MINIMAL CANTOR OMEGA-LIMIT SETS

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Abstract. This paper investigates unimodal maps f for which no iterate c_n of the turning point c is recurrent under f and the omega-limit set $\omega(c, f)$ is a minimal Cantor set. Given a non-periodic minimal sequence $r \in \mathcal{A}^{\mathbb{N}}$, we provide a characterization for when $u \in \mathcal{A}^{\mathbb{N}}$ is such that $\omega(u, \sigma) = \omega(r, \sigma)$. We then prove that the set of parameters for symmetric tent maps T_s for which $\omega(c, T_s)$ is a minimal Cantor set and $c_n \notin \omega(c, T_s)$ is dense in $[\sqrt{2}, 2]$. Modifications are provided that can be used to generate sequences $u \in \mathcal{A}^{\mathbb{N}}$ for which $\omega(u, \sigma) = X$, where $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a shift space with specific properties.

1. Introduction

There is a great deal of literature focusing on the dynamics of unimodal maps of an interval to itself [5, 9, 11]. Of interest within this family of maps is the behavior of the map restricted to the omega-limit set of the turning point. Many papers are dedicated to locating those unimodal maps f for which the omega-limit set $\omega(c, f)$ is a Cantor set and $f|_{\omega(c,f)}$ is a minimal homeomorphism [4, 6, 8], however there is no known combinatorial characterization. In this paper we focus on characterizing when $\omega(c, f)$ is a Cantor set, and the action $f|_{\omega(c,f)}$ is a minimal continuous map (though not necessarily a homeomorphism), and for brevity refer to this situation as one where $\omega(c, f)$ is a minimal Cantor set.

In [1], sufficient conditions on the kneading sequence of a map f were investigated to guarantee that $\omega(c, f)$ is a minimal Cantor set. A scheme was defined that could be used to generate uniformly and regularly recurrent sequences over a finite alphabet, and it was shown that if the kneading sequence of a unimodal map f is generated from one of these schemes, then $\omega(c, f)$ is a minimal Cantor set. This scheme characterized those unimodal maps f for which $\omega(c, f)$ is a minimal Cantor set and at most finitely many points in the orbit of c are non-recurrent. The question was posed whether

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it is possible to have $\omega(c, f)$ a minimal Cantor set when every point in the orbit of c is non-recurrent. In this paper we answer the question in the affirmative and show that the set of parameters in the tent family for which this behavior is exhibited is dense in $[\sqrt{2}, 2]$.

In Section 2 we establish the terminology and notation that will be used throughout this paper. In Theorem 3.1 we provide a characterization for when a non-eventually periodic sequence $u \in \mathcal{A}^{\mathbb{N}}$ is such that $\omega(u, \sigma)$ is a minimal Cantor set. Given a non-periodic minimal sequence $r \in \mathcal{A}^{\mathbb{N}}$, Theorem 3.3 provides a characterization of those sequences $u \in \mathcal{A}^{\mathbb{N}}$ such that $\omega(u, \sigma) = \omega(r, \sigma)$. This construction can then be used to generate the kneading sequences of unimodal maps for which no iterate of c is recurrent and the omega-limit set of the turning point is a minimal Cantor set. In Theorem 3.7 it is shown that the set of parameters in the tent family for which $\omega(c, T)$ is a minimal Cantor set and $c_n \notin \omega(c, T)$ for all $n \in \mathbb{N}$ is dense in $[\sqrt{2}, 2]$. Section 4 presents modifications to the construction in Theorem 3.3 that can be used to generate the kneading sequences of unimodal maps f for which $\omega(c, f)$ is either a non-minimal Cantor set or the union of a Cantor set and countable set. In Section 5 we provide a characterization of those shift spaces X such that $X = \omega(u, \sigma)$ for some $u \in \mathcal{A}^{\mathbb{N}}$ and conclude by classifying those shift spaces X that are topologically conjugate to $\omega(c, T)$ for some symmetric tent map T.

2. Background

2.1. **Sequences and Shift Spaces.** Let $\mathcal{A}$ be a finite set of letters called an *alphabet*. A finite string of letters from $\mathcal{A}$ is called a *word*, and the set of all finite words over $\mathcal{A}$ is denoted $\mathcal{A}^{\star}$; for completeness, we allow $\emptyset$ to denote the empty word. We set $\mathcal{A}^{\mathbb{N}}$ to be the set of all one-sided infinite strings of letters from $\mathcal{A}$. Given a sequence $x = x_1 x_2 x_3 \cdots \in \mathcal{A}^{\mathbb{N}}$, the *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by $\sigma(x) = x_2 x_3 x_4 \cdots$. Let $\mathcal{A}$ be given the discrete metric topology and assign the product topology on $\mathcal{A}^{\mathbb{N}}$ by $d(x, y) = 1/2^{n-1}$ where n is the least number such that $x_1 x_2 \cdots x_n \neq y_1 y_2 \cdots y_n$; hence $\mathcal{A}^{\mathbb{N}}$ is a compact metrizable space. A subset $X \subset \mathcal{A}^{\mathbb{N}}$ is called a *shift space* if X is closed and X is strongly invariant, i.e. $\sigma(X) = X$.

Given a shift space X, we let $\mathscr{L}$ be the set of all words from $\mathcal{A}^{\star}$ that appear in X and $\mathscr{F}$ be the set of all words from $\mathcal{A}^{\star}$ that never appear in X. We denote by $\mathscr{F}' \subset \mathscr{F}$ the set *first offender words*, i.e. those words $F \in \mathscr{F}$ such that every proper subword of F is in $\mathscr{L}$. We additionally denote by $\mathscr{L}_n$

the set of all words from $\mathscr{L}$ of length $n \in \mathbb{N}$. We say that a shift space X is *transitive* if for every $u, v \in \mathscr{L}$ there exists a $w \in \mathscr{L}$ such that $uwv \in \mathscr{L}$.

A *substitution* is a function $\theta : \mathcal{A} \to \mathcal{A}^\star \setminus \emptyset$ that is extended to $\mathcal{A}^\star$ or to $\mathcal{A}^\mathbb{N}$ by concatenation; that is, $\theta(xy) = \theta(x)\theta(y)$. A *fixed point* of a substitution is a sequence $u \in \mathcal{A}^\mathbb{N}$ such that $\theta(u) = u$. For more information on substitutions see [10]. We will use substitutions to define various examples in Sections 3 and 4.

We now define some terminology that is standard among arbitrary continuous maps on compact metric spaces.

Given $f : E \to E$, a continuous map on a compact metric space, and a point $x \in E$, the *omega-limit set of* x *under* f is the set $\omega(x, f) = \{y \in E \mid \text{there exists } n_1 < n_2 < \cdots \text{ with } f^{n_i}(x) \to y\}$. A point $x \in E$ is *recurrent* if for every open set U containing x, there exists $m \in \mathbb{N}$ such that $f^m(x) \in U$; equivalently, x is recurrent if and only if $x \in \omega(x, f)$. A point $x \in E$ is *uniformly recurrent* if for every open set U containing x, there exists an $M \in \mathbb{N}$ such that for all $j \geq 0$, $f^{j+k}(x) \in U$ for some $0 < k \leq M$. In terms of shift spaces, a sequence $w \in \mathcal{A}^\mathbb{N}$ is *recurrent* if every word u appearing in w appears infinitely often in w and is *uniformly recurrent* if for any word u appearing in w, there exists an M such that every word of length M in w contains at least one occurrence of u. A set $F \subseteq E$ is *minimal* provided F is nonempty, closed, invariant, and no proper subset of F has these properties.

We note the following well-known results about omega-limit sets, minimality, and recurrence (see, for example, [5]).

Theorem 2.1. *Let $f : E \to E$ be a continuous map of a compact metric space and $x \in E$. Suppose $x \in \omega(x)$, then $\omega(x, f)$ is minimal if and only if x is uniformly recurrent.*

Lemma 2.2. *Let $f : E \to E$ be a continuous map of a compact metric space. Then a non-empty set $F \subseteq E$ is minimal if and only if $\omega(x, f) = F$ for all $x \in F$.*

2.2. **Unimodal Maps.** A *unimodal map* is a continuous map $f : [0, 1] \to [0, 1]$ for which there exists a point $c \in (0, 1)$, called the *turning point* of the map, such that $f|_{[0,c)}$ is strictly increasing and $f|_{(c,1]}$ is strictly decreasing. For ease of notation we set $c_n = f^n(c)$. The two most common families of unimodal maps are the logistic and symmetric tent families. A logistic map is defined by $g_a(x) = ax(1 - x)$ where $a \in [0, 4]$, whereas a symmetric tent map is defined by $T_a(x) = \min\{ax, a(1-x)\}$ with $a \in [0, 2]$. We assume every unimodal map has no wandering intervals and no attracting periodic orbits;

thus we may assume our unimodal maps come from either the symmetric tent family or the logistic family[5].

