction x$; that is, for each standard $y \subseteq x$,

$$y \Vdash \dot{f}(y) = \bigcup\{\varphi(y \cap (i_{k_n}+1)^2) : n < \omega\}. \tag{9}$$

Proof. Suppose $s \subseteq s'$. Then $\pi_1[s] \subseteq \pi_1[s']$, so by monotonicity of ψ, we also have $\varphi(s) \subseteq \varphi(s')$.

Given $y \in \mathbb{P}_s$ with $y \subseteq x$, $y \Vdash \dot{f}(y) \supseteq \bigcup\{\varphi(y \cap (i_{k_n}+1)^2) : n < \omega\}$, by definition of φ and Claim 44. On the other hand, suppose $y \Vdash j \in \dot{f}(y)$. Taking n least such that $i_{k_n} \geq j$, Claim 44 and (†) imply $j \in \psi(y \cap (i_{k_n}+1)^2, \pi_1[y] \cap (i_{k_n}+1))$. Take m least such that $\pi_1[y \cap (i_{k_m}+1)^2] \supseteq \pi_1[y] \cap (i_{k_n}+1)^2$. Let s be $y \cap (i_{k_m}+1)^2$ restricted to indices in $\pi_1[y] \cap (i_{k_n}+1)^2$. That is, let $a \in \omega^2$ be in s if and only if $a \in y \cap (i_{k_m}+1)^2$ and $\pi_1[a] \in \pi_1[y] \cap (i_{k_n}+1)^2$. Monotonicity of ψ implies that $j \in \psi(s, \pi_1[y] \cap (i_{k_n}+1))$. Since $\pi_1[s] = \pi_1[y] \cap (i_{k_n}+1)$, we have $j \in \varphi(s)$. Since φ is monotone and $y \cap (i_{k_m}+1)^2 \supseteq s$, it follows that $j \in \varphi(y \cap (i_{k_m}+1)^2)$. □

It follows that in $V[\mathcal{G}]$, for every monotone map $f : \mathcal{G} \to \mathcal{P}(\omega)$, there is an $x \in \mathcal{G}_s$ such that $f \restriction (\mathcal{G}_s \restriction x)$ is represented by a monotone finitary map $\varphi : [x]^{<\omega} \to [\omega]^{<\omega}$.

If $\mathcal{V}$ is an ultrafilter on a countable base set, we may without loss of generality assume that base is ω. Thus, each monotone cofinal map $f : \mathcal{G} \to \mathcal{V}$ is represented on a base of the form $\mathcal{G}_s \restriction x$ by a finitary map φ on $[x]^{<\omega}$, for some $x \in \mathcal{G}$. □

Remark 46. We point out several key observations. First, the map φ in Theorem 42 actually generates a monotone map from $f^* : \mathcal{P}(\omega^2) \to \mathcal{P}(\omega)$, by defining $f^*(y) = \bigcup_{n<\omega} \varphi(y \cap (i_{k_n}+1)^2)$, for each $y \subseteq \omega^2$. It is easy to check that $f^* \restriction \mathcal{G}_s \restriction x = f \restriction \mathcal{G}_s \restriction x$. Thus, if f is a monotone cofinal map from $\mathcal{G}$ into some ultrafilter $\mathcal{V}$, then $f^* \restriction \mathcal{G}$ is also a monotone cofinal map from $\mathcal{G}$ into $\mathcal{V}$ which is moreover finitely represented.

Second, the map $f \restriction \mathcal{G}_s \restriction x$ is *almost continuous* in the following senses. First, being finitely represented, it quite similar to continuous maps on $\mathcal{P}(\omega^2)$, given the Cantor topology on $\mathcal{P}(\omega^2)$. Of more interest, though, is the property (†), that for each $y \in \mathcal{G}_s \restriction x$ and $n < \omega$, the map ψ can decide by time i_{k_n} whether or not n is in $f(y)$. ψ can do this because it not only looks at $y \cap i_{k_n}$ but also sees the future of whether or not the fibers $y(l)$ will be nonempty, for $l \leq i_{k_n}$.

We now answer a question of Blass, by applying Theorem 42 to show that $\mathcal{G}$ does not have maximal Tukey type.

Theorem 47. *For every $\mathcal{V} \leq_T \mathcal{G}$, the Tukey type of $\mathcal{V}$ has cardinality continuum. In particular, the Tukey type of $\mathcal{G}$ has cardinality continuum; hence, $\mathcal{G}$ does not have maximal Tukey type.*

Proof. By Theorem 42, every Tukey reduction $\mathcal{V} \leq_T \mathcal{G}$ is witnessed by a function f represented by a finitary monotone map $\varphi : [x]^{<\omega} \to [\omega]^{<\omega}$

for some $x \in \mathcal{G}$. Since there are only continuum such maps, it follows that for each $\mathcal{V} \leq_T \mathcal{G}$, the Tukey type of $\mathcal{V}$ has cardinality $\mathfrak{c}$. In particular, $\mathcal{G}$ does not have maximal Tukey type. □

In Theorem 49 we prove that $([\omega_1]^{<\omega}, \subseteq) \not\leq_T (\mathcal{G} \supseteq)$. This answers a question of Raghavan in [13], where he stated that it is easy to see that if in V, $\mathfrak{h}(\mathcal{P}(\omega^2)/(\mathrm{Fin}\times\mathrm{Fin})) > \omega_1$, then in $V[\mathcal{G}]$, $([\omega_1]^{<\omega}, \subseteq) \not\leq_T (\mathcal{G} \supseteq)$, but that he did not know whether this holds in general. This also gives a second proof that $\mathcal{G}$ has Tukey type strictly below the maximal type.

We first prove the following fact, which will be used in the proof of Theorem 49.

Proposition 48. *For any ultrafilter $\mathcal{U}$, if $(\mathcal{U}, \supseteq) \geq_T ([\omega_1]^{<\omega}, \subseteq)$, then there is a monotone convergent map $f : \mathcal{U} \to [\omega_1]^{<\omega}$ witnessing this.*

Proof. Suppose $\mathcal{U} \geq_T [\omega_1]^{<\omega}$. Then there is a Tukey map $g : [\omega_1]^{<\omega} \to \mathcal{U}$ mapping unbounded subsets of $([\omega_1]^{<\omega}, \subseteq)$ to unbounded subsets of $(\mathcal{U}, \supseteq)$. For each $u \in \mathcal{U}$, define $f(u) = \bigcup\{s \in [\omega_1]^{<\omega} : g(s) \supseteq u\}$. Note that $\{s \in [\omega_1]^{<\omega} : g(s) \supseteq u\} = g^{-1}(\{w \in \mathcal{U} : w \supseteq u\})$ which is the g-preimage of a bounded subset of $\mathcal{U}$. Since g is a Tukey map, $\{s \in [\omega_1]^{<\omega} : g(s) \supseteq u\}$ is bounded by some member of $[\omega_1]^{<\omega}$; so $f(u)$ is well-defined.

If $u \supseteq v$ are in $\mathcal{U}$, then $g^{-1}\{w \in \mathcal{U} : w \supseteq u\} \subseteq g^{-1}\{w \in \mathcal{U} : w \supseteq v\}$. Therefore, $\bigcup g^{-1}\{w \in \mathcal{U} : w \supseteq u\} \subseteq \bigcup g^{-1}\{w \in \mathcal{U} : w \supseteq v\}$; so $f(u) \subseteq f(v)$. Hence, f is monotone.

To see that f is a convergent map, it suffices to check that the f-image of $\mathcal{U}$ is cofinal in $[\omega_1]^{<\omega}$, since f is monotone. Let $s \in [\omega_1]^{<\omega}$. Then $s \in \{t \in [\omega_1]^{<\omega} : g(t) \supseteq g(s)\}$, so $s \subseteq \bigcup\{t \in [\omega_1]^{<\omega} : g(t) \supseteq g(s)\} = f(g(s))$, which is a member of $f[\mathcal{U}]$. □

Theorem 49. $(\mathcal{G}, \supseteq) \not\geq_T ([\omega_1]^{<\omega}, \subseteq)$.

