The Massive Coherent States of the Poincaré Group and Fuzzy Quantization of Space-time

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Abstract

In obtaining a fuzzy quantization of space-time, one works with the full Poincaré group $\mathcal{P}(1,3) = \mathcal{P}$, obtains a phase space Γ for a massive particle, and then takes $L^2(\Gamma)$. In order to obtain a purely quantum mechanical approach to measurement theory, one picks a wave function η for a particle in an irreducible representation space $\mathcal{H}$ of $\mathcal{P}$, and then intertwines $\mathcal{H}$ with $L^2(\Gamma)$. Every state has averages of momentum, position, and spin, which in turn produces a point in phase space. So, take the coherent state $\{U(g)\eta \mid g \in \mathcal{P}\}$ against which one measures every state ψ by computing the transition probabilities against the coherent state. That will be the quantity that one considers here. These coherent states result in a fuzzy quantization of space-time; i.e., this give the probability of measuring "distance" in the sense of the Poincaré group.

1 Introduction

Coherent states have formed the basis for many theoretical and applied applications of quantum physics. [1] From the group theory point of view in any particular representation space, $\mathcal{H}$, of the group, G, with representation, U, when first viewed, a coherent state is a vector, η, $||\eta|| = 1$, in that space along with the set of translates by means of the group, $\{U(g)\eta, g \in G\}$. This set is a coherent state iff

$$\int_G |U(g)\eta\rangle\langle U(g)\eta| d\mu(g) = \mathbf{1} \tag{1}$$

for μ the Haar measure suitably normalized. Because the "states" in $\{U(g)\eta, g \in G\}$ always occur in the form

$$T^\eta(g) = |U(g)\eta\rangle\langle U(g)\eta|,$$

we may write equation 1 in terms of the (1-dimensional) projections $T^\eta(g)$ and we may convert to the more conventional definition of $\{T^\eta(g), g \in G\}$ being the coherent state. This is in keeping with having a density operator (which is a sum of 1-dimensional projections with convex coefficients summed over a set of an orthonormal basis of vectors); for the density operator to be a mixed state then none of these coefficients may be 1.

This view was found to be insufficient for some groups as the integral in equation 1 may be infinite for all vectors in $\mathcal{H}$. For these groups, one may have $X = G/H$ and $\mathcal{H} = L^2_\mu(G/H)$ where now μ is a G-left-invariant measure on the space X. This agrees with the previous definition if H is normal in G, but is a natural extension otherwise. Now, if

$$U(h)\eta = \alpha(h)\eta$$

for all $h \in H$ and where α is a complex number, then the expression

$$\int_{G/H} |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| d\mu(x) = \mathbf{1} \tag{2}$$

for μ suitably normalized and $\sigma : G/H \to G$ is a Borel section, provides a generalization of the definition of "coherent state."

This latter case is the case when G is one of the Galilei or Poincaré groups. We will obtain the specific form(s) for coherent states of the Poincaré group in all massive representations. In fact, we will obtain these reults for any locally compact Lie group with a finite dimensional Lie algebra. This will include the Galilei group but we will not discuss that further here. For the Poincaré group, these forms of coherent states will appear at the end of sections 4 and 5. Then in section 6, we provide the fuzzy quantization of space-time, and in section 7 we give an example.

2 The Setting

First the general method for obtaining the phase spaces for any locally compact Lie group with a finite dimensional Lie algebra will be described. We have in mind taking the Lie group to be the Poincaré group, but we begin by obtaining the phase space in outline for any Lie group of this category. See [2] for a more detailed discussion.

Start with any locally compact Lie group, G. Form the Lie algebra, $\mathfrak{g}$, which we assume is finite dimensional, and take its dual, $\mathfrak{g}^*$. From the structure constants in any basis for $\mathfrak{g}$, construct the coboundary operator, δ, between the various alternating forms in $\wedge^n\mathfrak{g}^*$. Take one $\omega \in Z^2(\mathfrak{g}^*) = \{\omega \in \mathfrak{g}^* \wedge \mathfrak{g}^*)$ such that $\delta\omega = 0\}$. Define the (Lie sub-) algebra $\mathfrak{h}_\omega = \{X \in \mathfrak{g} \mid \omega(X, \cdot) = 0\}$. Exponentiate $\mathfrak{h}_\omega$ to obtain the