Given some iterate f^n of a unimodal map f and J a maximal subinterval on which $f^n|_J$ is monotone, then a *branch* $f^n : J \to [0,1]$ is called a *central branch* if c is an endpoint of J. An iterate n is called a *cutting time* if the image of a central branch of f^n contains c. We denote the cutting times $S_0 = 1, S_1 = 2, S_2, S_3, \cdots$. As the difference between two cutting times is again a cutting time, we may write $S_k - S_{k-1} = S_{Q(k)}$ where $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is a function called the *kneading map*. The kneading map and cutting times completely determine the combinatorics of the unimodal map f[7].

Given a unimodal map f and a point $x \in [0,1]$, the *itinerary of x under f* is the sequence $I(x) = I_0 I_1 I_2 \cdots$ where $I_j = 0$ if $f^j(c) < c$, $I_j = 1$ if $f^j(c) > c$, and $I_j = *$ if $f^j(c) = c$. The *kneading sequence* of f, denoted $\mathcal{K}(f)$, is the itinerary $I(c_1)$. If the turning point is periodic, then we terminate the kneading sequence after the first time $*$ appears; thus, if the turning point is not periodic, then $\mathcal{K}(f)$ is infinite.

Two sequences in $\{0,1,*\}^{\mathcal{A}^{\mathbb{N}}}$ can be compared using the *parity lexicographical ordering*: Find the first position two itineraries $v \neq w$ are different and use the ordering $0 \prec * \prec 1$ if the number of 1's preceding that position is even (*even parity*) and the ordering $0 \succ * \succ 1$ it the number of 1's preceding that position is odd (*odd parity*). An infinite sequence e of 1's and 0's (or a finite sequence of 1's and 0's ending in a $*$) is *shift maximal* if $\sigma^k(e) \preceq e$ for all $k \in \mathbb{N}$, where σ is the shift map. Every kneading sequence is shift maximal, and every shift maximal sequence is the kneading sequence for a unimodal map [3].

A unimodal map f is *renormalizable* provided there exists an interval J containing c and $n \geq 2$ such that $f^n(J) \subset J$ and $f^n|_J$ is a unimodal map (we note that we relax the definition of unimodal to allow for a map to be decreasing to the left of the turning point). If $f^n|_J$ is again renormalizable, then we say f is *twice renormalizable*; if this process can be continued forever, then f is *infinitely renormalizable*. If the unimodal map f is non-renormalizable, then we may assume that f is from the symmetric tent family with slope $a \in [\sqrt{2}, 2]$; otherwise we may assume that f is logistic [5].

We note that there is a strong relationship between renormalization and the structure of the kneading sequence of a unimodal map. Let $n, m \in \mathbb{N}$, $P = P_1 P_2 \cdots P_n \in \{0,1\}^n$, and $Q = Q_1 Q_2 \cdots Q_m \in \{0,1\}^m$. The *star* *product ($\star$-product)* of P and Q is defined by

$$P \star Q = \begin{cases} P\widetilde{Q}_1 P\widetilde{Q}_2 \cdots P\widetilde{Q}_m P & \text{if } P \text{ has odd parity,} \\ PQ_1PQ_2 \cdots PQ_mP & \text{if } P \text{ has even parity,} \end{cases}$$

where $\widetilde{Q}_i = 1 - Q_i$. If $P*$ and $Q*$ are both shift maximal, then $(P \star Q)*$ will also be shift maximal. Further, these definitions and results extend to sequences P and Q where P has finite length and $Q \in \{0,1\}^{\mathbb{N}}$. For additional details, see [9].

Remark 2.3. As is sometimes the approach, we may also define itinerary/ itineraries for a given point x via the rule $I(x) = I_0 I_1 I_2 \cdots$ where $I_j = 0$ if $f^j(x) \leq c$ and $I_j = 1$. Then every point in $\omega(\mathcal{K}(f), \sigma)$ corresponds to an itinerary of a point in $\omega(c, f)$. If c is recurrent, then $I(c)$ may be expressed as $0\mathcal{K}(f)$ or $1\mathcal{K}(f)$, and every preimage of c will also have two representations. If c is not recurrent, then this correspondence is one-to-one. Throughout this paper we are interested in the case where c is not recurrent, and thus to study $\omega(c, f)$ we can focus our attention only on $\omega(\mathcal{K}(f), \sigma)$; this is because the associated itineraries (in the usual sense) will correspond uniquely with the sequences in $\omega(\mathcal{K}(f), \sigma)$.

3. Minimal Cantor Set Construction

In this section we provide the main results of this paper. We begin by providing a characterization for a sequence $u \in \mathcal{A}^{\mathbb{N}}$ to be such that $\omega(u, \sigma)$ is a minimal Cantor set. In Theorem 3.3 we begin with a non-periodic minimal sequence $r \in \mathcal{A}^{\mathbb{N}}$ and provide a construction for a sequence $u \in \mathcal{A}^{\mathbb{N}}$ that is both necessary and sufficient for $\omega(u, \sigma) = \omega(r, \sigma)$. We show that this construction can be used to generate the kneading sequences of symmetric tent maps T for which no iterate of c is recurrent and $\omega(c, T)$ is a minimal Cantor set; in fact, this can be done densely within the parameter space $[\sqrt{2}, 2]$. Note that if $u \in \mathcal{A}^{\mathbb{N}}$ is an eventually periodic sequence then $\omega(u, \sigma)$ is a finite set, and thus not a Cantor set. As such the theorem below completes the desired characterization.

Theorem 3.1. *Let $u \in \mathcal{A}^{\mathbb{N}}$ be a sequence which is not eventually periodic. Then $\omega(u, \sigma)$ is a minimal Cantor set if and only if for each word v appearing in u, either v appears only finitely many times in u or there exists an M such that every block of length M in u contains a copy of v.*

Proof. First, suppose for each word v in u, either v appears only finitely often in u or v appears with bounded gap in u. Let $y \in \omega(u, \sigma)$ be arbitrarily chosen and fix $n \in \mathbb{N}$. Since a sequence of shifts of u converges to y, the initial word $y_1y_2\cdots y_n$ appears infinitely often in u. By our assumption this implies that there exists an $M \in \mathbb{N}$ such that every block of length M in u contains a copy of $y_1y_2\cdots y_n$. Let $z \in \omega(u,\sigma)$ also be arbitrarily chosen (possibly $y = z$). Since $z_1z_2\cdots z_{i+M}$ appears infinitely often in u for all i, thus each $z_iz_{i+1}\cdots z_{i+M}$ contains a copy of $y_1y_2\cdots y_n$. Thus $y_1y_2\cdots y_n$ appears in z infinitely often with bounded gaps. As n was arbitrarily fixed, it follows that $y \in \omega(z,\sigma)$. Further, as both y and z were arbitrarily chosen from $\omega(u,\sigma)$ (with $y = z$ permitted),then $z \in \omega(z,\sigma)$ and $y \in \omega(z,\sigma)$ for all $y, z \in \omega(u,\sigma)$. Hence $\omega(z,\sigma) = \omega(u,\sigma)$ for all $z \in \omega(u,\sigma)$, and therefore $\omega(u,\sigma)$ is minimal. As u was assumed to be non-eventually periodic, then $\omega(u,\sigma)$ is infinite and thus a minimal Cantor set.

Conversely, suppose $\omega(u,\sigma)$ is a minimal Cantor set. Then either $\sigma^k(u) \in \omega(u,\sigma)$ for some $k \geq 0$ or $\sigma^k(u) \notin \omega(u,\sigma)$ for all $k \geq 0$. In the first case $\omega(\sigma^k(u),\sigma) = \omega(u,\sigma)$ is minimal. Hence every word that appears in $\sigma^k(u)$ appears infinitely often in $\sigma^k(u)$ with bounded gap. Thus every word that appears in u either only appears finitely many times in u or appears with bounded gap in u.