Proof. Let $p \in \mathbb{P}$ and $\dot{f}$ be a $\mathbb{P}$-name such that $p \Vdash$ "$\dot{f}$ is a monotone map from $\dot{\mathcal{G}}$ into $[\omega_1]^{<\omega}$." We shall show that there is an $x \leq p$ such that $x \Vdash$ "$\dot{f}[\dot{\mathcal{G}}]$ is not cofinal in $[\omega_1]^{<\omega}$". By Fact 48 it will follow that, in $V[\mathcal{G}]$, $\mathcal{G} \not\geq_T [\omega_1]^{<\omega}$.

Case 1. Suppose that for all $p' \leq p$, $p' \Vdash \dot{f}(p') = 0$. Let $a \in \mathbb{P}$ with $a \subseteq p$ be given, and take any $r \in \mathbb{P}$ with $r \subseteq a$. Since $p \Vdash \dot{f}$ is monotone, it follows that $r \Vdash \dot{f}(r) \supseteq \dot{f}(a)$. $r \leq p$ implies $r \Vdash \dot{f}(r) = 0$. Thus, $r \Vdash \dot{f}(a) = 0$. Since the collection of all $r \in \mathbb{P}$ such that $r \subseteq a$ is dense below a, it follows that $a \Vdash \dot{f}(a) = 0$. Thus, $p \Vdash$ "$\forall a \in \dot{\mathcal{G}}(a \subseteq p \to \dot{f}(a) = 0)$". Suppose $p \in \mathcal{G}$. Then in $V[\mathcal{G}]$, for all $b \in \mathcal{G}$, $b \cap p \in \mathcal{G}$, so $f(b \cap p) = 0$; hence $f(b) = 0$, since f is monotone. Thus, p forces that $\dot{f}$ is constantly 0 on $\dot{\mathcal{G}}$. Letting $x = p$, the first case is finished.

Case 2. Suppose now that there is a $p' \leq p$ such that $p' \Vdash \dot{f}(p') \neq 0$. Without loss of generality, we may assume p has this property and $p \in \mathbb{P}_s$. For the rest of the proof, we restrict to using only standard conditions; that is, all conditions mentioned are assumed to be in $\mathbb{P}_s$. For each pair r, q such that $r \leq q \leq p$ and r decides $\dot{f}(q)$, let $\beta(r, q)$ denote the ordinal in ω_1 such that $r \Vdash \beta(r, q) = \min(\dot{f}(q))$. Define

$$\beta = \min\{\beta(r, q) : r \leq q \leq p \text{ and } r \text{ decides } \dot{f}(q)\}. \tag{10}$$

Fix a pair $p_{-1} \leq x_{-1} \leq p$ for which p_{-1} decides $\dot{f}(x_{-1})$ and such that $\beta(p_{-1}, x_{-1}) = \beta$.

Claim 50. For all $v \leq p_{-1}$ such that $v \subseteq x_{-1}$, $v \Vdash \beta = \min(\dot{f}(v))$.

Proof. Let $v \leq p_{-1}$ such that $v \subseteq x_{-1}$. For each $v' \leq v$, there is a $v'' \leq v'$ deciding $\dot{f}(v)$. Since $v'' \leq v \leq p$, we have that $\beta(v'', v) \geq \beta$. On the other hand, $v \subseteq x_{-1}$ and $p \Vdash$ "$\dot{f}$ is monotone" imply that $v'' \Vdash \dot{f}(v) \supseteq \dot{f}(x_{-1})$. Hence, $v'' \Vdash \beta \in \dot{f}(v)$. By minimality of $\beta(v'', v)$ in $\dot{f}(v)$, it must be the case that $\beta(v'', v) = \beta$. By density, we have that $v \Vdash \min \dot{f}(v) = \beta$. □

We now build a decreasing sequence $p \geq x_{-1} \geq p_{-1} \geq x_0 \geq p_0 \geq x_1 \geq p_1 \geq \dots$ of members of $\mathbb{P}_s$ such that each p_n decides everything we need to know about x_n. Diagonalizing the x_n, we will form an $x \subseteq p$ in $\mathbb{P}_s$ such that x forces the range of $\dot{f}$ to be countable. The construction follows the same general outline as the one given in the proof of Theorem 42.

Let $k \in \omega$ be given, and suppose we have chosen x_j, p_j for all $-1 \leq j < k$. If $k = 0$, let $i_0 = \min(\pi_1[x_{-1}])$; if $k > 0$, let $i_k = \min(\pi_1[x_{k-1}] \setminus (i_{k-1} + 1))$. Define S_k to be the set of all pairs (s, t) such that $s \in \mathcal{P}(\{i_0, \dots, i_k\} \times (i_k + 1)) \cap x_{-1}$, $t \in \mathcal{P}(\{i_0, \dots, i_k\})$, and $\pi_1[s] \subseteq t$. Fix an enumeration of S_k as (s_k^l, t_k^l), $l \leq l_k := |S_k| - 1$. Define $x_k^{-1} = x_{k-1}$ and $p_k^{-1} = p_{k-1}$.

Suppose $l \leq l_k$ and we have chosen x_k^{l-1}, p_k^{l-1}, and α_k^{l-1}. We choose $p_k^l \leq x_k^l$ in $\mathbb{P}_s$ and $\alpha_k^l \in \omega_1$ as follows.

If there are $v, q \in \mathbb{P}_s$ with $q \leq v \leq p_k^{l-1}$ and $\alpha < \omega_1$ such that

(i) q decides $\dot{f}(v)$;
(ii) $v \cap (i_k + 1)^2 = s_k^l$;
(iii) $\pi_1[v] \cap (i_k + 1) = t_k^l$;
(iv) For all $i \in t_k^l$, $v(i) \setminus (i_k + 1) \subseteq x_k^{l-1}(i)$;
(v) $v \cap ((i_k, \omega) \times \omega) \subseteq x_k^{l-1}$;
(vi) $q \Vdash \alpha \in \dot{f}(v) \setminus (\{\beta\} \cup \{\alpha_m^j : m < k,\ j \leq l_m,\ s_m^j = s_k^l \cap (i_m + 1)^2,$ and $t_m^j = t_k^l \cap (i_m + 1)\})$;

then take α_k^l to be the minimum of all α satisfying (vi) for some pair $q \le v \le p_k^{l-1}$ satisfying (i) - (vi), and take some pair $q_k^l \le v_k^l \le p_k^{l-1}$ satisfying (i) - (vi) with α_k^l. In this case, let

$$\begin{aligned} x_k^l = & \bigcup \{\{i\} \times (v_k^l(i) \setminus (i_k+1)) : i \in t_k^l\} \\ & \cup \bigcup \{\{i\} \times x_k^{l-1}(i) : i \in \{i_0, \dots, i_k\} \setminus t_k^l\} \\ & \cup (v_k^l \cap ((i_k, \omega) \times \omega)). \end{aligned} \tag{11}$$

Otherwise, let $p_k^l = p_k^{l-1}$, $x_k^l = x_k^{l-1}$, and $\alpha_k^l = \beta$; and define

$$v_k^l = s_k^l \cup \bigcup \{\{i\} \times x_k^l(i) : i \in t_k^l\} \cup (x_k^l \cap ((i_{k+1}, \omega) \times \omega)). \tag{12}$$

After the l_k steps, let $p_k = p_k^{l_k}$ and $x_k = x_k^{l_k}$. The construction guarantees that for each $k \in \omega$, $p_k \le x_k \le p_{k-1}$, $x_k \subseteq x_{k-1}$, and moreover for each $l \le l_k$, $v_k^l \subseteq x_{-1}$.