Lie subgroup, H_ω, of G. Assuming H_ω is a *closed* subgroup of G, form $\Gamma = G/H_\omega$, which by a theorem of Guillemin and Sternberg [3] is a phase space (symplectic space). We may take $\mu = \pi^*(\omega^{\wedge n})$ with the largest n such that $\omega^{\wedge n}$ is non-zero and π^* is the correspondent of the canonical map π from G to G/H_ω for elements of $Z^n(\mathfrak{g}^*)$. $\pi^*(\omega^{\wedge n})$ is (a multiple of) the volume measure on Γ. Thus we may define $L^2_\mu(\Gamma)$, a Hilbert space that hosts a left-regular representation $V(G) : [V(g)\Psi](\boldsymbol{x}) \equiv \Psi(g^{-1} \cdot \boldsymbol{x})$, $g \in G$ and $x \in G/H_\omega$.

Now take α equal to a one-dimensional unitary representation of H_ω with values in $\mathbb{C}$. Take $\sigma : G/H_\omega \to G$ to be a Borel section. In particular, for $\boldsymbol{x} \in G/H_\omega$, one has $\sigma(\boldsymbol{x})H_\omega = \boldsymbol{x}$**.** Therefore, for $g \in G$, $g \circ \sigma(\boldsymbol{x})H_\omega = g \cdot \boldsymbol{x}$; i.e., there exists a unique $h(g, \boldsymbol{x}) \in H_\omega$ such that $g \circ \sigma(\boldsymbol{x}) \circ h(g, \boldsymbol{x}) = \sigma(g \cdot \boldsymbol{x})$. Such an h is called a cocycle of G. One can show that V^α on $L^2_\mu(G/H_\omega)$ is a unitary representation of G, where

$$[V^\alpha(g)\Psi](\boldsymbol{x}) = \alpha(h(g^{-1}, \boldsymbol{x})\Psi(g^{-1} \cdot \boldsymbol{x}).$$

Now neither $V^\alpha(G)$ nor $V(G)$ is irreducible. So take an irreducible representation space $\mathcal{H}$ of functions over Γ, and hosting an irreducible representation, $U(G)$. [4] Then take $0 \neq \eta \in \mathcal{H}$ such that η is "square-integrable on G/H_ω" and is "α-admissible":

$$\int_{G/H_\omega} |\langle U(\sigma(\boldsymbol{x}))\eta, \eta\rangle_{\mathcal{H}}|^2 d\mu(\boldsymbol{x}) < \infty, \; U(h)\eta = \alpha(h)\eta. \tag{3}$$

Define $W^\eta : \mathcal{H} \to L^2_\mu(\Gamma)$ by

$$[W^\eta \varphi](\boldsymbol{x}) = \langle U(\sigma(\boldsymbol{x}))\eta, \varphi\rangle. \tag{4}$$

We have then that W^η intertwines $U(G)$ on $\mathcal{H}$ with $V^\alpha(G)$ on $L^2_\mu(G/H_\omega)$. Thus all irreducible representations are phase space representations.

Consider, for $||\eta|| = 1$,

$$T^\eta(\boldsymbol{x}) = U(\sigma(\boldsymbol{x}))P_\eta U(\sigma(\boldsymbol{x}))^\dagger \tag{5}$$

where P_η is the one dimensional projection onto the subspace generated by η. This is a coherent state in one sense because of the orthogonality condition

$$\begin{aligned}\int_{G/H_\omega} \langle \psi, T^\eta(\boldsymbol{x})\varphi\rangle_{\mathcal{H}} d\mu(\boldsymbol{x}) &= \int_{G/H_\omega} \langle \psi, U(\sigma(\boldsymbol{x}))\eta\rangle_{\mathcal{H}} \langle U(\sigma(\boldsymbol{x}))\eta, \varphi\rangle_{\mathcal{H}} d\mu(\boldsymbol{x}) \\ &= \langle \eta, C^2\eta\rangle_{\mathcal{H}} \langle \psi, \varphi\rangle_{\mathcal{H}}, \end{aligned} \tag{6}$$

where C is some positive self-adjoint operator in $\mathcal{H}$. So, letting ψ vary among an orthonormal basis $\{\psi_j\}$ of $\mathcal{H}$, the Fourier coefficients $\langle \psi_j, \varphi\rangle_{\mathcal{H}}$

are all determined, and hence φ is determined. This is what is frequently termed "the property of the coherent state $\{U(\sigma(\boldsymbol{x}))\eta \mid \boldsymbol{x} \in G/H_\omega\}$."