We thus consider the second case where no shift of u appears in $\omega(u,\sigma)$. Let $y \in \omega(u,\sigma)$. Then the orbit closure of y is minimal. Thus, if the word v appears in y, there exists some N such that every block of length N in y contains v. As v appears infinitely often in y, then v appears infinitely often in u. Suppose that there does not exist some M such that every block of length M in u contains a copy of v. Then there exist infinitely many arbitrarily large blocks in u that do not contain v. Let us denote these blocks as $\{A_n\}_{n=1}^{\infty}$ such that $|A_n| \to \infty$. Thus there exists an infinite sequence $x \in \omega(u,\sigma)$ such that x does not contain v. But then x is not in the orbit closure of y, a contradiction. It follows that if v appears infinitely often in u, then v appears in every block of length M for some $M \in \mathbb{N}$. □

Let r be any non-periodic minimal sequence. We provide a method to construct a sequence u such that $\omega(u,\sigma) = \omega(r,\sigma)$, and $\sigma^k(u) \notin \omega(u,\sigma)$ for any $k \in \mathbb{N}$.

Lemma 3.2. *Let r be a non-periodic minimal sequence in $\mathcal{A}^{\mathbb{N}}$, and let $\mathscr{F}$ be the set of finite words not appearing in r. Then there is an infinite set of words $\{w_1, w_2, \ldots\} \subset \mathscr{F}$ with the following property: For every $M > 0$ there is an n such that $|w_n| > M$ and every proper subword of w_n appears in r.*

Proof. Suppose this is not true. Then for some $M > 0$ every word $w \in \mathscr{F}$ with $|w| > M$ contains a proper subword in $\mathscr{F}$. Set $\mathscr{F}_M$ equal to the set of words in $\mathscr{F}$ with length M or less. The condition implies that any word $w \in \mathscr{F}$ is either in $\mathscr{F}_M$ or contains a subword in $\mathscr{F}_M$.

Notice that the set $\mathscr{F}_M$ is finite. We have that x is in the orbit closure of r if and only if no word appearing in x is in the set $\mathscr{F}$ if and only if no word appearing in x is in the set $\mathscr{F}_M$. But this means that the orbit closure of r is a set of sequences that avoids a finite set of forbidden words, i.e., a shift of finite type. The only minimal shifts of finite type are periodic orbits, which is a contradiction since r is non-periodic. □

Without loss of generality, we may assume that there is a non-decreasing unbounded sequence $\{l_n\}$ such that the collection $\{w_1, w_2, w_3, \cdots\}$ of forbidden words in the previous lemma satisfies $|w_n| \geq l_n$ for each $n \in \mathbb{N}$. Further, we can extend this to allow $w_i = w_j$ for finitely many $i \neq j$.

Fix $r \in \mathcal{A}^{\mathbb{N}}$, a non-periodic minimal sequence. Let $\{k_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ be sequences of integers which are non-decreasing and unbounded. Denote by $\mathscr{F}'$ the set of first offender words, i.e., the set of words that do not appear in r but every proper subword does appear in r. Let $\{u_n\}_{n=0}^{\infty}$ be a collection of finite words such that

- For each $i \geq 1$, u_i appears in r and $|u_i| \geq k_i$.
- For each $i \geq 1$, $u_i u_{i+1}$ is such that every word from $\mathscr{F}'$ appearing in $u_i u_{i+1}$ has length at least l_i, and it is possible that $u_i u_{i+1}$ contains no words from $\mathscr{F}'$.
- The word u_0 is in $\mathcal{A}^N$ for some $N \geq 0$; that is u_0 can be the empty word.

Set $u = u_0 u_1 u_2 u_3 \cdots$

Theorem 3.3. *Given a non-periodic minimal sequence $r \in \mathcal{A}^{\mathbb{N}}$ and $u \in \mathcal{A}^{\mathbb{N}}$, then $\omega(u, \sigma) = \omega(r, \sigma)$ if and only if u is constructed as above.*

Proof. Suppose first that u is constructed as above. A word v appears in $\omega(u, \sigma)$ if and only if v appears infinitely often in u. Suppose v appears infinitely often in u and has length M. If v appears in u_k for some $k \geq 1$, then v is a subword of r; otherwise v appears in $u_k u_{k+1}$ for infinitely many k where part of v lies in u_k and part of v lies in u_{k+1}. There exists an $N \in \mathbb{N}$ such that $l_n > M$ for all $n \geq N$. Hence, whenever $k \geq N$ and v appears in $u_k u_{k+1}$, then v is not in $\mathscr{F}'$, nor is any proper subword of v. Hence v appears in r, and thus v appears in $\omega(r, \sigma)$. Further, by construction, if w is a word appearing in $\omega(r, \sigma)$, then w also appears infinitely often in r. As

r is minimal and $|u_k| \to \infty$ as $k \to \infty$ where $\{u_k\}_{k\geq 1}$ are words from r, it follows that there exists a $K \in \mathbb{N}$ such that w appears in u_k for all $k \geq K$, and thus w appears in $\omega(u,\sigma)$. Therefore $\omega(u,\sigma) = \omega(r,\sigma)$.

Conversely, suppose $\omega(u,\sigma) = \omega(r,\sigma)$. Our choice of u_i, k_i and l_i will depend upon whether $\sigma^k(u) \in \omega(r,\sigma)$ for some $k \geq 0$ or not.

If $\sigma^{k_0}(u) \in \omega(r,\sigma)$ for some $k_0 \geq 0$, then every initial block of $\sigma^{k_0}(u)$ appears in r. In this case let $\{k_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ be any non-decreasing unbounded sequences. Set u_0 to be the initial block of u of length k_0. Then set u_1 to be the first block of length k_1 in $\sigma^{k_0}(u)$. For each $i \geq 2$, set u_i to be the first block of length k_i in $\sigma^{k_0+k_1+\cdots k_{i-1}}(u)$. Since $u = u_0u_1u_2\ldots$ and every word from u_iu_{i+1} is allowed for $i \geq 1$, the sequence u has the desired form.

Now consider the case where $\sigma^k(u) \notin \omega(r,\sigma)$ for all $k \geq 0$, then there exist infinitely many words in u that appear in $\mathscr{F}'$. Let u_0 be an initial block of u with length $N \geq 0$ such that $\sigma^N(u)$ begins with a word from r of length at least 2. Set u_1 be the longest initial block of $\sigma^N(u)$ that appears in r and set $k_1 = |u_1|$. For each $n \geq 2$, let u_n be the longest initial block of $\sigma^{N+k_1+\cdots+k_{n-1}}(u)$ that appears in r and set $k_n = |u_n|$. Then $u = u_0u_1u_2\cdots$ where $u_0 \in \mathcal{A}^N$ and u_i is a word from r for each $i \geq 1$.

Fix $M \in \mathbb{N}$ and suppose there exist infinitely many $i \geq 1$ such that $|u_i| < M$. Then by construction there exist infinitely many $i \geq 1$ such that a word from $\mathscr{F}'$ of length less than or equal to M appears in u_iu_{i+1}. Thus, there exists a word $v \in \mathscr{F}'$ with $|v| \leq M$ that appears infinitely often in u, a contradiction to $\omega(u,\sigma) = \omega(r,\sigma)$. It thus follows that $k_i \to \infty$ as $i \to \infty$.

Suppose there exists an $M \in \mathbb{N}$ such that u_iu_{i+1} has a word from $\mathscr{F}'$ of length M for infinitely many $i \in \mathbb{N}$. Then there exists a word $v \in \mathscr{F}'$ appearing infinitely often in u such that $|v| = M$, and as $\omega(u,\sigma) = \omega(r,\sigma)$, then v appears infinitely often in r. But since $v \in \mathscr{F}'$, this is a contradiction. Hence there must exist a non-decreasing and unbounded sequence $\{l_n\}_{n\geq 1}$ such that every word from $\mathscr{F}'$ appearing in u_iu_{i+1} has length at least l_i. □

Remark 3.4. If u_iu_{i+1} contains a word from $\mathscr{F}'$ for infinitely many $i \in \mathbb{N}$, then $\sigma^k(u) \notin \omega(r,\sigma)$ for all $k \geq 0$. Further, u may be constructed such that u_iu_{i+1} always contains a word from $\mathscr{F}'$. That is, given any minimal non-periodic sequence $r \in \mathcal{A}^{\mathbb{N}}$,there exists a sequence $u \in \mathcal{A}^{\mathbb{N}}$ such that $\sigma^k(u) \notin \omega(r,\sigma)$ for all $k \geq 0$, but $\omega(u,\sigma) = \omega(r,\sigma)$.