Claim 51. Suppose $\alpha_k^l = \beta$ and $v \subseteq v_k^l$ such that $v \le p_k^l$. For each $m \le k$, let $j_m \le l_m$ be the integer satisfying $s_m^{j_m} = v \cap (i_m+1)^2$ and $t_m^{j_m} = \pi_1[v] \cap (i_m+1)$. Then $v \Vdash \dot{f}(v) = \{\alpha_m^{j_m} : m \le k\}$. Moreover, $\alpha_n^j = \beta$ for each $n > k$ and $j \le l_n$ such that $s_n^j \cap (i_k+1)^2 = s_k^l$ and $t_n^j \cap (i_k+1) = t_k^l$.

Proof. Suppose $\alpha_k^l = \beta$, and suppose $v \le p_k^l$, $v \subseteq v_k^l$ and satisfies (ii) and (iii) for k, l. Since $v \subseteq v_k^l$, automatically v also satisfies (iv) and (v) for k, l. Then for each $q \le v$ satisfying (i) and each $\alpha < \omega_1$, (vi) does not hold, meaning that $q \Vdash \dot{f}(v) \subseteq \{\alpha_m^{j_m} : m \le k\}$. Since it is dense below v to have a q for which (i) holds, it follows that $v \Vdash \dot{f}(v) \subseteq \{\alpha_m^{j_m} : m \le k\}$.

On the other hand, for each $m \le k$, since $v \subseteq v_m^{j_m}$, $p_m^{j_m} \Vdash \alpha_m^{j_m} \in \dot{f}(v_k^{j_k})$, and $p \Vdash \dot{f}$ is monotone, it follows that $p_m^{j_m} \Vdash \alpha_m^{j_m} \in \dot{f}(v)$. Hence, $p_k^l \Vdash \{\alpha_m^{j_m} : m \le k\} \subseteq \dot{f}(v)$. Since $v \le p_k^l$, v also forces $\{\alpha_m^{j_m} : m \le k\} \subseteq \dot{f}(v)$.

The second half follows since $\alpha_k^l = \beta$ implies (vi) fails for the pair k, l. □

Next, diagonalize the x_k similarly as in the proof of Theorem 42. Let $k_0 = 0$. Given k_n, for each $j \le k_n$, choose an integer $a_{j,n} \in x_{k_n}(i_j) \setminus (i_{k_n}+1)$. Then take k_{n+1} so that $i_{k_{n+1}} > \max\{a_{j,n} : j \le k_n\}$. Let $x = \{(i_j, a_{j,n}) : n < \omega,\ j \le k_n\}$. Note that $x \subseteq x_{-1}$, $x \le p_{-1}$, and $x \subseteq^* x_k$ for all $k < \omega$. Moreover, for each $n < \omega$, we have $x \setminus (i_{k_n}+1)^2 \subseteq x_{k_n}$.

Claim 52. For each $y \subseteq x$ and each n, if $l \leq l_{k_n}$ satisfies $s^l_{k_n} = y \cap (i_{k_n}+1)^2$ and $t^l_{k_n} = \pi_1[y] \cap (i_{k_n}+1)$, then $x \Vdash \alpha^l_{k_n} \in \dot{f}(y)$.

Proof. Let $y \subseteq x$, and suppose n and l satisfy the hypothesis. Recall that $x \Vdash \alpha^l_{k_n} \in \dot{f}(v^l_{k_n})$. Since x forces $\dot{f}$ to be monotone, it follows that for each $v \subseteq v^l_{k_n}$, we have $x \Vdash \alpha^l_{k_n} \in \dot{f}(v)$. $y \subseteq x$ implies that $y \setminus (i_{k_n}+1)^2 \subseteq x_{k_n}$. This along with the fact that $y \cap (i_{k_n}+1)^2 = v^l_{k_n} \cap (i_{k_n}+1)^2$ and $\pi_1[y] \cap (i_{k_n}+1) = v^l_{k_n} \cap (i_{k_n}+1)$ implies $y \subseteq v^l_{k_n}$. It follows that $x \Vdash \alpha^l_{k_n} \in \dot{f}(y)$. □

Claim 53. Let $n < \omega$ and $l \leq l_n$. If $\alpha^l_{k_n} \neq \beta$, then for all $m < n$, $\alpha^l_{k_n} \neq \alpha^{j_m}_{k_m}$, where $j_m \leq l_m$ is the integer satisfying $s^{j_m}_{k_m} = s^l_{k_n} \cap (i_{k_m}+1)^2$ and $t^{j_m}_{k_m} = t^l_{k_n} \cap (i_{k_m}+1)$.

Proof. Suppose $\alpha^l_{k_n} \neq \beta$. In the construction of $v^l_{k_n}$, (vi) implies that $\alpha^l_{k_n} \notin \{\beta\} \cup \{\alpha^j_k : k < k_n,\ j \leq l_k,\ s^j_k = s^l_{k_n} \cap (i_k+1)^2, \text{ and } t^j_k = t^l_{k_n} \cap (i_k+1)\}$. In particular, $\alpha^l_{k_n} \notin \{\alpha^{j_m}_{k_m} : m < n\}$. □

Now suppose $y \subseteq x$ and y is a standard condition. For each $n < \omega$, let $j_n \leq l_{k_n}$ be the integer such that $y \cap (i_{k_n}+1)^2 = s^{j_n}_{k_n}$ and $\pi_1[y] \cap (i_{k_n}+1) = t^{j_n}_{k_n}$. Suppose that for all n, $\alpha^{j_n}_{k_n} \neq \beta$. Then by Claims 52 and 53, x forces that $\dot{f}(y)$ is infinite. This contradicts that p forces $\dot{f}$ to have range in $[\omega_1]^{<\omega}$. Thus, there is an $n < \omega$ for which $\alpha^{j_n}_{k_n} = \beta$. It follows from Claim 51 that $y \Vdash \dot{f}(y) = \{\alpha^{j_m}_m : m \leq k_n\}$, where $j_m \leq l_m$ is the integer such that $s^{j_m}_m = y \cap (i_m+1)^2$ and $t^{j_m}_m = \pi_1[y] \cap (i_m+1)$.

Thus, x forces that the range of $\dot{f}$ is countable: For each standard $y \subseteq x$, y forces $\dot{f}(y)$ to be a finite subset of $\{\alpha^j_m : m, j < \omega\}$. If $z \subseteq x$ is a nonstandard condition in $\mathbb{P}$, then for each $y \in \mathbb{P}_s$ such that $y \subseteq z$, we have $y \Vdash \dot{f}(y) \supseteq \dot{f}(z)$; so z forces $\dot{f}(z)$ is a finite subset of $\{\alpha^j_m : m, j < \omega\}$. Hence, if $x \in \mathcal{G}$, then in $V[\mathcal{G}]$, every $z \in \mathcal{G}$ has the property that $f(z)$ is a finite subset of $\{\alpha^j_m : m, j < \omega\}$. Therefore, x forces that $\dot{f}$ does not map $\dot{\mathcal{G}}$ cofinally into $[\omega_1]^{<\omega}$. □

5. Tukey Maps on Generic Ultrafilters

The results in Section 5 and in Section 6 are due to Raghavan. These results were obtained in mid-October 2012 during the Fields Institute's thematic program on Forcing and its applications.

Let $\mathcal{X} \subseteq \mathcal{P}(\omega)$. Recall that a map $\varphi : \mathcal{X} \to \mathcal{P}(\omega)$ is said to be *monotone* if $\forall a, b \in \mathcal{X}\, [b \subseteq a \implies \varphi(b) \subseteq \varphi(a)]$. Such a map is said to be *non-zero* if $\forall a \in \mathcal{X}\, [\varphi(a) \neq 0]$.

