

RAINBOW RAMSEY SIMPLE STRUCTURES

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ABSTRACT. A relational structure R is *rainbow Ramsey* if for every finite induced substructure C of R and every colouring of the copies of C with countably many colours, such that each colour is used at most k times for a fixed k , there exists a copy R^* of R so that the copies of C in R^* use each colour at most once.

We show that a class of homogeneous binary relational structures generalizing the Rado graph are rainbow Ramsey. Via compactness this then implies that for all finite graphs B and C and $k \in \omega$, there exists a graph A so that for every colouring of the copies of C in A such that each colour is used at most k times, there exists a copy B^* of B in A so that the copies of C in B^* use each colour at most once.

1. INTRODUCTION

There is an extensive literature concerning colouring problems of the following type: Given conditions on the colouring function, conclude that the restriction of the colouring function to a specific subset of its domain is injective. In particular, this has been widely studied for finite graphs, and then mostly complete graphs, and edge colouring; see [4] and [6] for two survey papers. Many of those problems and results are in analogy to standard Ramsey type problems and results, for which then the restriction of the colouring function to a specific subset of its domain is asked to be constant. Those investigations have given rise to the notions of anti-Ramsey numbers and restricted Ramsey numbers among others, and investigations on these anti-Ramsey numbers for certain pairs of graphs, in particular finding upper bounds, asymptotic behavior and complexity. As is outlined in the two surveys cited above, problems in the context of rainbow Ramsey have been studied extensively for k -bounded colouring of pairs of natural numbers, from finding exact numbers, to finding growth rates, to investigations of the relative strength of the statement to other Ramsey properties; see [2] for relative strength investigation. There are a few articles extending the work to hypergraphs and a few to infinite graphs. See [8] and [5].

However, the literature has not addressed the generalization from colouring of edges to colouring of copies of arbitrary finite graphs. In this work, we undertake this direction by investigating rainbow copies for k -bounded colouring of copies of a fixed finite graph within the Rado graph and other homogeneous structures.

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For B a relational structure we often denote by B the set of its elements and sometimes use B to denote the structure as well as the set of its elements. For $S \subseteq B$ let $B \downarrow S$ be the substructure of B induced by S . For B and C relational structures denote by

$$\binom{B}{C}$$

the set of induced substructures of B isomorphic to C . The elements of $\binom{B}{C}$ are the *copies of B in B* . An *embedding* of B is an isomorphism of B to a copy of B in B .

Since we will be concerned with countable structures then we make the following definitions using ω , but they can easily be generalized to higher cardinals. Let A and C be relational structures and k a natural number. A function $\gamma : \binom{A}{C} \rightarrow \omega$ is called *k -bounded* if $|\gamma^{-1}(i)| \leq k$ for every $i \in \omega$. A weaker notion and more appropriate for the arguments in this paper is that of a *k -delta function*. By a *δ -system of copies of C in A* we mean a subset \mathcal{S} of $\binom{A}{C}$ for which $X \setminus \bigcap_{Y \in \mathcal{S}} Y$ is a singleton set for all $X \in \mathcal{S}$ (This is a special case of the more general set-theoretic notion of δ -system.) Then we call a function $\gamma : \binom{A}{C} \rightarrow \omega$ a *k -delta function* if $\gamma^{-1}(n)$ does not contain a δ -system $\mathcal{S} \subseteq \binom{A}{C}$ having more than k elements. Thus clearly every k -bounded function is k -delta.

Following the notation of Fred Galvin, the delta and rainbow Ramsey properties we are interested are defined as follows.

Definition 1.1. Let A , B and C be relational structures and k a natural number. The (standard) arrow

$$(1) \quad A \rightarrow \binom{B}{C}_k$$

means that for every colouring $\gamma : \binom{A}{C} \rightarrow k$, there exists a $B^* \in \binom{A}{B}$ so that γ is constant on $\binom{B^*}{C}$.

The arrow

$$(2) \quad A \xrightarrow{\delta} \binom{B}{C}_k$$

means that for every k -delta colouring $\gamma : \binom{A}{C} \rightarrow \omega$, there exists a $B^* \in \binom{A}{B}$ so that γ is one-to-one on $\binom{B^*}{C}$.

The arrow

$$(3) \quad A \xrightarrow{*} \binom{B}{C}_k$$

means that for every k -bounded colouring $\gamma : \binom{A}{C} \rightarrow \omega$, there exists a $B^* \in \binom{A}{B}$ so that γ is one-to-one on $\binom{B^*}{C}$.

Thus clearly (2) implies (3), and it is easy to see that (1) implies (3), but we do not know whether (1) implies (2).

A countably infinite relational structure R is *rainbow Ramsey* if for every finite substructure C and every $k \in \omega$ the relation $R \xrightarrow{*} \binom{R}{C}_k$ holds; similarly R is *delta rainbow Ramsey* if for every finite substructure C and every $k \in \omega$ the relation $R \xrightarrow{\delta} \binom{R}{C}_k$ holds. A class \mathcal{F} of finite relational structures is *rainbow Ramsey* if for all structures B and C in \mathcal{F} and every $k \in \omega$, there exists a structure $A \in \mathcal{F}$ for

which $A \xrightarrow{*} (B)_k^C$ holds; similarly for *delta rainbow Ramsey*. If R is a countably infinite (delta) rainbow Ramsey relational structure, then the class of finite structures isomorphic to an induced substructure of R (the age of R , written $\text{Age}(R)$), is also (delta) rainbow Ramsey; see Corollary 5.1.

Relational structures A and B are called *equimorphic*, or said to be *siblings*, if there exists an isomorphic injection of A into B and an isomorphic injection of B into A . Note that A and B are siblings exactly when $\binom{A}{B} \neq \emptyset$ and $\binom{B}{A} \neq \emptyset$. If A and B are equimorphic relational structures, then A is (delta) rainbow Ramsey if and only if B is (delta) rainbow Ramsey; see Lemma 5.2.