Now $\sigma(\boldsymbol{x})$ only differs from any other vector in $\sigma(\boldsymbol{y})$ by the multiplication by some $h \in H_\omega$ on the right. Hence η is invariant under $U(H_\omega)$ modulo a factor of α. But this does not change any of (1) and (3)-(4). So one may also express $\{U(\sigma(\boldsymbol{x}))\eta \mid \boldsymbol{x} \in G/H_\omega\}$ as $\{U(g)\eta \mid g \in G\}$. Consequently, one has a "coherent state" also in the group sense.

We also have

$$\begin{aligned} U(g)U(\sigma(\boldsymbol{x}))\eta &= U(g \circ \sigma(\boldsymbol{x}))\eta \\ &= U(\sigma(g \cdot \boldsymbol{x}) \circ h(g, \boldsymbol{x})^{-1})\eta \\ &= U(\sigma(g \cdot \boldsymbol{x}))U(h(g, \boldsymbol{x})^{-1})\eta \\ &= U(\sigma(g \cdot \boldsymbol{x}))\alpha(h(g, \boldsymbol{x})^{-1})\eta. \end{aligned}$$

Now, $U(g)$ acting on any function will just replace the variables of the function with g of the variables. Hence, $U(\sigma(g \cdot \boldsymbol{x}))\alpha(h(g, \boldsymbol{x})^{-1})\eta =$ some $\alpha' U(\sigma(g \cdot \boldsymbol{x}))\eta$ with α' a phase. Then

$$\begin{aligned} U(g)T^\eta(\boldsymbol{x})U(g)^\dagger &= U(g)U(\sigma(\boldsymbol{x}))P_\eta U(\sigma(\boldsymbol{x}))^\dagger U(g)^\dagger \\ &= T^\eta(g \cdot \boldsymbol{x}). \end{aligned}$$

Again one has, for $g \in H_\omega$, $T^\eta(g \cdot \boldsymbol{x}) = T^\eta(\boldsymbol{x})$. Consequently, the set of group coherent states corresponding to η is $\{T^\eta(\boldsymbol{x}) \mid \boldsymbol{x} \in G/H_\omega\}$.

Now for any state (i.e., density operator) ρ on $\mathcal{H}$, with η satisfying in addition

$$\langle U(\sigma(\boldsymbol{x}))\eta, \eta\rangle_{\mathcal{H}} > 0 \tag{7}$$

for almost all $\boldsymbol{x} \in G/H_\omega$, one can prove [5] that ρ is completely determined by the set of expectation values $\{Tr(\rho T^\eta(\boldsymbol{x})T^\eta(\boldsymbol{y})) \mid \boldsymbol{x}, \boldsymbol{y} \in G/H_\omega\}$. This is termed "the informational completeness of the $\{T^\eta(\boldsymbol{x})\}$." We can show that there is an η which is α-admissible and for which $\{T^\eta(\boldsymbol{x})\}$ is informationally complete in any irreducible, unitary representation of G.

Hence, in either the vector or the group interpretation, one has coherent states on all states ρ that have the informational completeness property.

This has the following physical interpretation. One starts with an η at "the origin" and uses it to obtain the transition overlap with any other vector ψ by means of some measuring device. Then one carries the device, i.e., the T^η, to any other point in the group space. But that means rotating, boosting, and translating the measuring device by the Poincaré group. Hence, we have constructed a basis of frames for the group. This is equivalent to having one measuring device somewhere and rotating, boosting, and translating the vector ψ by the inverse: $\langle U(g)\eta, \psi\rangle = \langle \eta, U(g^{-1})\psi\rangle$.