We will show in this section that any monotone maps defined on the generic ultrafilter $\dot{\mathcal{G}}$ have a "nice" canonical form similar to what is obtained in Section 4 of [14]. This will imply that if $\mathcal{G}$ is $(\mathbf{V}, \mathbb{P})$-generic, then in $\mathbf{V}[\mathcal{G}]$ there are only $\mathfrak{c}$ many ultrafilters that are Tukey below $\mathcal{G}$. This gives yet another proof that the generic ultrafilter is not of the maximal cofinal type for directed sets of size continuum. The proof will go through the corresponding result for the sum of selective ultrafilters indexed by a selective ultrafilter. Recall the following definitions and results which appear in [14].

Definition 54. Let $\mathcal{X} \subseteq \mathcal{P}(\omega)$ and let $\varphi : \mathcal{X} \to \mathcal{P}(\omega)$. Define $\psi_\varphi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by $\psi_\varphi(a) = \{k \in \omega : \forall b \in \mathcal{X}\,[a \subseteq b \implies k \in \varphi(b)]\} = \bigcap\{\varphi(b) : b \in \mathcal{X} \wedge a \subseteq b\}$, for each $a \in \mathcal{P}(\omega)$.

Lemma 55 (Lemma 16 of [14]). *Let $\mathcal{U}$ be basically generated by $\mathcal{B} \subseteq \mathcal{U}$. Suppose moreover that $\forall b_0, b_1 \in \mathcal{B}\,[\, b_0 \cap b_1 \in \mathcal{B}]$. Let $\varphi : \mathcal{B} \to \mathcal{P}(\omega)$ be a monotone map such that $\varphi(b) \neq 0$ for every $b \in \mathcal{B}$. Let $\psi = \psi_\varphi$. Then for every $b \in \mathcal{B}$, $\bigcup_{s\in[b]^{<\omega}} \psi(s) \neq 0$.*

Clearly Definition 54 and Lemma 55 apply to $\mathcal{P}(\omega^2)$ as well with the obvious modifications.

Let $\mathcal{E}$ and $\langle \mathcal{V}_n : n \in \omega\rangle$ be selective ultrafilters. Put $\mathcal{V} = \mathcal{E}\text{-}\sum_n \mathcal{V}_n$. Consider $\mathcal{B}_\mathcal{V} = \{b \subseteq \omega^2 : \pi_1(b) \in \mathcal{E}$ and $\forall n \in \pi_1(b)\,[b(n) \in \mathcal{V}_n]\}$. Then the following is easy to prove. For a more general statement see [14].

Lemma 56. *$\mathcal{V}$ is basically generated by $\mathcal{B}_\mathcal{V}$. Moreover*

$$\forall b_0, b_1 \in \mathcal{B}_\mathcal{V}\,[\, b_0 \cap b_1 \in \mathcal{B}_\mathcal{V}]\,.$$

Thus Lemma 55 can be applied to any sum over a selective ultrafilter of selective ultrafilters. We will do this below to some selective ultrafilters that are generically added to a ground model.

Theorem 57. *Let $\mathcal{G}$ be $(\mathbf{V}, \mathbb{P})$-generic. In $\mathbf{V}[\mathcal{G}]$, let $\varphi : \mathcal{G} \to \mathcal{P}(\omega)$ be a monotone non-zero map. Then there exist $P \subseteq [\omega^2]^{<\omega}$ and $\psi : P \to \omega$ such that*

(1) $\forall a \in \mathcal{G}\left[P \cap [a]^{<\omega} \neq 0\right]$.
(2) $\forall a \in \mathcal{G} \exists b \in \mathcal{G} \cap [a]^\omega \forall s \in P \cap [b]^{<\omega}\,[\psi(s) \in \varphi(b)]$.

Proof. Suppose that the theorem fails. Fix $\dot\varphi \in \mathbf{V}^\mathbb{P}$ such that

$$\Vdash \dot\varphi : \dot{\mathcal{G}} \to \mathcal{P}(\omega) \text{ is a monotone non-zero map.}$$

Fix a standard $p_0 \in \mathbb{P}$ such that for any $P \subseteq [\omega^2]^{<\omega}$ and $\psi : P \to \omega$,

$$p_0 \Vdash \text{“either } \exists a \in \dot{\mathcal{G}}\left[P \cap [a]^{<\omega} = 0\right] \\ \text{or } \exists a \in \dot{\mathcal{G}} \forall b \in \dot{\mathcal{G}} \cap [a]^\omega \exists s \in P \cap [b]^{<\omega}\,[\psi(s) \notin \dot\varphi(b)]\text{''}.$$

Let $\{\langle p_\alpha, A_\alpha, \psi_\alpha\rangle : \alpha < \mathfrak{c}^\mathbf{V}\}$ enumerate all triples $\langle p, A, \psi\rangle$ such that $p \in \mathbb{P}$ and $p \leq p_0$, $A \subseteq [\omega^2]^{<\omega}$, and $\psi : A \to \omega$. Define $\chi : \mathbb{P} \to \mathcal{P}(\omega)$

by $\chi(p) = \{k \in \omega : \exists q \leq p\, [q \Vdash k \in \dot{\varphi}(p)]\}$. Observe that if $q \leq p$, then $q \Vdash p \in \dot{\mathcal{G}}$, and hence $q \Vdash \dot{\varphi}(p)$ is defined. Next, it is easy to check that χ is monotone. Moreover, $p \Vdash \dot{\varphi}(p) \neq 0$. Therefore, for some $q \leq p$ and $k \in \omega$, $q \Vdash k \in \dot{\varphi}(p)$, whence $k \in \chi(p)$. Thus χ is monotone and non-zero. Now build a sequence $\langle q_\alpha : \alpha < \mathfrak{c}^{\mathbf{V}} \rangle$ with the following properties:

(3) $q_\alpha \in \mathbb{P}$, q_α is standard, and $q_\alpha \subseteq p_\alpha$.
(4) either $A_\alpha \cap [q_\alpha]^{<\omega} = 0$ or for some $s \in A_\alpha \cap [q_\alpha]^{<\omega}$, $\psi_\alpha(s) \notin \chi(q_\alpha)$.

To see how to build such a sequence, fix $\alpha < \mathfrak{c}^{\mathbf{V}}$. Let $\mathcal{G}$ be $(\mathbf{V}, \mathbb{P})$-generic with $p_\alpha \in \mathcal{G}$. Since $p_\alpha \leq p_0$, in $\mathbf{V}[\mathcal{G}]$ either there is $a \in \mathcal{G}$ such that $A_\alpha \cap [a]^{<\omega} = 0$ or there is $a \in \mathcal{G}$ such that for all $b \in \mathcal{G} \cap [a]^\omega$, there exists $s \in A_\alpha \cap [b]^{<\omega}$ such that $\psi_\alpha(s) \notin \dot{\varphi}[\mathcal{G}](b)$. Suppose that the first case happens. Let q_α be a standard element of $\mathbb{P}$ such that $q_\alpha \subseteq p_\alpha \cap a$. Then $[q_\alpha]^{<\omega} \cap A_\alpha \subseteq [a]^{<\omega} \cap A_\alpha = 0$.

Now suppose that the second case happens in $\mathbf{V}[\mathcal{G}]$. Working in $\mathbf{V}[\mathcal{G}]$ fix $a \in \mathcal{G}$ as in the second case. Let $b \in \mathcal{G}$ be standard such that $b \subseteq p_\alpha \cap a$. Since $b \in \mathcal{G} \cap [a]^\omega$ there is $s \in A_\alpha \cap [b]^{<\omega}$ such that $\psi_\alpha(s) \notin \dot{\varphi}[\mathcal{G}](b)$. Find $q^* \in \mathcal{G}$ such that (in $\mathbf{V}$) $q^* \Vdash \psi_\alpha(s) \notin \dot{\varphi}(b)$. Let $q \in \mathcal{G}$ be standard so that $q \subseteq b \cap q^*$. Back in $\mathbf{V}$, define q_α as follows. For $n \in \pi_1(s)$, put $q_\alpha(n) = b(n)$. If $n \in \omega - \pi_1(s)$, then $q_\alpha(n) = q(n)$. Note that $q_\alpha \in \mathbb{P}$, it is standard, and $s \subseteq q_\alpha \subseteq b \subseteq p_\alpha$. Moreover, if $\langle n, m \rangle \in q_\alpha - q$, then $n \in \pi_1(s)$. As $\pi_1(s)$ is finite, $q_\alpha - q \in (\mathcal{F}^{\otimes 2})^*$. Therefore, $q_\alpha \leq q$ and $q_\alpha \Vdash \psi_\alpha(s) \notin \dot{\varphi}(b)$. Note that $s \in A_\alpha \cap [q_\alpha]^{<\omega}$. To see that $\psi_\alpha(s) \notin \chi(q_\alpha)$, suppose for a contradiction that there is $r \leq q_\alpha$ such that $r \Vdash \psi_\alpha(s) \in \dot{\varphi}(q_\alpha)$. As $q_\alpha \subseteq b$, $r \Vdash \psi_\alpha(s) \in \dot{\varphi}(b)$, which is impossible. This completes the construction of q_α.