We call a binary relational structure a *simple* structure if for each of its symmetric relations the structure is a simple graph and for each of its directed relations the structure is the orientation of a simple graph; see Definition 3.1 for a more precise definition. Thus every simple graph is an example of a simple binary structure, but so is every ordered simple graph.

A structure is *homogeneous*, also often called *ultrahomogeneous*, if every isomorphism between finite substructures extends to an automorphism of the entire structure; see [3] for an introduction to homogeneous structures. In this paper we prove that countable simple homogeneous structures are (delta) rainbow Ramsey, as are other simple binary relational structures; see Theorem 5.2 for a more precise statement. It follows that the age of any such countable simple structure is also (delta) rainbow Ramsey, see Corollary 5.1. In particular this shows that countable homogeneous relational structures without the Ramsey property may still be rainbow Ramsey, for example the Rado graph.

That such classes of finite simple binary relational structures are rainbow Ramsey also follows via a similar construction as given here from the Nešetřil-Rödl partition theorem; see [9]. In fact, any Fraïssé class of finite relational structures with free amalgamation and the Ramsey property is rainbow Ramsey, by a similar argument.

To compare we conclude the introduction by pointing out an interesting independent rainbow Ramsey result in the case of the first uncountable cardinal. First it was shown by Sierpiński that the well-known Ramsey Theorem does not generalize to ω_1 , and later Fred Galvin showed that the Continuum Hypothesis implies $\omega_1 \not\xrightarrow{*} (\omega_1)_2^2$. Finally, in [12], Todorćević established the relative consistency of a strong form of rainbow Ramsey; namely $\omega_1 \xrightarrow{*} (\omega_1)_{<\aleph_0}^2$, that is for every colouring $\gamma : [\omega_1]^2 \rightarrow \omega_1$ such that $\gamma^{-1}(i)$ is finite for all i , there exists an uncountable $B \subseteq \omega_1$ so that γ is one-to-one on $[B]^2$.

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2. TREES OF SEQUENCES

Let \mathfrak{T}_ω be the tree consisting of all finite sequences with entries in ω . The order of the tree \mathfrak{T}_ω , denoted by \subseteq , is given by sequence extension. That is $x = \langle x(0), x(1), \dots, x(n-1) \rangle \subseteq y = \langle y(0), y(1), \dots, y(m-1) \rangle$ if $n \leq m$ and $x(i) = y(i)$ for all $i \in n$. The length $|x|$ of the sequence $x = \langle x(0), x(1), \dots, x(n-1) \rangle$ is n . The *intersection* $x \wedge y$ of two sequences is the longest sequence z with $z \subseteq x$ and $z \subseteq y$.

For $|x| < |y|$ the number $y(|x|)$ is the *passing number* of y at x . Let $S \subseteq \mathfrak{T}_\omega$. The sequence $y \in S$ is an *immediate S -successor* of the sequence $x \in S$ if $x \subseteq y$ and there is no sequence $z \in S$ with $x \subsetneq z \subsetneq y$. The S -degree of $x \in S$ is the number of immediate S -successors of x . Finally we denote $\text{levels}(S) = \{|x| : x \in S\}$.

We further write $x \prec y$ if x and y are incomparable under \subseteq and the passing number of x is smaller than the passing number of y at their intersection. That is:

$$x(|x \wedge y|) < y(|x \wedge y|).$$

For \subseteq incomparable sequences the relation \prec agrees with the lexicographic order on the tree \mathfrak{T}_ω .

As in the third paragraph of the preliminaries section of [11] and Definition 4.1 of [11], we define:

Definition 2.1. A subset T of \mathfrak{T}_ω is an ω -tree if it is not empty, closed under initial segments, has no endpoints and the T -degree of every sequence $x \in T$ is finite.

For $2 \leq \mathfrak{d} \in \omega$, let $\mathfrak{T}_\mathfrak{d} \subseteq \mathfrak{T}_\omega$ be the ω -tree consisting of all finite sequences with entries in \mathfrak{d} . Then $\mathfrak{T}_\mathfrak{d}$ is actually a wide ω -tree according to Definition 4.1 of [11], but this will not be needed here. The root of $\mathfrak{T}_\mathfrak{d}$ is the empty sequence \emptyset .

Let $S \subseteq \mathfrak{T}_\mathfrak{d}$ be a set of sequences. Following Definitions 2.2 and 3.2 of [11] we define: The set S is *transversal* if no two different elements of S have the same length and it is an *antichain* if no two different elements of S are ordered under the \subseteq relation. The closure of S is the set of intersections of elements of S . A subset $T \subseteq \mathfrak{T}_\mathfrak{d}$ is cofinal if for every $x \in \mathfrak{T}_\mathfrak{d}$ there is a $y \in T$ with $x \subseteq y$.

The set S is *diagonal* if it is an antichain and the closure of S is transversal and the degree of every element of closure S within the closure of S is at most 2. The set S is *strongly diagonal* if it is diagonal and if for all $x, y, z \in S$ with $x \neq y$:

- (1) If $|x \wedge y| < |z|$ and $x \wedge y \not\subseteq z$ then $z(|x \wedge y|) = 0$.
- (2) $x(|x \wedge y|) \in \{0, 1\}$.