As $\omega(u,\sigma)$ is a minimal Cantor set, every word appearing in u appears either finitely many times or appears with bounded gap. Since $\omega(u,\sigma)$ contains no fixed points, there exists an $N \in \mathbb{N}$ such that if 0^n appears in u, then $n < N$. One can then check that $e = 10^N u \in \{0,1\}^{\mathbb{N}}$ is a shift maximal sequence and therefore there exists a unimodal map f such that $\mathcal{K}(f) = e$. Since 10^N never again appears in e and e is not eventually periodic, the map f is also non-renormalizable and may be assumed to be from the symmetric tent family. Further, if we assume u was constructed such that $u_i u_{i+1}$ contains a word from $\mathscr{F}'$ for infinitely many $i \in \mathbb{N}$, then $\sigma^k(e) \notin \omega(r,\sigma)$ for any $k \geq 0$. That is, no iterate of c is recurrent. Thus there exists a unimodal map for which every point in the orbit of the turning point is isolated with respect to the orbit, but the omega-limit set of the turning point is a minimal Cantor set. Combining this fact with the aforementioned construction, we have the following result.

Theorem 3.5. *Let $r \in \{0,1\}^{\mathbb{N}}$ be any non-periodic sequence with minimal orbit closure (X,σ). There is a unimodal map in the symmetric tent family f such that (X,σ) is topologically conjugate to $(\omega(c,f),f)$ and $c_n \notin \omega(c,f)$ for all $n \geq 0$.*

We now provide an explicit example of a unimodal map f for which $\omega(c,f)$ is a minimal Cantor set and $c_n \notin \omega(c,f)$ for each $n \geq 0$ to illustrate the construction.

Example 3.6. *Consider the Morse substitution given by $\theta(0) = 01$ and $\theta(1) = 10$. Consider the fixed point of this substitution, which we will denote r, generated by $r = \lim_{n\to\infty} \theta^n(0) = 0110100110010110100101100 1\cdots$.*

The sequence r is a well-known example of a non-periodic sequence with minimal orbit closure [10]. Let $\{l_n\} = \{2^{n+1}+1\}$ and $\{k_n\} = \{2^{n+2}\}$. Fix u_1 to be an initial word from r of length at least k_1 that ends with $\theta(0) = 01$. For each $n \geq 2$, let u_n be a word from r of length at least k_n beginning with $\theta^{n-1}(0)\theta^{n-1}(0)$ and ending with $\theta^n(0)$. Then based on the construction, whenever 0^n appears in $u_1u_2u_3\cdots$, then $n \leq 2$. Thus, let $u_0 = 1000$. Set $u \in \{0,1\}^{\mathbb{N}}$ be such that $u = u_0u_1u_2u_3\cdots$. Hence u is a shift maximal sequence in $\{0,1\}^{\mathbb{N}}$ and there exists a unimodal map f such that $\mathcal{K}(f) = u$.

For each $n \geq 1$, u_nu_{n+1} contains a copy of $\theta^n(0)\theta^n(0)\theta^n(0)$; that is, the only words from $\mathscr{F}'$ appearing in u_nu_{n+1} are of length greater than or equal to l_n for all $n \geq 1$. Hence, by Theorem 3.3 we have $\omega(u,\sigma) = \omega(r,\sigma)$. Further, because each u_nu_{n+1} contains a word from $\mathscr{F}'$ that never appears again in u, $\sigma^k(u) \notin \omega(r,\sigma)$ for any $k \in \mathbb{N}$. It thus follows that $\omega(c,f)$ is a minimal Cantor set for which $c_n \notin \omega(c,f)$ for all $n \geq 0$.

Following this example, one kneading sequence we can obtain is

$$\mathcal{K}(f) = u = 1000011010101100110011001101001011010010110100101 10\cdots$$

We now show that not only do there exist symmetric tent maps for which $\omega(c, f)$ is a minimal Cantor set and no iterate of the turning point is recurrent, but that this behavior actually occurs densely in the parameter space $[\sqrt{2}, 2]$.

Theorem 3.7. *The set of parameters of symmetric tent maps T_s for which $\omega(c, T_s)$ a minimal Cantor set and $c_n \notin \omega(c, T_s)$ for all $n \geq 0$ is dense in $[\sqrt{2}, 2]$.*

Proof. It suffices to show that if f_s is a tent map with $s > \sqrt{2}$ whose kneading sequence begins with $i_1 i_2 \cdots i_N$, then there is a tent map T such that $\mathcal{K}(T)$ also begins with $i_1 i_2 \cdots i_N$, $\omega(c, T)$ is a minimal Cantor set, and $c_n \notin \omega(c, T)$ for all $n \in \mathbb{N}$.

Let g be a non-renormalizable tent map with an embedded adding machine such that $\mathcal{K}(g) = e_1 e_2 \cdots$ is infinite and begins with $i_1 i_2 \cdots i_N$. Let $h \neq g$ be another non-renormalizable tent map with an embedded adding machine such that $\mathcal{K}(h)$ begins with $i_1 i_2 \cdots i_N$. It was shown in [4] that the set of parameters for which the restriction to the omega-limit set of the turning point is conjugate to any given adding machine is dense in $[\sqrt{2}, 2]$, so we know that such symmetric tent maps g and h exist. Let M be the first place that $\mathcal{K}(g)$ and $\mathcal{K}(h)$ disagree. Without loss of generality suppose $\mathcal{K}(g) \succ \mathcal{K}(h)$. Then $e_1 e_2 \cdots e_M$ never appears in $\mathcal{K}(h)$, which is a non-periodic minimal sequence.

Create a list $\mathscr{F}'$ of words of length longer than $M+1$ that never appear in $\mathcal{K}(h)$ such that each proper subword does appear in $\mathcal{K}(h)$. Let $\{l_n\}_{n\geq 1}$ and $\{k_n\}_{n\geq 1}$ be non-decreasing unbounded sequences of integers where each $l_i \geq M+1$. Construct u as in the Theorem 3.3 such that $u_0 = e_1 \cdots e_M$, u_1 begins with $e_1 e_2 \cdots e_{m-1} \widetilde{e}_M$, and for each $i \geq 1$ $u_i u_{i+1}$ has a word from $\mathscr{F}'$ of length longer than l_i.

We now show that u is shift maximal. We first note that

$$e_1 e_2 \cdots e_{M-1} e_M e_1 e_2 \cdots e_{M-1} * = (e_1 e_2 \cdots e_{M-1} \star 1) *$$

is shift maximal because both $e_1 e_2 \cdots e_{M-1} *$ and $1*$ are shift maximal. Thus

$$e_1 e_2 \cdots e_{M-1} e_M e_1 e_2 \cdots e_{M-1} \widetilde{e}_M \succeq \sigma^k (e_1 e_2 \cdots e_{M-1} e_M e_1 e_2 \cdots e_{M-1} \widetilde{e}_M)$$

for all $k \geq 0$. Note that $e_1 e_2 \cdots e_M$ never appears in $u_1 u_2 u_3 \cdots$. Thus every shift of $\sigma^k(u)$ with $k \geq M$ will disagree with $e_1 e_2 \cdots e_M$ in the first M positions, but no word of length less than $M+1$ is in $\mathscr{F}$. Thus, if the disagreement is in the Mth position we have $e_1 e_2 \cdots e_{M-1} e_M \succeq e_1 e_2 \cdots e_{M-1} \widetilde{e}_M$. If the disagreement is in a position prior to the Mth, then the disagreement happens within $e_1e_2\cdots e_{M-1}$ and the block appears in $\mathcal{K}(h)$. The shift maximality of $\mathcal{K}(h)$ gives that $e_1e_2\cdots e_{M-1}$ is larger than the disagreeing block. Thus u is shift maximal and is the kneading sequence of a unimodal map T. It remains to show that T is non-renormalizable.

Note that by construction, M is a cutting time of T. We set $M = S_t$. Then $Q(k) \leq t$ for all $k \geq t$. Hence T cannot be infinitely renormalizable. Thus suppose that T is finitely renormalizable. Then there exists a finite word w and a non-renormalizable shift maximal sequence $v \in \{0,1\}^{\mathbb{N}}$ such that $u = w \star v$. As $|w| + 1 = s_l$ is a cutting time such that $Q(k) \geq l$ for all $k \geq l$, $l \leq t$ and therefore $|w| < M$. Note that as this is a renormalizable map, the cutting times S_k with $k \geq l$ are all such that $S_k = n \cdot S_l$ for some $n \in \mathbb{N}$. Hence $e_1e_2\cdots e_{M-1} = w\Delta_1 w\Delta_2 \cdots \Delta_n w$ where each $\Delta_i \in \{0,1\}$ (and for completeness, it is possible that $e_1e_2\cdots e_{M-1} = w$), $u = w\Delta_1 w\Delta_2 \cdots \Delta_n w \cdots$, and $u_1u_2\cdots = w\Delta_{n+1}w\Delta_{n+2}w\cdots$.