We wish to apply Lemma 55 to a sum of selective ultrafilters indexed by another selective ultrafilters. These selective ultrafilters are obtained generically as follows. Let $\mathcal{E}$ be $(\mathbf{V}, \mathcal{P}(\omega)/\mathcal{F})$-generic with $\pi_1(p_0) \in \mathcal{E}$. In $\mathbf{V}[\mathcal{E}]$ consider the poset $\mathbb{Q} = (\mathcal{P}(\omega)/\mathcal{F})^\omega$. Define $x_0 \in \mathbb{Q}$ as follows. For any $n \in \pi_1(p_0)$, $x_0(n) = p_0(n)$. For any $n \notin \pi_1(p_0)$, $x_0(n) = \omega$. Let G be $(\mathbf{V}[\mathcal{E}], \mathbb{Q})$-generic with $x_0 \in G$. In $\mathbf{V}[\mathcal{E}][G]$ define for each $n \in \omega$, $\mathcal{V}_n = \{x(n) : x \in G\}$. It is clear that $\mathcal{E}$ and the $\mathcal{V}_n$'s are selective ultrafilters in $\mathbf{V}[\mathcal{E}][G]$. Put $\mathcal{V} = \mathcal{E}\text{-}\sum_n \mathcal{V}_n$. Then by Lemma 56 $\mathcal{V}$ is basically generated by $\mathcal{B}_\mathcal{V}$ and $\forall b_0, b_1 \in \mathcal{B}_\mathcal{V}\, [\, b_0 \cap b_1 \in \mathcal{B}_\mathcal{V}]$. Note that $\mathcal{B}_\mathcal{V} \subseteq \mathcal{V} \subseteq \mathbb{P}$. Put $\varphi = \chi \restriction \mathcal{B}_\mathcal{V}$. The hypotheses of Lemma 55 are satisfied. Put $A = \{s \in [\omega^2]^{<\omega} : \psi_\varphi(s) \neq 0\}$. Define $\psi : A \to \omega$ by $\psi(s) = \min(\psi_\varphi(s))$ for any $s \in A$. We claim that there exists $\alpha < \mathfrak{c}^{\mathbf{V}}$ such that $q_\alpha \in \mathcal{B}_\mathcal{V}$ and $A_\alpha = A$ and $\psi_\alpha = \psi$. Suppose for a moment that this claim is true. Applying Lemma 55 to q_α find $s \in [q_\alpha]^{<\omega}$ such that $\psi_\varphi(s) \neq 0$. So $s \in A_\alpha \cap [q_\alpha]^{<\omega}$. Moreover, by the definition of ψ_φ, for any $t \in A_\alpha \cap [q_\alpha]^{<\omega}$, $\psi_\varphi(t) \subseteq \varphi(q_\alpha) = \chi(q_\alpha)$. This means that for every $t \in A_\alpha \cap [q_\alpha]^{<\omega}$, $\psi_\alpha(t) \in \chi(q_\alpha)$. But this contradicts the way q_α was constructed.

To prove the claim first note that A and ψ are in $\mathbf{V}$. In $\mathbf{V}[\mathcal{E}]$ define $D(A, \psi)$ to be the collection of all $y \in \mathbb{Q}$ such that

$$\exists a \in \mathcal{E} \exists \alpha < \mathfrak{c}^{\mathbf{V}} \left[\pi_1(q_\alpha) = a, y \restriction a = q_\alpha, A_\alpha = A, \text{ and } \psi_\alpha = \psi\right].$$

We argue that $D(A, \psi)$ is dense below x_0. Fix $x \in \mathbb{Q}$ with $x \leq x_0$. Note that $x \in \mathbf{V}$. Working in $\mathbf{V}$ define $D(x, A, \psi) = \{\pi_1(q_\alpha) : \alpha < c^{\mathbf{V}}, q_\alpha \subseteq x \restriction \omega, A_\alpha = A, \text{ and } \psi_\alpha = \psi\}$. To see that $D(x, A, \psi)$ is dense below $\pi_1(p_0)$ fix $a \in [\pi_1(p_0)]^\omega$. Put $p = x \restriction a$ and note that $p \in \mathbb{P}$ and that $p \leq p_0$. Therefore, there exists $\alpha < \mathfrak{c}^{\mathbf{V}}$ such that $p_\alpha = p$, $A_\alpha = A$, and $\psi_\alpha = \psi$. Thus $q_\alpha \subseteq x \restriction a \subseteq x \restriction \omega$. Also $\pi_1(q_\alpha) \subseteq a$. Therefore $\pi_1(q_\alpha)$ is as needed. Back in $\mathbf{V}[\mathcal{E}]$, fix $a \in \mathcal{E}$ and $\alpha < \mathfrak{c}^{\mathbf{V}}$ such that $\pi_1(q_\alpha) = a$, $q_\alpha \subseteq x \restriction \omega$, $A_\alpha = A$, and $\psi_\alpha = \psi$. For $n \in a$, put $y(n) = q_\alpha(n)$. For $n \in \omega - a$, put $y(n) = x(n)$. Then $y \in \mathbb{Q}$ and $y \leq x$. It is clear that $y \in \mathbb{Q}$ and that $y \leq x$. Also $y \restriction a = q_\alpha$ and so it is clear that y is as needed. So in $\mathbf{V}[\mathcal{E}][G]$, there is $y \in G$, $a \in \mathcal{E}$, and $\alpha < \mathfrak{c}^{\mathbf{V}}$ such that $\pi_1(q_\alpha) = a$, $y \restriction a = q_\alpha$, $A_\alpha = A$, and $\psi_\alpha = \psi$. Since $\pi_1(q_\alpha) = a \in \mathcal{E}$ and for all $n \in \pi_1(q_\alpha)$, $q_\alpha(n) = y(n) \in \mathcal{V}_n$, $q_\alpha \in \mathcal{B}_{\mathcal{V}}$, and we are done. □

Now we show that the conclusion of Theorem 17 of [14], which was proved there to hold for all ultrafilters that are basically generated by a base that is closed under finite intersections, also holds for the generic ultrafilter $\dot{\mathcal{G}}$.

Definition 58. Let $\mathcal{U}$ be an ultrafilter on ω^2, and let $P \subseteq [\omega^2]^{<\omega} - \{0\}$. We define $\mathcal{U}(P) = \{A \subseteq P : \exists a \in \mathcal{U}\left[P \cap [a]^{<\omega} \subseteq A\right]\}$.

If $\forall a \in \mathcal{U}\left[\left|P \cap [a]^{<\omega}\right| = \omega\right]$, then $\mathcal{U}(P)$ is a proper, non-principal filter on P. The following theorem says that any Tukey reduction from $\dot{\mathcal{G}}$ is given by a Rudin-Keisler reduction from $\dot{\mathcal{G}}(P)$ for some P.