Definition 2.2. A function f mapping a set $R \subseteq \mathfrak{T}_\mathfrak{d}$ of sequences into $\mathfrak{T}_\mathfrak{d}$ has the properties of Order (order bipreservation), Level (level bipreservation), Level-imp (level preservation), Pnp (passing number preservation), Pnp-strong (strong passing number preservation) or Lexico (lexicographic order preservation) respectively, if for all $x, y, z, u \in R$:

- Order:** $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
- Level:** $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
- Level-imp:** $|x \wedge y| < |z \wedge u|$ implies $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
- Pnp:** If $|z| > |x|$ then $z(|x|) = f(z)(|f(x)|)$.
- Pnp-strong:** If $|z| > |x \wedge y|$ then $z(|x \wedge y|) = f(z)(|f(x) \wedge f(y)|)$.
- Lexico:** If $x \prec y$ then $f(x) \prec f(y)$.

According to Definition 3.1 of [11], we define:

Definition 2.3. Let R and S be two subsets of \mathfrak{T}_ω . A bijection f of R to S is a *strong similarity* of R to S if it satisfies the properties of Order and Level and Pnp-strong (and also Lexico which is implied by Pnp-strong).

The sets R and S are strongly similar, written $R \stackrel{s}{\sim} S$, if there is a (necessarily unique) strong similarity of R to S .

Then for T a set of sequences, the set $\text{Sims}_T(R)$ is the equivalence class of all subsets S of T with $R \stackrel{s}{\sim} S$.

Observe that a strong similarity maps sequences of the same length to sequences of the same length. It can be viewed as “stretching” or in the inverse then “compressing” length of sequences, but preserving all shapes and passing numbers.

3. BINARY STRUCTURES ENCODED BY $\mathfrak{T}_\mathfrak{d}$

We now define *simple* structures as generalizations of simple graphs. The context should not make any confusion with the model theory notion of simple theories. The main focus will be rainbow Ramsey properties of these simple structures.

Definition 3.1. A binary relational structure R on a set R with binary relations E_0, E_1, \dots, E_{n-1} and $n \geq 2$ is called *simple* if:

- (1) Each of the binary relations E_i is irreflexive.
- (2) There is a number $m \leq n$, the *symmetry number* of R so that:
 - (a) $E_i(x, y)$ implies $E_i(y, x)$ for all $x, y \in R$ and all $i \in m$.
 - (b) $E_i(x, y)$ implies $\neg E_i(y, x)$ for all $x, y \in R$ and all $m \leq i \in n$.
- (3) For all $i, j \in n$ with $i \neq j$: $E_i(x, y)$ implies $\neg E_j(x, y)$ and $\neg E_j(y, x)$.
- (4) For all (x, y) , there is an (unique) $i \in n$ with $E_i(x, y)$ or $E_i(y, x)$.

That is, the set R together with only one of the relations E_i is a simple graph if $i \in m$, and is an orientation of a simple graph if $m \leq i \in n$. Any two different such graphs do not have overlapping edges. Note that every binary relational structure having only irreflexive relations can be encoded as a simple binary relational structure.

For $m \leq n$ in ω , let $\mathcal{S}(n, m)$ denote the class of all simple binary relational structures with set of binary relations $\{E_0, E_1, \dots, E_n\}$ and symmetry number m . Let $\mathcal{A}(n, m) \subseteq \mathcal{S}(n, m)$ be the subclass of the finite structures in $\mathcal{S}(n, m)$. The class $\mathcal{A}(n, m)$ is an age with amalgamation whose Fraïssé limit is denoted by $\mathbb{U}(n, m)$, and the homogeneous structures of the form $\mathbb{U}(n, m)$ are the *random simple binary homogeneous structures*. Thus note for example that $\mathbb{U}(2, 2)$ is the random graph (it has adjacent and non-adjacent two element subsets as set of relations), $\mathbb{U}(1, 0)$ is the random Tournament, $\mathbb{U}(2, 1)$ is the random oriented graph, $\mathbb{U}(1, 1)$ is the complete graph K_ω , $\mathbb{U}(0, 0)$ is the relational structure having no relation, $\mathbb{U}(3, 3)$ may be seen as a graph having heavy and light edges, and finally $\mathbb{U}(3, 2)$ as a graph having symmetric and oriented edges. The cases $(n, m) = (0, 0)$ and $(n, m) = (1, 1)$ do not quite fit into the general notational framework developed subsequently and will be dealt with later on. Note for later convenience that for $m \leq n$, the inequality $(0, 0) \neq (n, m) \neq (1, 1)$ holds if and only if $m + 2(n - m) \geq 2$.

Let $m \leq n \in \omega$ and $\mathfrak{d} = m + 2(n - m) \geq 2$. The tree $\mathfrak{T}_\mathfrak{d}$ encodes a simple binary relational structure $\mathbb{T}(n, m) \in \mathcal{S}(n, m)$ on the set of sequences in $\mathfrak{T}_\mathfrak{d}$, in which for all $x, y \in \mathfrak{T}_\mathfrak{d}$ with $x \neq y$:

- (1) If $|x| < |y|$ then let for the passing number $p = y(|x|)$:
 - (a) If $p < m$ then $E_p(x, y)$ and $E_p(y, x)$.
 - (b) If $p \geq m$ and $p - m$ even then $E_{m + \frac{p-m}{2}}(x, y)$.
 - (c) If $p \geq m$ and $p - m$ odd then $E_{m + \frac{p-m-1}{2}}(y, x)$.
- (2) If $|x| = |y|$ and $x \prec y$, then $E_0(x, y)$ if $m = 0$, and $E_0(x, y)$ and $E_0(y, x)$ if $m > 0$.

For R a relational structure and F a subset of its domain R , let $R \downarrow F$ be the substructure of R induced by F . Note that if f is a similarity of a subset F of $\mathfrak{T}_\mathfrak{d}$ to a subset G of $\mathfrak{T}_\mathfrak{d}$, then f is an isomorphism of $\mathbb{T}(n, m) \downarrow F$ to $\mathbb{T}(n, m) \downarrow G$.