Note that $\mathcal{K}(h)$ contains words of the form $w\Delta'_1 w\Delta'_2 w \cdots \Delta'_m w$ for arbitrarily large $m \in \mathbb{N}$ where each $\Delta_i \in \{0,1\}$; also, for each initial block I of $\mathcal{K}(h)$, there exists a $j \in \mathbb{N}$ such that $w\Delta_{i+1}w\Delta_{i+2}\cdots\Delta_{i+j}w$ contains a copy of I for each $i \in \mathbb{N}$. Hence $\mathcal{K}(h) = \sigma^k(w)\Delta''_1 w\Delta''_2 w\cdots$ for some $0 \leq k \leq |w|$ and $\Delta''_i \in \{0,1\}$ for all $i \in \mathbb{N}$. If $k = 0$, then $\mathcal{K}(h) = w\Delta''_1 w\Delta''_2 w \cdots$ and is thus either periodic, not shift maximal, or is obtained from a star product. If $k \geq 1$, then $\Delta''_i = \Delta''_j$ for all $i, j \geq 1$ and $\mathcal{K}(h)$ is eventually periodic. As h was taken to be a non-renormalizable tent map with an embedded adding machine, we obtain a contradiction in each case. It thus follows that T is non-renormalizable. □

4. Modifications to the Construction

In this section we present modifications to the construction in Theorem 3.3. We first show that if we begin with a non-periodic and minimal $r \in \mathcal{A}^{\mathbb{N}}$ and allow for the same word from $\mathscr{F}'$ to appear in u_iu_{i+1} for infinitely many $i \in \mathbb{N}$, then the corresponding omega-limit set $\omega(u,\sigma)$ will properly contain $\omega(r,\sigma)$ and can either be a non-minimal Cantor set or the union of a Cantor set and a countable set. We then provide examples of unimodal maps f for which $\omega(c,f)$ is either a non-minimal Cantor set or the union of a Cantor set and a countable set.

Proposition 4.1. *Let $r \in \mathcal{A}^{\mathbb{N}}$ be a non-periodic minimal sequence. Let $\mathscr{F}$ be the set of finite words not appearing in r and set $\mathscr{F}' \subset \mathscr{F}$ to be the set of first offender words ($F \in \mathscr{F}'$ implies every proper subword of F appears in r). Let $\{k_i\}_{i\geq 1}$ be a non-decreasing sequence of integers that is unbounded.*

Let $\{u_i\}_{i\geq 1}$ *be a collection of finite words from* r *such that* $|u_i| > k_i$ *for all* $i \in \mathbb{N}$ *and suppose there exists an* $F \in \mathscr{F}'$ *such that* F *appears in* $u_i u_{i+1}$ *for infinitely many* $i \in \mathbb{N}$*. Let* $u_0 \in \mathcal{A}^N$ *for some* $N \geq 0$ *and set* $u = u_0 u_1 u_2 \cdots$*. Then* $\omega(u,\sigma) \supset \omega(r,\sigma)$ *and* $\omega(u,\sigma)$ *is either a non-minimal Cantor set or the union of a Cantor set and a countable set.*

Proof. Recall that a word w appears in $\omega(u,\sigma)$ if and only if w appears infinitely often in u. Let w be a word that appears in $\omega(r,\sigma)$. Then w appears infinitely often in r, and since r is a minimal sequence, there exists an $N \in \mathbb{N}$ such that every word of length N in r contains a copy of w. Thus there exists an $L \in \mathbb{N}$ such that $k_i \geq N$ for all $i \geq L$, and thus w appears in u_i for all $i \geq L$. Since w appears infinitely often in u, w appears in $\omega(u,\sigma)$ and $\omega(r,\sigma) \subseteq \omega(u,\sigma)$.

As there exists $F \in \mathscr{F}'$ that appears in infinitely many $u_i u_{i+1}$, F appears infinitely often in u, and thus F appears in $\omega(u,\sigma)$. By construction, F does not appear in r, and thus F does not appear in $\omega(r,\sigma)$. It follows that $\omega(u,\sigma)$ properly contains $\omega(r,\sigma)$, and thus $\omega(u,\sigma)$ is uncountable and non-minimal.

Further, since $|u_i| \to \infty$ as $i \to \infty$, the minimum distance between two consecutive occurrences of F in $\sigma^k(u)$ must go to infinity with k. It follows that if F appears in a sequence $y \in \omega(u,\sigma)$, then F can only appear once in y. Since u contains infinitely many copies of F, $\sigma^k(u) \notin \omega(u,\sigma)$ for all $k \in \mathbb{N}$. Therefore, $\omega(u,\sigma)$ is either a Cantor set or the union of a Cantor set and a countable set [5, Section 10.2]. □

We now prove the following proposition, which provides the distinguishing characterization for whether $\omega(u,\sigma)$ will be a Cantor set or the union of a Cantor set and a countable set when $\omega(u,\sigma)$ is uncountable that does not contain every shift of u.

Proposition 4.2. *Let* $u \in \mathcal{A}^{\mathbb{N}}$ *be such that* $\sigma^k(u) \notin \omega(u,\sigma)$ *for some* $k \in \mathbb{N}$ *and* $\omega(u,\sigma)$ *is uncountable. Then* $\omega(u,\sigma)$ *is a Cantor set if and only if for every word* w *that appears infinitely often in* u *there exist words* v *and* v' *such that* $|v| = |v'|$*,* $v \neq v'$*, and both* wv *and* wv' *appear infinitely often in* u*. Otherwise* $\omega(u,\sigma)$ *is the union of a Cantor set and countable set.*

Proof. In this setting, $\omega(u,\sigma)$ is either a Cantor set or the union of a Cantor set and a countable set. The set $\omega(u,\sigma)$ is a Cantor set if and only if every point in $\omega(u,\sigma)$ is a limit point of $\omega(u,\sigma)$. A point $y \in \omega(u,\sigma)$ is a limit point if and only if for all $n \in \mathbb{N}$ there exists a $z \in \omega(u,\sigma)$ such that $z \neq y$ and z and y agree for at least n positions. Recall that $y \in \omega(u,\sigma)$ if and only if every word that appears in y appears infinitely often in u. Thus a point $y \in \omega(u,\sigma)$ is a limit point if and only if every word w that appears infinitely often in y can be extended in two different ways into w' and w'' such that w' and w'' both begin with w, both appear infinitely often in u, and they disagree in at least one position. Hence every point in $\omega(u,\sigma)$ is a limit point if and only if for each word w appearing infinitely often in u there exist v and v' such that $|v| = |v'|$, $v \neq v'$, and wv and wv' both appear infinitely often in u. We note we require $|v| = |v'|$ so that we eliminate the case where v is a prefix of v'. □

We now provide examples of kneading sequences $\mathcal{K}(f)$ that can be constructed as in Proposition 4.1 such that $c_n \notin \omega(c,f)$ for each $n \in \mathbb{N}$ and $\omega(c,f)$ is either a non-minimal Cantor set or the union of a Cantor set and a countable set.

Example 4.3. *The construction in Proposition 4.1 can be used to generate the kneading sequence of a symmetric tent map f such that $\omega(c,f)$ is the union of a countable set and a minimal Cantor set.*

Consider the substitution given by $\theta(1) = 10$ and $\theta(0) = 11$ and the fixed point of this substitution given by

$$r = \lim_{n\to\infty} \theta^n(1) = 1011101010111011101110101011101010111010101011101\cdots.$$

Let $u_0 = 100$, $u_1 = 01110$, and for each $n \geq 2$ let u_n be a word appearing in r that begins with u_{n-1} and ends with 0 and is such that $|u_n| > |u_{n-1}|$. Set $u = u_0u_1u_2\cdots$. Then u is a shift maximal sequence and is thus the kneading sequence for a unimodal map f. By Proposition 4.1, $\omega(r,\sigma) \subset \omega(u,\sigma)$.

Let $y \in \omega(u,\sigma)\setminus\omega(r,\sigma)$. Because every word in u that does not appear in r has 00 within it, it follows that y contains a copy of 00. Further, because $|u_n| > |u_{n-1}|$, it follows that y cannot contain two copies of 00. Hence, by setting $v = \lim_{n\to\infty} u_n$, $y = y_1y_2\cdots y_n0v$ (where $y_1\cdots y_n$ could be the empty word). But note that there are only countably many such y. Hence $\omega(u,\sigma)\setminus\omega(r,\sigma)$ is a countable set, and it follows that $\omega(u,\sigma)$ is the union of a countable set and the minimal Cantor set $\omega(r,\sigma)$.