Theorem 59. *Let $\mathcal{G}$ be $(\mathbf{V}, \mathbb{P})$-generic. In $\mathbf{V}[\mathcal{G}]$, let $\mathcal{V}$ be an arbitrary ultrafilter so that $\mathcal{V} \leq_T \mathcal{G}$. Then there is $P \subseteq [\omega^2]^{<\omega} - \{0\}$ such that*

(1) $\forall t, s \in P\, [t \subseteq s \implies t = s]$
(2) $\mathcal{G}(P) \equiv_T \mathcal{G}$
(3) $\mathcal{V} \leq_{RK} \mathcal{G}(P)$

Proof. The proof is almost the same as the proof of Theorem 17 of [14]. Work in $\mathbf{V}[\mathcal{G}]$. Fix an ultrafilter $\mathcal{V}$ and a map $\varphi : \mathcal{G} \to \mathcal{V}$ which is monotone and cofinal in $\mathcal{V}$. Since φ is monotone and non-zero, fix $A \subseteq [\omega^2]^{<\omega}$ and $\psi : A \to \omega$ as in Theorem 57. First we claim that $0 \notin A$. Indeed suppose for a contradiction that $0 \in A$ and let $k = \psi(0)$. Let $e \in \mathcal{V}$ be such that $k \notin e$ and let $a \in \mathcal{G}$ be such that $\varphi(a) \subseteq e$. By (2) of Theorem 57 there is $b \in \mathcal{G} \cap [a]^{\omega}$ such that for all $s \in A \cap [b]^{<\omega}$, $\psi(s) \in \varphi(b)$. However, $0 \in A \cap [b]^{<\omega}$, and so $k = \psi(0) \in \varphi(b) \subseteq \varphi(a) \subseteq e$, a contradiction. Thus $0 \notin A$. Define

$$P = \{s \in A : s \text{ is minimal in } A \text{ with respect to } \subseteq\}.$$

It is clear that $P \subseteq [\omega^2]^{<\omega} - \{0\}$ and that P satisfies (1) by definition.

Next, for any $a \in \mathcal{G}$, $\bigcup\left(P \cap [a]^{<\omega}\right) \in \mathcal{G}$. To see this, fix $a \in \mathcal{G}$, and suppose that $a - \left(\bigcup\left(P \cap [a]^{<\omega}\right)\right) \in \mathcal{G}$. By (1) of Theorem 57, fix $s \in A$ with $s \subseteq a - \left(\bigcup\left(P \cap [a]^{<\omega}\right)\right)$. However there is $t \in P$ with $t \subseteq s$, whence $t = 0$, an impossibility. It follows from this that for each $a \in \mathcal{G}$, $P \cap [a]^{<\omega}$ is infinite.

Next, verify that $\mathcal{G}(P) \equiv_T \mathcal{G}$. Define $\chi : \mathcal{G} \to \mathcal{G}(P)$ by $\chi(a) = P \cap [a]^{<\omega}$, for each $a \in \mathcal{G}$. This map is clearly monotone and cofinal in $\mathcal{G}(P)$. So χ is a convergent map. On the other hand, χ is also Tukey. To see this, fix $\mathcal{X} \subseteq \mathcal{G}$, unbounded in $\mathcal{G}$. Assume that $\{\chi(a) : a \in \mathcal{X}\}$ is bounded in $\mathcal{G}(P)$. So there is $b \in \mathcal{G}$ such that $P \cap [b]^{<\omega} \subseteq P \cap [a]^{<\omega}$ for each $a \in \mathcal{X}$. However $c = \bigcup\left(P \cap [b]^{<\omega}\right) \in \mathcal{G}$. Now, it is clear that $c \subseteq a$, for each $a \in \mathcal{X}$, a contradiction.

Next, check that $\mathcal{V} \leq_{RK} \mathcal{G}(P)$. Define $f : P \to \omega$ by $f = \psi \restriction P$. Fix $e \subseteq \omega$, and suppose first that $f^{-1}(e) \in \mathcal{G}(P)$. Fix $a \in \mathcal{G}$ with $P \cap [a]^{<\omega} \subseteq f^{-1}(e)$. If $e \notin \mathcal{V}$, then $\omega - e \in \mathcal{V}$, and there exists $c \in \mathcal{G}$ with $\varphi(c) \subseteq \omega - e$. By (2) of Theorem 57 fix $b \in \mathcal{G} \cap [a \cap c]^{\omega}$ such that for all $s \in A \cap [b]^{<\omega}$, $\psi(s) \in \varphi(b)$. By (1) of Theorem 57, fix $s \in A \cap [b]^{<\omega}$. Fix $t \subseteq s$ with $t \in P$. Let $k = f(t) = \psi(t)$. As $t \subseteq s \subseteq b \subseteq a$, $t \in P \cap [a]^{<\omega} \subseteq f^{-1}(e)$. Thus $k \in e$. On the other hand, since $t \in A \cap [b]^{<\omega}$, $\psi(t) \in \varphi(b)$. So $k \in \varphi(b) \subseteq \varphi(c) \subseteq \omega - e$, a contradiction.

Next, suppose that $e \in \mathcal{V}$. By cofinality of φ, there is $a \in \mathcal{G}$ such that $\varphi(a) \subseteq e$. Applying (2) of Theorem 57, fix $b \in \mathcal{G} \cap [a]^{\omega}$ such that for all $s \in A \cap [b]^{<\omega}$, $\psi(s) \in \varphi(b)$. Now, if $s \in P \cap [b]^{<\omega}$, then $f(s) = \psi(s) \in \varphi(b) \subseteq \varphi(a) \subseteq e$. Therefore, $P \cap [b]^{<\omega} \subseteq f^{-1}(e)$, whence $f^{-1}(e) \in \mathcal{G}(P)$. □

An immediate corollary of Theorem 59 is that if $\mathcal{G}$ is $(\mathbf{V}, \mathbb{P})$-generic, then in $\mathbf{V}[\mathcal{G}]$, $\{\mathcal{V} : \mathcal{V} \text{ is an ultrafilter on } \omega \text{ and } \mathcal{V} \leq_T \mathcal{G}\}$ has size $\mathfrak{c}$. This is because there are only $\mathfrak{c}$ many sets $P \subseteq [\omega^2]^{<\omega} - \{0\}$, and for each such P there are only $\mathfrak{c}$ many ultrafilters that are RK below $\mathcal{G}(P)$.

6. Generic Ultrafilters are not Basically Generated

In this section, we show that the generic ultrafilter is not basically generated. This is the first consistent example of an ultrafilter that is not basically generated and whose cofinal type is not maximal. Thus our result establishes the consistency of the statement

$$\text{“}\exists\mathcal{U}\left[\mathcal{U}\text{ is not basically generated and }[\omega_1]^{<\omega}\not\leq_T\mathcal{U}\right]\text{''}.$$

Cardinal invariants of the Boolean algebra $\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}$ were considered in [16] and [9]. Recall that for a Boolean algebra $\mathcal{B}$, $\mathfrak{t}(\mathcal{B})$ is the least κ such that there is a sequence $\langle b_\alpha : \alpha < \kappa\rangle$ of elements of $\mathcal{B}\setminus\{0\}$ such that for all $\alpha < \beta < \kappa\,[b_\alpha \geq b_\beta]$ and there does not exist $b \in \mathcal{B}\setminus\{0\}$ such that $\forall\alpha < \kappa\,[b \leq b_\alpha]$. $\mathfrak{h}(\mathcal{B})$ is the distributivity number of $\mathcal{B}$. Szymański and Zhou showed in [16] that $\mathfrak{t}\left(\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}\right)$ is provably equal to ω_1 in ZFC. Hernández-Hernández showed in [9] that it is consistent to have $\mathfrak{h}\left(\mathcal{P}(\omega^2)/\mathcal{F}^{\otimes 2}\right) < \mathfrak{h}\left(\mathcal{P}(\omega)/\mathrm{Fin}\right)$. Our proof of Theorem 60 is partly inspired by these well-known facts, though our construction is different.