In accordance with Definition 3.3 of [11]:

Definition 3.2. An injection $f : \mathfrak{T}_\mathfrak{d} \rightarrow \mathfrak{T}_\mathfrak{d}$ is a *strong diagonalization* of $\mathfrak{T}_\mathfrak{d}$ if the image of f is a strongly diagonal subset of $\mathfrak{T}_\mathfrak{d}$ and if f satisfies the properties Level-imp, Pnp and Lexico.

According to Theorem 4.1 of [11], there exists a strong diagonalization of $\mathfrak{T}_\mathfrak{d}$ into $\mathfrak{T}_\mathfrak{d}$. We fix for every $2 \leq \mathfrak{d} \in \omega$ such a strong diagonalization $\Delta_\mathfrak{d}$ and denote by $\mathfrak{D}_\mathfrak{d}$ the image of the strong diagonalization $\Delta_\mathfrak{d}$. For $\mathfrak{d} = m + 2(n - m) \geq 2$, let $\mathbb{D}(n, m) = \mathbb{T}(n, m) \downarrow \mathfrak{D}_\mathfrak{d}$. Note that $\Delta_\mathfrak{d}$ is an isomorphism of $\mathbb{T}(n, m)$ to $\mathbb{D}(n, m)$. Hence $\mathbb{D}(n, m)$ is equimorphic to $\mathbb{T}(n, m)$, but observe it is itself not a homogeneous structure.

Theorem 3.1. *Let $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$. The simple binary relational structures $\mathbb{U}(n, m)$, $\mathbb{T}(n, m)$ and $\mathbb{D}(n, m)$ are all equimorphic.*

Proof. Because $\mathbb{T}(n, m)$ is equimorphic to $\mathbb{D}(n, m)$ it suffices to show that $\mathbb{T}(n, m)$ and $\mathbb{U}(n, m)$ are equimorphic.

Every finite induced substructure of $\mathbb{T}(n, m)$ is an element of $\mathcal{A}(n, m)$. Hence, and because the homogeneous structure $\mathbb{U}(n, m)$ is universal, there exists an isomorphism of $\mathbb{T}(n, m)$ into $\mathbb{U}(n, m)$.

Let $\mathfrak{d} = m + 2(n - m)$ and S be a transversal and cofinal subset of $\mathfrak{T}_\mathfrak{d}$. Let F be a finite subset of S . Then for every function $f : F \rightarrow \mathfrak{d}$ there exists a sequence $y \in S$ so that the passing number $y(|x|) = f(x)$ for every $x \in F$. This then translates to the fact that the substructure of $\mathbb{T}(n, m)$ induced by S has the mapping extension property with respect to the age $\mathcal{A}(n, m)$. Hence the structure induced by V is homogeneous with age $\mathcal{A}(n, m)$ and hence isomorphic to $\mathbb{U}(n, m)$. \square

4. APPLICATION OF MILLIKEN'S THEOREM

As in [11], the notions of strong subtrees and Milliken's Theorem will play a crucial role.

Definition 4.1. The set $T \subseteq \mathfrak{T}_\mathfrak{d}$ is *closed by levels* if $s \in T$ whenever there exist $t \in T$ with $s \subseteq t$ and $y \in T$ with $|s| = |y|$. Let T be a meet closed set of sequences which is also closed by levels and let $n \in \omega + 1$. The set $S \subseteq T$ is an element of $\text{Str}^n(T)$ if:

- (1) $|\text{levels}(S)| = n$.
- (2) S is meet closed and closed by levels.
- (3) For all $s \in S$, the degree of s in S is equal to the degree of s in T .

The elements of $\bigcup_{n \in \omega+1} \text{Str}^n(T)$ are the strong subtrees of T .

Let $T \in \text{Str}^\omega(\mathfrak{T}_\mathfrak{d})$ and $S \in \text{Str}^\omega(T)$. Then one obtains $\mathfrak{T}_\mathfrak{d}$ back by removing from every sequence $s \in S$ the entries whose indices are not in $\text{levels}(S)$. Moreover there exists a unique strong similarity $f : T \rightarrow S$, and this similarity f maps every level of T into a level of S , preserving the order, via length, of the levels. Conversely, for every strong similarity f of T into T , the set $f[T] \in \text{Str}^\omega(T)$. If further $R \in \text{Str}^\omega(S)$ (and $S \in \text{Str}^\omega(T)$), then $R \in \text{Str}^\omega(T)$. Finally, if we let $T^* := \{x \in \mathfrak{T}_\mathfrak{d} \mid \exists y \in T (x \subseteq y)\}$, then T^* is an ω -tree, and if $T^* = S^*$ then $T = S$.

Hence the following Lemma 4.1 is a direct consequence of Theorem 5.2 of [11], which is in itself an application of Milliken's Theorem on partitioning strong subtrees.

Lemma 4.1. *Let f be a strong similarity of $\mathfrak{T}_\mathfrak{d}$ to $T = f[\mathfrak{T}_\mathfrak{d}] \subseteq \mathfrak{T}_\mathfrak{d}$, F be a finite meet closed subset of T , and $m \in \omega$.*

Then for any colouring $\gamma : \text{Sims}_T(F) \rightarrow m \in \omega$, there exists a strong similarity function f of T to a tree $S \subseteq T$ so that γ is constant on $\text{Sims}_S(F)$.

The following thus follows from the fact that if F and G are two finite and strongly diagonal sets with $F \overset{s}{\sim} G$ and with $\text{closure}(F) = \text{closure}(G)$, then $F = G$.

Corollary 4.1. *Let f be a strong similarity of $\mathfrak{T}_\mathfrak{d}$ to $T = f[\mathfrak{T}_\mathfrak{d}] \subseteq \mathfrak{T}_\mathfrak{d}$, F be a finite and strongly diagonal subset of T , and $m \in \omega$.*

Then for any colouring $\gamma : \text{Sims}_T(F) \rightarrow m \in \omega$, there exists a strong similarity function f of T to a tree $S \subseteq T$ so that γ is constant on $\text{Sims}_S(F)$.