Letting $\mathcal{K}(f) = u$, note that $\sigma^k(u) \notin \omega(u,\sigma)$ for all $k \in \mathbb{N}$. Hence the turning point c for the map f is not recurrent, and neither is any c_k. There is a one-to-one correspondence between the sequences in $\omega(u,\sigma)$ and the itineraries of the points in $\omega(c,f)$; hence $\omega(c,f)$ is the union of a countable set and a Cantor set.

Example 4.4. *The construction in Proposition 4.1 can be used to generate the kneading sequence of a symmetric tent map f such that $\omega(c,f)$ is a non-minimal Cantor set.*

Again consider the substitution $\theta(1) = 10$ and $\theta(0) = 11$ and the fixed point given by $r = \lim_{n\to\infty} \theta^n(1)$. Note that r can also be constructed in the following manner. Let $r_1 = 1$, and for each $n \geq 2$ set

$$r_n = \begin{cases} r_{n-1}0r_{n-1} & \text{if } n \text{ is even,} \\ r_{n-1}1r_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Then $r = \lim_{n\to\infty} r_n$. Further

$$r_{n+1} = \begin{cases} r_{n-1}0r_{n-1}1r_{n-1}0r_{n-1} & \text{if } n \text{ is even,} \\ r_{n-1}1r_{n-1}0r_{n-1}1r_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Let w be a word appearing in r. Then w is contained in r_{n-1} for some $n \in \mathbb{N}$. Since both $r_{n-1}0$ and $r_{n-1}1$ appear in r, it follows that there exist words v and v' in r such that $|v| = |v'|$, v and v' agree in all but the last position, and both wv and wv' appear in r. We now construct a sequence u from r.

Let $u_{0,0} = 100$
$u_{0,1} = 010111010$
$u_{0,2} = 01011101110$
$u_{1,3} = 01011101010111010$
$u_{1,4} = 0101110101011101110$
$u_{2,5} = 0101110111011101010111010$
$u_{2,6} = 010111011101110101011101110$
$\vdots$

where for each $k \geq 1$, $u_{k,j}$ and $u_{k,j+1}$ are words from r that extend $u_{m,k}$, where $j = 2k+1$, and $k = 2m+1$ or $k = 2m+2$. Further, without loss of generality assign $|u_{k,j+1}| > |u_{k,j}|$ such that $u_{k,j+1}$ and $u_{k,j}$ differ in position $|u_{k,j}|$ and $u_{k,j}$ and $u_{k,j+1}$ both end in 0. Set

$$u = u_{0,0}u_{0,1}u_{0,2}u_{1,3}u_{1,4}u_{2,5}u_{2,6}u_{3,7}u_{3,8}\cdots.$$

First note that $u = \mathcal{K}(f)$ for some symmetric tent map f, since u is shift maximal and cannot be generated from a star product. Further, by Proposition 4.1, $\omega(r,\sigma) \subset \omega(u,\sigma)$, and thus $\omega(u,\sigma)$ will be non-minimal.

Next we prove that for every $k \in \mathbb{N}$, $\sigma^k(u) \notin \omega(u,\sigma)$; further, $c_n \notin \omega(c,f)$ for any $n \in \mathbb{N}$. Let $y \in \omega(u,\sigma) \setminus \omega(r,\sigma)$. Then y contains 00. Note that y cannot contain two copies of 00 since the lengths of the $u_{m,k}$ increase as k increases. Hence every point in $\omega(u,\sigma)$ has at most one copy of 00 appearing in it. It follows that no shift of u can appear in $\omega(u,\sigma)$. Because $I(c_n) = \sigma^{n-1}(u)$, no iterate of c is recurrent under f.

It remains to show that the set $\omega(u,\sigma)$ is a Cantor set. We will show that every point in $\omega(u,\sigma)$ is a limit point. Note that it suffices to check that $y \in \omega(u,\sigma) \setminus \omega(r,\sigma)$ is a limit point where y begins with 00.

Let $y = 00y_3y_4y_5\cdots \in \omega(u,\sigma)$. We show that for each $n \in \mathbb{N}$ there exists a sequence $z \in \omega(u,\sigma)$, $z \neq y$, such that z and y agree for at least n positions. Thus fix $n \in \mathbb{N}$. As $y \in \omega(u,\sigma)$, there exists a subsequence $\{u_{m_l,k_l}\}$ such that $0y_3y_4y_5\cdots y_n$ is the initial word of u_{m_l,k_l} for all $l \in \mathbb{N}$. Further, there exists an L such that u_{m_L,k'_L} begins with $0y_3y_4y_5\cdots y_n$ and

$$k'_L = \begin{cases} k_L + 1 & \text{if } k_L \text{ is odd,} \\ k_L - 1 & \text{if } k_L \text{ is even.} \end{cases}$$

As each $u_{m,k}$ is extended in the construction, there exists a sequence $z \in \omega(u,\sigma)$ such that z begins with $0u_{m_L,k'_L}$. As u_{m_L,k'_L} and u_{m_L,k_L} agree for at least n positions but are not identical, it follows that $z \neq y$ and z and y agree for at least n positions. As n was arbitrarily fixed, y is a limit point of $\omega(u,\sigma)$. Thus every point of the form $00y_3y_4y_5\cdots$ is a limit point of $\omega(u,\sigma)$, and hence every point in $\omega(u,\sigma)\setminus\omega(r,\sigma)$ is a limit point. As $\omega(r,\sigma)$ was a minimal Cantor set, it follows that every point in $\omega(u,\sigma)$ is a limit point. We conclude that $\omega(u,\sigma)$ is a non-minimal Cantor set. Therefore $\omega(c,f)$ is a non-minimal Cantor set such that no iterate of c is recurrent.

5. Generalizations of the Construction

In each of the prior sections we began with a non-periodic minimal sequence $r \in \mathcal{A}^{\mathbb{N}}$ and then generated a point $u \in \mathcal{A}^{\mathbb{N}}$ such that $\omega(u,\sigma) = \omega(r,\sigma)$ or $\omega(u,\sigma) \supset \omega(r,\sigma)$. The next natural question is to begin with a shift space X and determine when there exists a point $u \in \mathcal{A}^{\mathbb{N}}$ such that $\omega(u,\sigma) = X$. This question was addressed in [2], where the authors prove that given a shift space $X \subseteq \mathcal{A}^{\mathbb{N}}$, there exists a point $u \in \mathcal{A}^{\mathbb{N}}$ such that $\omega(u,\sigma) = X$ if and only if X is internally chain transitive. Given a continuous function f on a metric space X and a closed, invariant set $\Lambda \subseteq X$, Λ is *internally chain transitive* if for every pair of points $x, y \in \Lambda$ and for all $\epsilon > 0$ there is a finite sequence of points $x = x_0, x_1, x_2, \ldots, x_n = y \in \Lambda$ and a sequence of integers $t_1, t_2, \ldots, t_n \geq 1$ such that $d(f^{t_i}(x_{i-1}), x_i) < \epsilon$.

We provide an equivalent definition of internally chain transitive in the case of shift spaces below. Theorem 5.3 then provides a characterization for $u \in \mathcal{A}^{\mathbb{N}}$ to be such that $\omega(u,\sigma) = X$, where X is a shift space that is internally chain transitive.

Definition 5.1. Let X be a shift space with language $\mathscr{L}$. Then X is *N-chain transitive* if for every $v, w \in \mathscr{L}$ there exists $u \in \mathcal{A}^{\star}$ such that every subword of length N in vuw is in $\mathscr{L}$.

Lemma 5.2. *X is N-chain transitive for every $N \in \mathbb{N}$ if and only if X is internally chain transitive.*

Proof. Suppose that X is N-chain transitive for every $N \in \mathbb{N}$. Fix $x, y \in X$ and $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that if $z, z' \in X$ agree for at least the first N positions, then $d(z, z') < \epsilon$. Let v and w be the initial words of x and y of length $2N$, respectively. Choose $u \in \mathcal{A}^\star$ such that every subword of vuw of length $2N$ is in $\mathscr{L}$ and vu and x agree for at least $2N$ positions.

Let u_1 be the longest initial block of vuw that agrees with $x_0 = x$; then $|u_1| \geq 2N$. Let u_2 be the longest word from $\mathscr{L}$ beginning in the $|u_1| - N$ position of vuw; then $|u_2| \geq 2N$. For each $i = 2, 3, \cdots n$, let u_i be the longest word from $\mathscr{L}$ beginning in the $|u_1| + |u_2| + \cdots + |u_{i-1}| - (i-1)N$ position of vuw, where u_n ends with w.