For ease of notation, throughout this section, we use $\mathcal{I}$ to denote the dual ideal to $\mathcal{F}^{\otimes 2}$. In other words, $\mathcal{I} = \left(\mathcal{F}^{\otimes 2}\right)^*$. Also, let $a \in [\omega]^\omega$. If $\mathscr{A} \subseteq \mathcal{P}(a)$, $\mathcal{I}(\mathscr{A})$ denotes the ideal on a generated by $\mathscr{A}$ together with the Fréchet ideal on a.

Theorem 60. $\Vdash \dot{\mathcal{G}}$ *is not basically generated.*

Proof. Let $\dot{\mathcal{B}} \in \mathbf{V}^{\mathbb{P}}$ be such that

(1) $\Vdash \dot{\mathcal{B}} \subseteq \dot{\mathcal{G}}$
(2) $\Vdash \forall a \in \dot{\mathcal{G}}\exists b \in \dot{\mathcal{B}}\,[b \subseteq a]$

Let $p^* \in \mathbb{P}$ be standard such that

$$p^* \Vdash \text{“every convergent sequence from } \dot{\mathcal{B}}$$
$$\text{contains an infinite sub-sequence bounded in } \dot{\mathcal{G}}\text{''}$$

Now build two sequences $\{p_\alpha : \alpha < \omega_1\}$ and $\{x_\alpha : \alpha < \omega_1\}$ with the following properties.

(3) $p_\alpha \subseteq p^*$, both p_α and x_α are elements of $\mathbb{P}$, p_α is standard, $p_\alpha \subseteq x_\alpha$, and $p_\alpha \Vdash x_\alpha \in \dot{\mathcal{B}}$.
(4) $\forall\xi < \alpha\,[x_\alpha \leq p_\xi]$ (therefore, $\forall\xi < \alpha\,[p_\alpha \leq x_\alpha \leq p_\xi]$).
(5) $\forall n \in \omega\,[\{\xi < \alpha : |(x_\alpha \cap x_\xi)(n)| = \omega\}$ is finite$]$.
(6) for each $\alpha < \omega_1$ and $n \in \pi_1(p_\alpha)$ let $F(\alpha, n) = \{\xi \leq \alpha : p_\alpha(n) \subseteq^* x_\xi(n)\}$. Note that $\alpha \in F(\alpha, n)$. Let $G(\alpha, n) = \{p_\alpha(n) \cap x_\xi(n) : \xi \in \omega_1 - F(\alpha, n)\}$. Then $\mathcal{I}(G(\alpha, n))$ is a proper ideal on $p_\alpha(n)$ for each $\alpha < \omega_1$ and $n \in \pi_1(p_\alpha)$.

Suppose for a moment that such sequences can be constructed. Let $\delta < \omega_1$ and $\{\alpha_0 < \alpha_1 < \cdots\} \subseteq \delta$ be such that $\{\langle x_{\alpha_i}, p_{\alpha_i}\rangle : i \in \omega\}$ converges to $\langle x_\delta, p_\delta\rangle$ (with respect to the usual topology on $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$). Note that for each $i \in \omega$, $p_\delta \leq p_{\alpha_i}$. Therefore $p_\delta \Vdash \{x_{\alpha_i} : i < \omega\} \cup \{x_\delta\} \subseteq \dot{\mathcal{B}}$. Since $p_\delta \leq p^*$ and since $\mathbb{P}$ does not add any countable sets of ordinals, there exist $X \in [\omega]^\omega$ and a standard $q \in \mathbb{P}$ such that $q \subseteq x_\delta$ and $\forall i \in X\, [q \subseteq x_{\alpha_i}]$. Fix $n \in \pi_1(q)$. Then for each $i \in X$, $q(n) \subseteq x_\delta(n) \cap x_{\alpha_i}(n)$. So $\{\alpha_i : i \in X\} \subseteq \{\alpha < \delta : |(x_\delta \cap x_\alpha)(n)| = \omega\}$, contradicting (5).

To see how to build such sequences, note first that if $\delta \leq \omega_1$ is a limit ordinal and if for each $\beta < \delta$ the sequences $\langle x_\alpha : \alpha < \beta\rangle$ and $\langle p_\alpha : \alpha < \beta\rangle$ do not contain any witnesses violating clauses (3)-(6), then the sequences $\langle x_\alpha : \alpha < \delta\rangle$ and $\langle p_\alpha : \alpha < \delta\rangle$ do not contain any such witnesses either. We may thus concentrate on extending two such given sequences by one step. Therefore, fix $\alpha < \omega_1$ and assume that $\langle x_\xi : \xi < \alpha\rangle$ and $\langle p_\xi : \xi < \alpha\rangle$ are given to us. We only need to worry about finding x_α and p_α. First if $\alpha = 0$, then fix a $(\mathbf{V}, \mathbb{P})$-generic $\mathcal{G}$ with $p^* \in \mathcal{G}$. In $\mathbf{V}[\mathcal{G}]$ fix $x_0 \in \dot{\mathcal{B}}[\mathcal{G}]$ with $x_0 \subseteq p^*$. In $\mathbf{V}$, fix a standard $p_0 \in \mathbb{P}$ such that $p_0 \subseteq x_0$ and $p_0 \Vdash x_0 \in \dot{\mathcal{B}}$. It is clear that (3) is satisfied, and (4)-(6) are trivially true. So assume $\alpha > 0$. Let $\{\xi_n : n \in \omega\}$ enumerate α, possibly with repetitions. For each $n \in \omega$, let $\zeta_n = \max\{\xi_i : i \leq n\}$. Note that for each $i \leq n$, $p_{\zeta_n} \leq p_{\xi_i}$. So it is possible to find a sequence of elements of ω, $\{k_0 < k_1 < \cdots\}$, such that for each $n \in \omega$, $k_n \in \pi_1(p_{\zeta_n})$ and for each $i \leq n$, $p_{\zeta_n}(k_n) \subseteq^* p_{\xi_i}(k_n)$. Define $p \subseteq \omega \times \omega$ as follows. If $m \notin \{k_0 < k_1 < \cdots\}$, then $p(m) = 0$. Suppose $m = k_n$. Put $G(\zeta_n, m, \alpha) = \{p_{\zeta_n}(m) \cap x_\xi(m) : \xi \in \alpha - F(\zeta_n, m)\}$. By (6) $\mathcal{I}(G(\zeta_n, m, \alpha))$ is a proper ideal on $p_{\zeta_n}(m)$. Since this ideal is countably generated, it is possible to find $p(m) \in [p_{\zeta_n}(m)]^\omega$ such that

(7) for all $a \in \mathcal{I}(G(\zeta_n, m, \alpha))$, $|p(m) \cap a| < \omega$
(8) for all $a \in \mathcal{I}(G(\zeta_n, m, \alpha))$, $|(\omega - a) \cap (\omega - p(m))| = \omega$.

Note that $p \in \mathbb{P}$. Furthermore, note that if $i \in \omega$, then for any $n \geq i$, $p(k_n) \subseteq p_{\zeta_n}(k_n) \subseteq^* p_{\xi_i}(k_n)$. Hence for all $\xi < \alpha$, $p \leq p_\xi$. Next, fix $m \in \omega$ and suppose that $(p \cap x_\xi)(m)$ is infinite for some $\xi < \alpha$. Then $m = k_n$ for some (unique) n and $\xi \in F(\zeta_n, m)$. However $F(\zeta_n, m)$ must be a finite set. This is because if $\xi \in F(\zeta_n, m)$, then $\xi \leq \zeta_n$ and $p_{\zeta_n}(m) \subseteq^* x_\xi(m) \cap x_{\zeta_n}(m)$, and so since $m \in \pi_1(p_{\zeta_n})$, $F(\zeta_n, m) \subseteq \{\zeta_n\} \cup \{\xi < \zeta_n : |(x_{\zeta_n} \cap x_\xi)(m)| = \omega\}$, which is a finite set. So for any $m \in \omega$, $\{\xi < \alpha : |(p \cap x_\xi)(m)| = \omega\}$ is finite. Finally, note that $p \subseteq p^*$.