By repeated application of the above Lemma 4.1 and Corollary 4.1 we obtain:

Theorem 4.1. *Let $\{F_i \mid i \in p \in \omega\}$ be a finite set of finite meet closed subsets of $\mathfrak{T}_\mathfrak{d}$ so that $F_i \not\overset{s}{\sim} F_j$ for $i \neq j$. Let $m_i \in \omega$ for all $i \in p$.*

Then for any set $\gamma_i : \text{Sims}_{\mathfrak{T}_\mathfrak{d}}(F_i) \rightarrow m_i$ of colouring functions, there exists a strong similarity function f of $\mathfrak{T}_\mathfrak{d}$ to a tree $T \subseteq \mathfrak{T}_\mathfrak{d}$ so that each one of the colouring functions γ_i is constant on $\text{Sims}_T(F_i)$.

Corollary 4.2. *Let $\{F_i \mid i \in p \in \omega\}$ be a finite set of strongly diagonal subsets of $\mathfrak{T}_\mathfrak{d}$ so that $F_i \not\overset{s}{\sim} F_j$ for $i \neq j$. Let $m_i \in \omega$ for all $i \in p$.*

Then for any set $\gamma_i : \text{Sims}_{\mathfrak{T}_\mathfrak{d}}(F_i) \rightarrow m_i$ of colouring functions, there exists a strong similarity function f of $\mathfrak{T}_\mathfrak{d}$ to a tree $T \subseteq \mathfrak{T}_\mathfrak{d}$ so that each one of the colouring functions γ_i is constant on $\text{Sims}_T(F_i)$.

Hence we can apply these results to $\mathfrak{D}_\mathfrak{d}$.

Theorem 4.2. *Let $\{F_i \mid i \in p \in \omega\}$ be a finite set of finite subsets of $\mathfrak{D}_\mathfrak{d}$ so that $F_i \not\overset{s}{\sim} F_j$ for $i \neq j$. Let $m_i \in \omega$ for all $i \in p$.*

Then for any set $\gamma_i : \text{Sims}_{\mathfrak{D}_\mathfrak{d}}(F_i) \rightarrow m_i$ of colouring functions, there exists a strong similarity function g of $\mathfrak{D}_\mathfrak{d}$ to a tree $D \subseteq \mathfrak{D}_\mathfrak{d}$ so that each one of the colouring functions γ_i is constant on $\text{Sims}_D(F_i)$.

Proof. For every $i \in p$, let $\delta_i : \text{Sims}_{\mathfrak{T}_\mathfrak{d}}(F_i) \rightarrow m_i$ be given by $\delta_i(F'_i) = \gamma_i(\Delta_\mathfrak{d}(F'_i))$ for $F'_i \in \text{Sims}_{\mathfrak{T}_\mathfrak{d}}(F_i)$. Observe that $F_i \overset{s}{\sim} F'_i$ if and only if $F_i \overset{s}{\sim} \Delta_\mathfrak{d}(F'_i)$ since F_i and hence F'_i are strongly diagonal.

According to Corollary 4.2 there exists a strong similarity function f of $\mathfrak{T}_\mathfrak{d}$ to a tree $T \subseteq \mathfrak{T}_\mathfrak{d}$ so that each one of the colouring functions δ_i is constant on $\text{Sims}_T(F_i)$. The restriction of Δ_k to a strongly diagonal subset S of $\mathfrak{T}_\mathfrak{d}$ is a strong similarity of S . Hence the function g , which is the restriction of $\Delta_\mathfrak{d} \circ f$ to $\mathfrak{D}_\mathfrak{d}$, is a strong similarity of $\mathfrak{D}_\mathfrak{d}$ to a subset D of $\mathfrak{D}_\mathfrak{d}$. \square

Let $n \geq 2$ and $\mathfrak{d} = m + 2(n - m) \geq 2$. Let R and S be two finite induced substructures of $\mathbb{D}(n, m)$ with domains $R, S \subseteq \mathfrak{D}_\mathfrak{d}$. Thus is $R = \mathbb{D}(n, m) \downarrow R$ and $S = \mathbb{D}(n, m) \downarrow S$. Then R and S are *strongly similar* (as substructures), written again $R \overset{s}{\sim} S$, if their domains are *strongly similar*, that is $R \overset{s}{\sim} S$. Note that if $R \overset{s}{\sim} S$ then the strong similarity of R to S is an isomorphism of the binary structure R to the binary structure S . For S an induced substructure of $\mathbb{D}(n, m)$ and F a finite induced substructure of S let $\text{Sims}_S(F)$ be the set of all induced substructures F' of S with $F \overset{s}{\sim} F'$. Then $\text{Sims}_S(F)$ is the set of all $F' \in \binom{S}{F}$ with $F \overset{s}{\sim} F'$. Hence:

Corollary 4.3. *Let $n \geq 2$ and $\mathfrak{d} = m + 2(n - m)$. Let $\{F_i \mid i \in p \in \omega\}$ be a finite set of finite induced substructures of $\mathbb{D}(n, m)$ so that $F_i \not\cong F_j$ for $i \neq j$. Let $m_i \in \omega$ for all $i \in p$.*

Then for any set $\gamma_i : \text{Sims}_{\mathbb{D}(n, m)}(F_i) \rightarrow m_i$ of colouring functions, there exists a strong similarity function g of $\mathbb{D}(n, m)$ to a copy D of $\mathbb{D}(n, m)$ so that each one of the colouring functions γ_i is constant on $\text{Sims}_D(F_i)$.

5. $\mathbb{T}(n, m)$ AND $\mathbb{U}(n, m)$ AND $\mathbb{D}(n, m)$ ARE RAINBOW RAMSEY.