For each $i = 1, 2, \ldots, n-1$, let $x_i \in X$ be a sequence beginning with u_{i+1}. Then $\sigma^{|u_i|-N}(x_i)$ begins with the first N terms of u_{i+1} for each $i = 0, 1, \ldots, n-1$, and by construction there is a shift of x_{n-1} that begins with w. Hence we have a sequence of points $x_0 = x, x_1, \ldots, x_n = y \in X$ and a sequence of positive integers $t_1, t_2, \cdots, t_n$ such that $d(\sigma^{t_i}(x_{i-1}), x_i) < \epsilon$ for all $i = 0, 1, \ldots, n-1$. It follows that X is internally chain transitive.

Conversely, suppose X is internally chain transitive. Fix $N \in \mathbb{N}$. Then there exists an $\epsilon > 0$ such that if $d(z, z') < \epsilon$, then z and z' must agree for at least N positions. Let v and w be words from $\mathscr{L}$ and let $x, y \in X$ begin with v and w, respectively. We may find sequences $x = x_0, x_1, x_2, \ldots, x_n = y \in X$ and $t_1, t_2, \cdots t_n \geq 1$ such that $d(\sigma^{t_i}(x_{i-1}), x_i) < \epsilon$ for all $i = 0, 1, \ldots, n-1$. Thus for each $i = 0, 1, \ldots, n-1$, $\sigma^{t_i}(x_{i-1})$ and x_i agree for at least N positions. Let $u_1 = x_{0_{[1+|v|, t_1+N]}}$, $u_i = x_{i-1_{[1+N, t_i+N]}}$ for $i = 2, 3, \ldots, n-1$, and $u_n = x_{n-1_{[1+N, t_{n-1}]}}$ (here $x_{[i,j]} = x_i x_{i+1} \cdots x_j$). Set $u = u_1 u_2 \cdots u_n$. It remains to show that vuw is such that every word of length N is in $\mathscr{L}$.

Note that $x_{i_{[1, t_{i+1}+N]}} \in \mathscr{L}$ for all $i = 0, 1, \ldots, n-1$ and $x_{i_{[1, t_{i+1}+N]}}$ and $x_{i-1_{[1, t_i+N]}}$ overlap for at least N positions. Thus every subword of length N in $u_i u_{i+1}$ is in $\mathscr{L}$ for each $i = 1, 2, \cdots, n-1$, and hence every subword of length N in u is in $\mathscr{L}$. Since very subword of length N in x_0 is in $\mathscr{L}$, then every word of length N in vu_1 is in $\mathscr{L}$; similarly, every word of length N in $u_n w$ is in $\mathscr{L}$. It follows that every word of length N in vuw is in $\mathscr{L}$, so X is N-chain transitive. □

In the following proofs, for a given shift space X, let $\mathscr{L}_n$ denote the set of words of length n in $\mathscr{L}$, the language of X.

Theorem 5.3. *Consider the shift space $X \subseteq \mathcal{A}^{\mathbb{N}}$ with alphabet $\mathcal{A}$, language $\mathscr{L}$, forbidden words $\mathscr{F}$, and first offender forbidden words $\mathscr{F}'$. Then $u \in \mathcal{A}^{\mathbb{N}}$ is such that $\omega(u, \sigma) = X$ if and only if $u = u_0 u_1 u_2 \cdots$ where*

- *$u_0 \in \mathcal{A}^N$ for some $N \geq 0$;*
- *$\{u_i\}_{i \geq 1}$ are words from $\mathscr{L}$;*
- *there exists an increasing sequence of integers $\{k_i\}_{i \geq 0}$ with $k_0 = 1$ such that every word in $\mathscr{L}_n$ appears between $u_{k_{n-1}}$ and u_{k_n};*
- *there exists a non-decreasing, unbounded sequence of integers $\{l_i\}_{i \geq 1}$ such that no word from $\mathscr{F}'$ of length n appears after u_{l_n}.*

Further, such a word $u \in \mathcal{A}^{\mathbb{N}}$ exists if and only if X is internally chain transitive.

Proof. First let $u = u_0 u_1 u_2 \cdots$ be constructed as above. Then each word from $\mathscr{L}$ appears infinitely often in u, and thus $X \subseteq \omega(u, \sigma)$. Suppose w is a word appearing in u but not in $\mathscr{L}$. Then $w \in \mathscr{F}$ with $|w| = n$ for some $n \in \mathbb{N}$. There exists a subword of w that is in $\mathscr{F}'$ and is of length less than or equal to n. Hence w does not appear in u to the right of u_{l_n}, and thus w does not appear infinitely often in u. Therefore every word appearing infinitely often in u is in $\mathscr{L}$, and it follows that $\omega(u, \sigma) \subseteq X$.

Conversely suppose that $\omega(u, \sigma) = X$. Then every word appearing infinitely often in u is in $\mathscr{L}$ and every word in $\mathscr{L}$ appears infinitely often in u. Further, no word from $\mathscr{F}'$ appears infinitely often in u. We consider two cases: $\sigma^k(u) \notin X$ for all $k \in \mathbb{N}$ and $\sigma^k(u) \in X$ for some $k \geq 0$.

First suppose that no shift of u is in X. Since $\omega(u, \sigma) = X$, there is an $N \geq 0$ such that $\sigma^N(u)$ begins with a symbol that appears in X. Let u_0 be the initial block of u of length N. Then let u_1 be the longest initial word of $\sigma^N(u)$ that is in $\mathscr{L}$, let u_2 be the longest initial word of $\sigma^{|u_1|+N}(u)$ that is in $\mathscr{L}$, let u_3 be the longest initial block of $\sigma^{|u_1|+|u_2|+N}(u)$ that is in $\mathscr{L}$; continue in this manner. Clearly each $u_i \in \mathscr{L}$ for $i \geq 1$. Since every word of $\mathscr{L}_1$ appears infinitely often in u, there exists a $k_1 \in \mathbb{N}$ such that every word of $\mathscr{L}_1$ appears in $u_1 u_2 \cdots u_{k_1}$. Similarly, for each $n \geq 2$ there exists a $k_n \in \mathbb{N}$ such that every word of $\mathscr{L}_n$ appears in $u_{k_{n-1}+1} u_{k_{n-1}+2} \cdots u_{k_n}$. Thus we have found our desired strictly increasing sequence of integers $\{k_n\}_{n \geq 1}$. Because no word from $\mathscr{F}'$ can appear infinitely often in u, there exists a largest position l_n such that $u_{l_n - 1} u_{l_n} u_{l_n + 1} \cdots$ contains a word of length less than or equal to n from $\mathscr{F}'$. Thus $\{l_n\}_{n \geq 1}$ is a non-decreasing sequence. Note that if this sequence is ever such that $l_i = l_j$ for all $j \geq i$, then we may set $l'_j = l_j$ for all $j \leq i$ and $l'_j = l_i + j$ for all $j > i$. Then $\{l'_n\}_{n \geq 1}$ is our desired non-decreasing and unbounded sequence of integers.

Now suppose $\sigma^k(u) \in X$ for some $k \in \mathbb{N}$, but $\sigma^{k-1}(u) \notin X$. Thus every word in $\sigma^k(u)$ is in $\mathscr{L}$. Let $u_0 = u_{[1,k]}$. Rewrite $\sigma^k(u) = u_1u_2u_3\cdots$ where the $|u_n| = n$. Since every word of $\mathscr{L}_1$ appears infinitely often in $\sigma^k(u)$, there exists a k_1 such that each word of $\mathscr{L}_1$ appears in $u_1u_2\cdots u_{k_1}$. Similarly, for each $n \geq 2$, there exists a k_2 such that every word of $\mathscr{L}_n$ appears in $u_{k_{n-1}+1}u_{k_{n-1}+2}\cdots u_{k_n}$. Hence we have defined our increasing sequence of integers $\{k_n\}_{n\geq 1}$. Further, set $\{l_n\}_{n\geq 1}$ to be any increasing sequence of integers with $l_1 > k$.

In [2] it was shown that $\omega(u,\sigma) = X$ for some $u \in \mathcal{A}^{\mathbb{N}}$ if and only if X is internally chain transitive. Thus, by Lemma 5.2, it follows that this construction can be done only for those X that are N-chain transitive for all $N \in \mathbb{N}$. We note that this property appears naturally in the construction of u. □

In the next two lemmas we make some observations about which points can generate the shift space X when X is either a shift of finite type or is not a shift of finite type. Recall that a shift space X is a shift of finite type if and only if the list of first offender words is finite.