Let $\mathcal{G}$ be $(\mathbf{V}, \mathbb{P})$-generic with $p \in \mathcal{G}$. In $\mathbf{V}[\mathcal{G}]$, let $x_\alpha \in \dot{\mathcal{B}}[\mathcal{G}]$ with $x_\alpha \subseteq p$. In $\mathbf{V}$, let $p_\alpha \in \mathbb{P}$ be standard such that $p_\alpha \subseteq x_\alpha \subseteq p$ and $p_\alpha \Vdash x_\alpha \in \dot{\mathcal{B}}$. It is clear that (3)-(5) are satisfied by $\langle x_\xi : \xi \leq \alpha\rangle$ and $\langle p_\xi : \xi \leq \alpha\rangle$.

We only need to check that (6) is satisfied. There are several cases to consider here. First fix $m \in \omega$ and suppose that $m \in \pi_1(p_\alpha)$. As $p_\alpha \subseteq p$, $m = k_n$ for some (unique) $n \in \omega$. Now if $\xi < \alpha$ and $p_\alpha(m) \subseteq^* x_\xi(m)$, then $p(m) \cap x_\xi(m)$ is infinite and so $\xi \in F(\zeta_n, m)$. On the other hand if $\xi \in F(\zeta_n, m)$, then $p_\alpha(m) \subseteq p(m) \subseteq p_{\zeta_n}(m) \subseteq^* x_\xi(m)$, whence $\xi \in F(\alpha, m)$. Therefore, $F(\alpha, m) = \{\alpha\} \cup F(\zeta_n, m)$. Put $G(\alpha, m, \alpha+1) = \{p_\alpha(m) \cap x_\xi(m) : \xi \in (\alpha+1) - F(\alpha, m)\}$. By (7) it is clear that $\mathcal{I}(G(\alpha, m, \alpha+1))$ is the Frechet ideal on $p_\alpha(m)$. This takes care of α. Next, suppose $\xi < \alpha$ and $m \in \pi_1(p_\xi)$. Put $G(\xi, m, \alpha) = \{p_\xi(m) \cap x_\zeta(m) : \zeta \in \alpha - F(\xi, m)\}$ and put $G(\xi, m, \alpha+1) = \{p_\xi(m) \cap x_\zeta(m) : \zeta \in (\alpha+1) - F(\xi, m)\}$. We know that $\mathcal{I}(G(\xi, m, \alpha))$ is a proper ideal on $p_\xi(m)$ and it is clear that $\mathcal{I}(G(\xi, m, \alpha)) = \mathcal{I}(G(\xi, m, \alpha+1))$ unless $p_\xi(m) \cap x_\alpha(m) \notin \mathcal{I}(G(\xi, m, \alpha))$. Suppose this is the case. In particular, $p_\xi(m) \cap x_\alpha(m)$ is infinite. Since $x_\alpha(m) \subseteq p(m)$ and $p_\xi(m) \subseteq x_\xi(m)$, it follows that $m = k_n$ for some (unique) n and $\xi \in F(\zeta_n, m)$. Moreover, if $\xi < \zeta_n$, then since $\zeta_n \in \alpha - F(\xi, m)$ and since $x_\alpha(m) \subseteq p(m) \subseteq p_{\zeta_n}(m) \subseteq x_{\zeta_n}(m)$, we have that $p_\xi(m) \cap x_\alpha(m) \subseteq p_\xi(m) \cap x_{\zeta_n}(m) \in \mathcal{I}(G(\xi, m, \alpha))$. Therefore, $\xi = \zeta_n$. Thus we need to show that $\mathcal{I}(G(\zeta_n, m, \alpha+1))$ is a proper ideal on $p_{\zeta_n}(m)$. For this it suffices to show that $\omega - (p_{\zeta_n}(m) \cap x_\alpha(m)) \notin \mathcal{I}(G(\zeta_n, m, \alpha))$. Note that since $x_\alpha(m) \subseteq p(m) \subseteq p_{\zeta_n}(m)$, $p_{\zeta_n}(m) \cap x_\alpha(m) = x_\alpha(m)$. However it is clear from (8) that $\omega - x_\alpha(m) \notin \mathcal{I}(G(\zeta_n, m, \alpha))$ and we are done. □

References

[1] Andreas Blass, "A model-theoretic view of some special ultrafilters," in *Logic Colloquium '77* (ed. A. Macintyre, L. Pacholski, and J. Paris) North-Holland, Studies in Logic and the Foundations of Mathematics **96** (1978) 79–90.

[2] Andreas Blass, "Selective ultrafilters and homogeneity," *Ann. Pure Appl. Logic* **38** (1988) 215–255.

[3] Carlos Di Prisco and Stevo Todorčević, "Souslin partitions of products of finite sets," *Adv. Math.*, **176** (2003) 145–173.

[4] Natasha Dobrinen, "Continuous cofinal maps on ultrafilters," (2010) 25 pp., submitted.

[5] Natasha Dobrinen and Stevo Todorčević, "Tukey types of ultrafilters," *Illinois J. Math.* **55** 3 (2011), 907–951.

[6] Natasha Dobrinen and Stevo Todorčević, "A new class of Ramsey-classification theorems and their applications in the Tukey theory of ultrafilters, Part 1," *Trans. Amer. Math. Soc.* 27 pp., to appear.

[7] Natasha Dobrinen and Stevo Todorčević, "A new class of Ramsey-classification theorems and their applications in the Tukey theory of ultrafilters, Part 2," *Trans. Amer. Math. Soc.* 34 pp., to appear.

[8] Ilijas Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), no. 702, xvi+177 pp.

[9] Fernando Hernández-Hernández, "Distributivity of quotients of countable products of Boolean algebras," *Rend. Istit. Mat. Univ. Trieste*, **41** (2009) 27–33.

[10] Kenneth Kunen, "Weak P-points in N^*," in *Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978*, Colloq. Math. Soc. János Bolyai **23**, North-Holland (1980) 741–749.

[11] Krzysztof Mazur, "F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$," *Fund. Math.* **138** (1991) 103–111.

[12] José G. Mijares and Jesús Nieto, "Local Ramsey theory. An abstract approach," arXiv:0712.2393v1 (2013) 11pp.

[13] Dilip Raghavan, "The generic ultrafilter added by $(\mathrm{Fin}\times\mathrm{Fin})^+$" arXiv:1210.7387 (2012) 7 pp.

[14] Dilip Raghavan and Stevo Todorčević, "Cofinal types of ultrafilters," *Ann. Pure Appl. Logic* **163** (2012) 185–199.

[15] Sławomir Solecki and Stevo Todorčević, "Cofinal types of topological directed orders," *Ann. Inst. Fourier (Grenoble)* **54** (2004) 1877–1911.

[16] Andrzej Szymański and Hao Xua Zhou, "The behavior of ω^{2*} under some consequences of Martin's axiom," in *General Topology and Its Relations to Modern Analysis and Algebra. V*, (Prague, 1981), (ed. J. Novák) Sigma Ser. Pure Math. **3**, Heldermann Verlag (1983) 577–584.

[17] Stevo Todorčević, "Gaps in analytic quotients," *Fund. Math.*, **156** (1998) 85–97.

[18] The Online Encyclopedia of Integer Sequences, `http://oeis.org`.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109–1043, U.S.A.

E-mail address: `ablass@umich.edu`

URL: `http://www.math.lsa.umich.edu/~ablass`

Department of Mathematics, University of Denver, 2360 Gaylord St., Denver, CO 80208 U.S.A.

E-mail address: `natasha.dobrinen@du.edu`

URL: `http://web.cs.du.edu/~ndobrine`

Department of Mathematics, National University of Singapore, Singapore 119076

E-mail address: `raghavan@math.nus.edu.sg`

URL: `http://www.math.toronto.edu/~raghavan`