In this section we show that simple relational structures are k -delta rainbow Ramsey for any k , and thus rainbow Ramsey.

First we show that it is sufficient to show the result for $k = 2$.

Lemma 5.1. *Let R be a relational structure and let $C \in \text{Age}(R)$.*

If $R \xrightarrow{\delta} (R)_2^C$, then $R \xrightarrow{\delta} (R)_k^C$, for all $k \geq 2$.

Proof. Let $C \in \text{Age}(R)$. The proof is by induction on $k \geq 2$, where the base case is given by the hypothesis.

Thus assume that $R \xrightarrow{\delta} (R)_k^C$, and let $f : \binom{R}{C} \rightarrow \omega$ be a $k+1$ -delta colouring. Note that if \mathcal{S} is a δ -system of copies of C , then $|\bigcap_{Y \in \mathcal{S}} Y| = |C| - 1$. Thus, each maximal δ -system \mathcal{S} is uniquely determined by the structure $\bigcap_{Y \in \mathcal{S}} Y$; for \mathcal{S} is the collection of all copies X of C in $\text{Age}(R)$ for which $\bigcap_{Y \in \mathcal{S}} Y$ is an induced substructure of X .

Enumerate the members of $\binom{R}{C}$ as C_n , $n \in \omega$, enumerate the maximal δ -systems of copies of C as \mathcal{S}_i , $i \in \omega$, and let $\ell \in \omega$ be fixed. If there are $k+1$ members of \mathcal{S}_0 with f -colour ℓ , then choose $n(0, \ell)$ to be the least n such that $C_n \in \mathcal{S}_0$ and $f(C_n) = \ell$, and let $N_{0, \ell} = \{n(0, \ell)\}$. Otherwise, let $N_{0, \ell} = \emptyset$.

Suppose we have chosen $N_{i, \ell}$. If either (a) there are $k+1$ members of \mathcal{S}_{i+1} with f -colour ℓ and there is an $n \in N_{i, \ell}$ such that $C_n \in \mathcal{S}_{i+1}$ and $f(C_n) = \ell$, or (b) there are at most k members of \mathcal{S}_{i+1} with f -colour ℓ , then let $N_{i+1, \ell} = N_{i, \ell}$. Otherwise, there are $k+1$ members of \mathcal{S}_{i+1} with f -colour ℓ and none of these members have an index in $N_{i, \ell}$. Then let $n(i+1, \ell)$ be the least n such that $C_n \in \mathcal{S}_{i+1}$ with $f(C_n) = \ell$, and set $N_{i+1, \ell} = N_{i, \ell} \cup \{n(i+1, \ell)\}$.

At the end of the inductive construction, let $N_\ell = \bigcup_{i \in \omega} N_{i, \ell}$. Note that this construction ensures that for each $i \in \omega$ for which \mathcal{S}_i has $k+1$ members with f -colour ℓ , there is exactly one $n \in N_i$ such that $C_n \in \mathcal{S}_i$. Note further that for $\ell \neq \ell'$, $N_\ell \cap N_{\ell'} = \emptyset$.

Let $N = \bigcup_{\ell \in \omega} N_\ell$ and define a k -delta colouring g on $\binom{R}{C}$ as follows. For each $n \in \omega \setminus N$, let $g(C_n) = f(C_n)$; and for $n \in N$, let $g(C_n) = \omega + n$. Then g is k -delta, since if \mathcal{S}_i has $k+1$ members with f -colour ℓ , then there is exactly one $n \in N_i$ such that $C_n \in \mathcal{S}_i$, and $g(C_n)$ is defined to be $\omega + n$. By the induction hypothesis, there is a copy $R' \in \binom{R}{C}$ such that g is one-to-one on $\binom{R'}{C}$. Then f is 2-delta on $\binom{R'}{C}$. By the induction hypothesis applied again, there is another copy R^* in $\binom{R'}{C}$ such that f is one-to-one on $\binom{R^*}{C}$. \square

We are now ready to show that the structures $\mathbb{D}(n, m)$ are delta rainbow Ramsey.

Proposition 5.1. *Let $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$, and let C be a finite induced substructure of $\mathbb{D}(n, m)$. Then*

$$\mathbb{D}(n, m) \xrightarrow{\delta} (\mathbb{D}(n, m))_2^C$$

Proof. Let $\mathfrak{d} = m + 2(n - m)$ and $C \subseteq \mathfrak{D}_{\mathfrak{d}}$ with $C = \mathbb{D}(n, m) \downarrow C$. Let \mathcal{B} be the set of triples (B, X, Y) of subsets of $\mathfrak{D}_{\mathfrak{d}}$ for which $B = X \cup Y$ and $X \neq Y$ and $X \overset{s}{\sim} C \overset{s}{\sim} Y$. To easily identify the order of the copies we require further that the longest sequence x in $B \setminus (X \cap Y)$ is an element of X . Write $(B, X, Y) \overset{s}{\sim} (B', X', Y')$ if $B \overset{s}{\sim} B'$, and moreover write $(B, X, Y) \equiv (B', X', Y')$ if $B \overset{s}{\sim} B'$ and if $f[X] = X'$ and $f[Y] = Y'$ for the (unique) strong similarity f of B to B' . Note that if g is a strong similarity function of $\mathfrak{D}_{\mathfrak{d}}$ onto a subset of $\mathfrak{D}_{\mathfrak{d}}$ and if $(B, X, Y) \equiv (B', X', Y')$ then $(B, X, Y) \equiv (g(B), g(X), g(Y)) \equiv (g(B'), g(X'), g(Y'))$. Also, if $(B, X, Y) \equiv (B, X', Y')$ then $X = X'$ and $Y = Y'$.