Lemma 5.4. *If $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a shift of finite type and if $\omega(u,\sigma) = X$, then $\sigma^k(u) \in X$ for some $k \in \mathbb{N}$.*

Proof. Let $X \subseteq \mathcal{A}^{\mathbb{N}}$ be a shift of finite type and suppose that u is generated as in Theorem 5.3 such that $\omega(u,\sigma) = X$. Then $\mathscr{F}'$ is a finite set, and if $\sigma^k(u) \notin X$ for all $k \in \mathbb{N}$, then that means that there are infinitely many occurrences of words from $\mathscr{F}'$ in u. Thus there exists an $F \in \mathscr{F}'$ such that F appears in u infinitely often, and hence F appears in $\omega(u,\sigma)$, a contradiction. Thus there exists some $k \in \mathbb{N}$ such that $\sigma^k(u) \in X$. □

Lemma 5.5. *If $X \subset \mathcal{A}^{\mathbb{N}}$ is an internally chain transitive shift space that is not a shift of finite type, then $\omega(u,\sigma) = X$ for some $u \in \mathcal{A}^{\mathbb{N}}$ such that $\sigma^k(u) \notin X$ for all $k \in \mathbb{N}$.*

Proof. Suppose $X \subset \mathcal{A}^{\mathbb{N}}$ is an internally chain transitive shift space that is not a shift of finite type. Then by Theorem 5.3 there exists some $u \in \mathcal{A}^{\mathbb{N}}$ such that $\omega(u,\sigma) = X$. We now show that u may be constructed such that $\sigma^k(u) \notin X$ for all $k \in \mathbb{N}$. Since $\mathscr{F}'$ is an infinite set, for each $n \in \mathbb{N}$ there exists an $F \in \mathscr{F}'$ such that $|F| \geq n$.

For each $n \in \mathbb{N}$ order the words in $\mathscr{L}_n$ as $\{v_{n,1}v_{n,2},\cdots,v_{n,m_n}\}$ such that if there exists an $F \in \mathscr{F}'$ with $|F| = 2n$ or $|F| = 2n-1$, then $v_{n,m_n}v_{n+1,1}$ contains F for one such F; if no such F exists, then the first word of $\mathscr{L}_{n+1}$

and the last word of $\mathscr{L}_n$ can be arbitrarily chosen. Because X is internally chain transitive, for each $n \in \mathbb{N}$ we may find words $y_{n,1}, y_{n,2}, \cdots y_{n,m_{n-1}} \in \mathscr{L}$ such that every subword of length n of $v_{n,1}y_{n,1}v_{n,2}y_{n,2}\cdots v_{n,m_n-1}y_{n,m_n-1}v_{n,m_n}$ is in $\mathscr{L}_n$.

Set $k_0 = 0$. For all $n \in \mathbb{N}$ set $v_{n,1}y_{n,1}v_{n,2}y_{n,2}\cdots v_{n,m_n-1}y_{n,m_n-1}v_{n,m_n} = u_{k_{n-1}+1}u_{k_{n-1}+2}\cdots u_{k_n}$ where each $u_i \in \mathscr{L}$. Let $u = u_1u_2u_3\cdots$. Then for infinitely many $n \in \mathbb{N}$, $u_{k_{n-1}}u_{k_{n-1}+1}\cdots u_{k_{n+1}}$ contains a word from $\mathscr{F}'$, but no words of length less than or equal to n from $\mathscr{F}'$ will appear after u_{k_n}. Let $\{k_n\}_{n\geq 1}$ be the sequence determined above, and set $\{l_n\}_{n\geq 1} = \{k_n\}_{n\geq 1}$. By Theorem 5.3 we have $\omega(u,\sigma) = X$, but by construction $\sigma^k(u) \notin X$ for all $k \in \mathbb{N}$. ☐

Recall that X is transitive if whenever $u, v \in \mathscr{L}$ there exists $w \in \mathscr{L}$ such that $uwv \in \mathscr{L}$. We now use the concept of transitivity to characterize those shift spaces X that can be generated as the omega-limit set of a point from inside X.

Lemma 5.6. *A shift space X is such that $X = \omega(u,\sigma)$ for some $u \in X$ if and only if X is transitive.*

Proof. Suppose $X = \omega(u,\sigma)$ for some $u \in X$. Then every word in $\mathscr{L}$ appears infinitely often in u, and no word from $\mathscr{F}$ appears in u. Thus, let $w, v \in \mathscr{L}$. Then w and v both appear in u. Let x be a word such that wxv appears in u. Then $wxv \notin \mathscr{F}$, and thus $wxv \in \mathscr{L}$.

Suppose X is transitive. Let $\mathscr{L}_n = \{v_{n,1}, v_{n,2}, \cdots, v_{n,m_n}\}$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ there exists $x_{n,1}, x_{n,2}, \cdots, x_{n,m_n-1} \in \mathscr{L}$ such that $u_n = v_{n-1}x_{n,1}v_{n,2}x_{n,2}\cdots v_{n,m_n-1}x_{n,m_n-1}v_{n,m_n} \in \mathscr{L}$. Then there exists $y_1 \in \mathscr{L}$ such that $u_1y_1u_2 \in \mathscr{L}$ and for all $n \geq 2$ there exists $y_n \in \mathscr{L}$ such that $u_1y_1u_2y_2\cdots u_ny_nu_{n+1} \in \mathscr{L}$. Let $u = u_1y_1u_2y_2\cdots u_ny_nu_{n+1}\cdots \in \mathcal{A}^{\mathbb{N}}$. Then u contains every word in $\mathscr{L}$ infinitely many times, and u does not contain any words from $\mathscr{F}$. Thus $u \in X$ and $\omega(u,\sigma) = X$. ☐

We now conclude by characterizing those shift spaces X that can be obtained as $\omega(c,T)$ for some symmetric tent map T.

Proposition 5.7. *Suppose $X \subseteq \{0,1\}^{\mathbb{N}}$ is internally chain transitive that is not a shift of finite type. Then there exists a symmetric tent map T with $c \notin \omega(c,T)$ such that $T : \omega(c,T) \to \omega(c,T)$ is topologically conjugate to $\sigma : X \to X$ if and only if X has at most one fixed point.*

Proof. Given an internally chain transitive shift space $X \subseteq \{0,1\}^{\mathbb{N}}$ that is not a shift of finite type. By Lemma 5.5 there exists $u \in \mathcal{A}^{\mathbb{N}}$ such that

$\omega(u,\sigma) = X$. Suppose X has at most 1 fixed point; then there exists an $N \in \mathbb{N}$ such that 0^N or 1^N is in $\mathcal{F}$.

If $0^N \in \mathscr{F}$ for some $N \in \mathbb{N}$, then there exists some $M > N$ such that of 0^n appears in u, then $n < M$. Set $v = 10^{M+1}u$; then v is shift maximal and not generated by a star product; thus $10^{M+1}u = \mathcal{K}(T)$ for some symmetric tent map T. Further, since $v \notin \omega(u,\sigma) = \omega(v,\sigma)$, there is a homeomorphism between $\omega(c,T)$ and $\omega(u,\sigma) = X$ such that the itineraries of the points in $\omega(c,T)$ are exactly the sequences in X.

If $1^N \in \mathscr{F}$ for some $N \in \mathbb{N}$, then there exists some $M > N$ such that if 1^n appears in u, then $n < M$. Consider $v = 01^{M+1}u$; then v', given by converting each letter v_i in v to $(1-v_i)$, is shift maximal and is not generated by a star product. Thus $v' = \mathcal{K}(T)$ for some symmetric tent map T. The shift spaces defined by $\omega(v,\sigma)$ and $\omega(v',\sigma)$ are conjugate via the map that switches 0's and 1's. Further, since $v \notin \omega(u,\sigma) = \omega(v,\sigma)$, there exists a homeomorphism between $\omega(c,T)$ and $\omega(u,\sigma) = X$ such that the itineraries of the points in $\omega(c,T)$ are the sequences in X with the letters 0 and 1 switched. In both cases, $T : \omega(c,T) \to \omega(c,T)$ is topologically conjugate to $\sigma : X \to X$.

Now suppose that T is a symmetric tent map such that $T : \omega(c,T) \to \omega(c,T)$ is topologically conjugate to $\sigma : X \to X$ and $c \notin \omega(c,T)$. Then there exists a homeomorphism $\pi : \omega(c,T) \to X$ such that $\pi \circ \sigma = T \circ \pi$. Hence, if X has two fixed points, then $\omega(c,T)$ contains two fixed points. This is a contradiction, since $\omega(c,T)$ can contain at most one fixed point. $\square$

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