Enumerate the finitely many \equiv equivalence classes as \mathcal{P}_i for $i \in r \in \omega$. Let \mathcal{B}_0 denote the set of all B for which there exist X and Y such that $(B, X, Y) \in \mathcal{B}$. Observe that for every $B \in \mathcal{B}_0$ and every $i \in r$, either there is no pair (X, Y) of copies of C with $(B, X, Y) \in \mathcal{P}_i$, or else by the remark above there exists exactly one such pair.

Let $i \in r$ and $(B, X, Y) \in \mathcal{P}_i$ be given. We claim that there are at least three distinct (B_j, X_j, Y_j) , $j \leq 2$, such that:

- (1) each $(B_j, X_j, Y_j) \equiv (B, X, Y)$, and hence are in \mathcal{P}_i ;
- (2) X_0, X_1, X_2 are distinct and form a δ -system;
- (3) and $Y_0 = Y_1 = Y_2$.

The three distinct sets will be obtained by stretching some appropriate elements. First since B is a subset of $\mathfrak{D}_{\mathfrak{d}}$, B is also a subset of $\mathfrak{T}_{\mathfrak{d}}$ and is strongly diagonal. Let x denote the longest element in B with $x \notin X \cap Y$, thus $x \in X$ following our convention. Now for $z \in B$, define $\Lambda_x(z) =$

$$\begin{cases} z & \text{if } |z| < |x|; \\ (z(0), \dots, z(|x| - 1), z(|x|), z(|x|), z(|x| + 1), \dots, z(|z| - 1)) & \text{if } |z| > |x|. \end{cases}$$

Now define $x^0 = x \frown 0$, $x^1 = x \frown 1$, and $x^2 = x$. For each $j \leq 2$, let

$$\begin{cases} B^j = \{\Lambda_x(z) : z \in B \setminus \{x\}\} \cup \{x^j\} \\ X^j = \Lambda_x(X \setminus \{x\}) \cup \{x^j\} \\ Y^j = \Lambda_x(Y). \end{cases}$$

Then each B^j is strongly diagonal, $B^j \overset{s}{\sim} B$, and moreover, $(B^j, X^j, Y^j) \equiv (B, X, Y)$. Let B_j, X_j, Y_j denote $\Delta_{\mathfrak{d}}(B^j), \Delta_{\mathfrak{d}}(X^j), \Delta_{\mathfrak{d}}(Y^j)$, respectively. Since $\Delta_{\mathfrak{d}}$ is a strong diagonalization of $\mathfrak{T}_{\mathfrak{d}}$, each (B_j, X_j, Y_j) , $j \in 2$ is in \mathcal{P}_i . The sets X_0, X_1, X_2 form a δ -system of copies of C . It follows that for any strong similarity f , $f(\mathcal{P}_i)$ contains three different \equiv related elements with the same third entry whose three second entries form a δ -system of copies of C .

Let $\delta : \text{Sims}_{\mathbb{D}(n, m)}(C) \rightarrow \omega$ be a 2-delta colouring. Let $\lambda : \text{Sims}_{\mathfrak{D}_{\mathfrak{d}}}(C) \rightarrow \omega$ be given by $\lambda(C) = \delta(C)$ for $C = \mathbb{D}(n, m) \downarrow C$. Then λ is a 2-delta colouring of $\text{Sims}_{\mathfrak{D}_{\mathfrak{d}}}(C)$. Associate with every $B \in \mathcal{B}_0$ the r -tuple $\gamma(B) = (\sigma_i(B); i \in r)$, with entries one of the elements in the set $\{=, \neq, \notin\}$, so that for $i \in r$:

$$\sigma_i(B) = \begin{cases} = & \text{if } (B, X, Y) \in \mathcal{P}_i \text{ for sets } X \text{ and } Y \text{ with } \lambda(X) = \lambda(Y), \\ \neq & \text{if } (B, X, Y) \in \mathcal{P}_i \text{ for sets } X \text{ and } Y \text{ with } \lambda(X) \neq \lambda(Y), \\ \notin & \text{if there are no sets } X \text{ and } Y \text{ with } (B, X, Y) \in \mathcal{P}_i. \end{cases}$$

According to Theorem 4.2 there exists a strong similarity function g of \mathfrak{D}_δ to a tree $D \subseteq \mathfrak{D}_\delta$ so that the colouring function γ is constant on every $\overset{s}{\sim}$ -equivalence class of subsets in \mathfrak{B}_0 . Let \mathcal{M} denote the set of all the copies of B in D .

If there is an $i \in r$ and a $B \in \mathcal{M}$ for which $\sigma_i(B)$ is equal to $=$, then there are sets X and Y with $(B, X, Y) \in \mathcal{P}_i$ and with $\lambda_i(X) = \lambda_i(Y)$. By the above the \equiv -equivalence class containing (B, X, Y) contains (at least) three elements (B_j, X_j, Y_j) , $j \leq 2$, with the X_j , $j \leq 2$, distinct and moreover forming a δ -system of copies of C . Then $(g[B_j], g[X_j], g[Y_j])$, $j \leq 2$, are three triples for which $g[X_0], g[X_1], g[X_2]$ form a δ -system of copies of C with $\lambda(g[X_0]) = \lambda(g[X_1]) = \lambda(g[X_2])$, contradicting that λ is 2-delta.

Thus $\sigma_i(B)$ is not equal to $=$ for every $B \in \mathcal{M}$ and every $i \in r$. Now suppose X and Y are copies of C in D , the g -image of \mathfrak{D}_δ . Letting $B = X \cup Y$, the triple (B, X, Y) is in \mathcal{P}_i for some $i \in r$, and thus $\lambda(X) \neq \lambda(Y)$.

This completes the proof. \square

We now observe that (delta) rainbow Ramsey is preserved among equimorphic structures.

Lemma 5.2. *Let A and B be two equimorphic structures. Then*

$$A \overset{\delta}{\rightarrow} (A)_k^C \quad \text{if and only if} \quad B \overset{\delta}{\rightarrow} (B)_k^C.$$

And similarly for the rainbow Ramsey property.

Proof. By symmetry it suffices to assume that A is an induced substructure of B .

Let $A \overset{\delta}{\rightarrow} (A)_k^C$ and $\gamma : \binom{B}{C} \rightarrow \omega$ be k -delta. The restriction of γ to $\binom{A}{C}$ is k -delta and hence there a copy $A^* \in \binom{A}{C}$ so that γ is one to one on $\binom{A^*}{C}$. Let $B^* \in \binom{A^*}{B}$. Then γ is one to one on $\binom{B^*}{C}$. \square

We now come to the main result of this paper.

Theorem 5.2. *The binary homogeneous structure $\mathbb{U}(n, m)$ is (delta) rainbow Ramsey for all m and n in ω with $m \leq n$.*

If $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$, then the tree structure $\mathbb{T}(n, m)$ and its diagonal substructure $\mathbb{D}(n, m)$ of $\mathbb{T}(n, m)$ are (delta) rainbow Ramsey as well.

Proof. If $m \leq n \in \omega$ and $m + 2(n - m) \geq 2$, the result follows from Proposition 5.1 in conjunction with Lemma 5.1 and then Theorem 3.1 in conjunction with Lemma 5.2.

The special cases $\mathbb{U}(0, 0)$ and $\mathbb{U}(1, 1)$ follow from the standard Ramsey theorem in a similar but much simpler way as in the proof of Proposition 5.1; see [2]. \square

By a standard compactness, the general (delta) rainbow Ramsey result implies a finite version.

Lemma 5.3. *Let R be a countably infinite relational structure for which the relation $R \overset{\delta}{\rightarrow} (R)_k^C$ holds.*

Then for each $B \in \text{Age}(R)$ there is an $A \in \text{Age}(R)$ such that the relation $A \overset{\delta}{\rightarrow} (B)_k^C$ holds.

The corresponding result holds for rainbow Ramsey.

Proof. Let $(r_i; i \in \omega)$ be an ω -enumeration of R and for $n \in \omega$ let R_n be the induced substructure of R on the set $\{r_i \mid i \in n\}$. Let $B \in \text{Age}(R)$ and assume for a contradiction that for every $n \in \omega$ there is a k -delta colouring $\gamma_n : \binom{R_n}{C} \rightarrow \omega$ for which the restriction of γ_n to $\binom{B^*}{C}$ is not one-to-one for every $B^* \in \binom{R_n}{B}$. We will construct a k -delta colouring $\gamma : \binom{R}{C} \rightarrow \omega$ so that for all $B^* \in \binom{R}{B}$ the colouring γ is not injective on $\binom{B^*}{C}$; contradicting $R \xrightarrow{\delta} (R)_k^C$.

Let \mathcal{U} be an ultrafilter on ω which contains all co-finite subsets of ω . For $C^* \in \binom{R}{C}$ and $C^\circ \in \binom{R}{C}$, call C^* and C° equivalent, written $C^* \sim C^\circ$, if $\{n \in \omega \mid \gamma_n(C^*) = \gamma_n(C^\circ)\} \in \mathcal{U}$. Let $\gamma : \binom{R}{C} \rightarrow \omega$ be a function which is constant on every \sim -equivalence class and assigns different colours to elements in different \sim -equivalence classes.

We claim that the function γ is delta. For assume that \mathcal{S} is a δ -system of structures containing k structures in $\binom{R}{C}$ and that γ is constant on \mathcal{S} . But this is not possible because there would then be an $n \in \omega$ so that γ_n is constant on \mathcal{S} .

Finally we must show that there is no $B^* \in \binom{R}{B}$ for which γ is injective on $\binom{B^*}{C}$. To do so assume otherwise there is such a B^* ; but because $\binom{B^*}{C}$ is finite, there is an $n \in \omega$ for which γ_n is injective on B^* , a contradiction. \square

Corollary 5.1. *If R is a countably infinite (delta) rainbow Ramsey relational structure, then the class of finite structures isomorphic to an induced substructure of R is also (delta) rainbow Ramsey.*

6. CONCLUSION

In this paper, we proved rainbow Ramsey results for what we defined as simple binary relational structures, leaving open a wide spectrum of relational structures. Thus we can easily ask the following.

- Questions 6.1.** (1) Which homogeneous relational structures have the rainbow Ramsey property?
 (2) Are there some ages which have the rainbow Ramsey property, but their Fraïssé limit does not? Or are the two equivalent?

In particular we do not know whether the homogeneous triangle-free graph has the rainbow Ramsey property, and we can ask for which finite triangle-free graph C , does the countable triangle-free homogeneous graph have the rainbow Ramsey property.

Moreover we do not know the implications between Ramsey and delta rainbow Ramsey.

- Questions 6.2.** Does the Ramsey property imply the delta rainbow Ramsey property?

In [2], Csima and Mileti proved that the reverse mathematical strength of the Rainbow Ramsey Theorem for pairs of natural numbers is strictly weaker over RCA_0 than Ramsey's Theorem. We may ask similar questions for the Rado graph and other homogeneous relational structures, replacing Ramsey's Theorem with the existence of canonical partitions in the sense of Corollary 4.3.

- Questions 6.3.** (1) For homogeneous binary simple structures, is the statement of Corollary 4.3 stronger over RCA_0 than the statement of Theorem 5.2?

- (2) In particular, what is the reverse mathematical strength of the Rado graph being rainbow Ramsey for k -bounded colouring of edges; how does its strength compare with those of Ramsey's Theorem and the Rainbow Ramsey Theorem for k -bounded colouring of pairs of natural numbers?

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