Now one considers moving to a different frame, say by premultiplying the vectors in $\{U(\sigma(\boldsymbol{x}))\eta \mid \boldsymbol{x} \in G/H_\omega\}$ by $U(g)$. In particular, one should look at what the effect of moving to a different frame is on the value of $\alpha(h)$. One may do this by exactly the same computations as before.

In moving to a different frame, one has gone to a different ω' in the orbit of ω, hence $\mathfrak{h}_{\omega'}$, hence $H_{\omega'}$, and hence $G/H_{\omega'}$. In particular, one has $\omega \longrightarrow g^*\omega$, $\mathfrak{h}_\omega \longrightarrow g_*\mathfrak{h}_\omega$, $H_\omega \longrightarrow g \circ H_\omega \circ g^{-1} = H_{(g^{-1})^*\omega}$ for a left-invariant ω, and $G/H_\omega \longrightarrow G/H_{(g^{-1})^*\omega}$. Thus the "origin" may be taken as any point in G.

Now, having the coherent states in hand, one considers the quantization of any classical observable (a μ-measureable, real valued function on G/H_ω). Start with a vector, ψ, in $\mathcal{H}$ and then, by a theorem [2], move to $L^2_\mu(\Gamma)$ by means of W^η. Then one multiplies $W^\eta\psi$ by the classical observable, f, as a multiplication function, $M(f)$. However, the result is not in general in the image of W^η, but is a new vector in $L^2_\mu(\Gamma)$. So, one defines P^η as the canonical projection (see [2]) from $L^2_\mu(\Gamma)$ to $W^\eta\mathcal{H}$. Thus $P^\eta M(f)W^\eta\psi$ is in $W^\eta\mathcal{H}$ and so

$$A^\eta(f) \equiv [W^\eta]^{-1}P^\eta M(f)W^\eta \tag{8}$$

is taken as the (quantized) operator corresponding to the classical observable f. It has been proven [2] that

$$A^\eta(f) = \int_\Gamma f(\boldsymbol{x})T^\eta(\boldsymbol{x})d\mu(\boldsymbol{x}) \tag{9}$$

where

$$T^\eta(\boldsymbol{x}) = |U(\sigma(\boldsymbol{x}))\eta\rangle\langle U(\sigma(\boldsymbol{x}))\eta| \tag{10}$$

and where μ is the invariant measure on Γ. Consequently, the quantum expectation $Tr(\rho A^\eta(f))$ is equal to the classical expectation $\int_\Gamma f(\boldsymbol{x})\rho_{cl}(\boldsymbol{x})d\mu(\boldsymbol{x})$, where $\rho_{cl}(\boldsymbol{x}) = Tr(\rho T^\eta(\boldsymbol{x})) = \rho^\eta_{cl}(\boldsymbol{x})$. But

$$\begin{aligned} Tr(\rho A^\eta(f)) &= \sum_j \rho_j Tr(P_{\psi_j}A^\eta(f)) \\ &= \sum_j \rho_j \int_\Gamma f(\boldsymbol{x})|\langle U(\sigma(\boldsymbol{x}))\eta, \psi_j\rangle|^2 d\mu(\boldsymbol{x}) \end{aligned}$$

where P_{ψ_j} is the one-dimensional projection onto the vector ψ_j and $\rho = \sum_j \rho_j P_{\psi_j}$ for some orthonormal basis $\{\psi_j\}$, $\rho_j \geq 0, \sum_j \rho_j = 1$.

Hence, one has a fuzzy view of measurement with the fuzz being a consequence of using the coherent state $\{U(\sigma(\boldsymbol{x}))\eta$ for $\boldsymbol{x} \in \Gamma\}$. This qantum mechanical view of measurement is also referred to as the measurement of f on ρ with respect to the coherent state $\{T^\eta(\boldsymbol{x}) \mid \boldsymbol{x} \in G/H_\omega\}$.

One stresses that it is the quantum probability of the measurement operator $A^{\eta}(f)$ on any quantum density state that is all that matters, and this is given by the transition probability to another quantum particle η. It also gives one interpretation of the η in $A^{\eta}(f)$.

We could also generalize this by replacing $|\eta\rangle\langle\eta|$ with a density matrix, and obtaining the $T(\boldsymbol{x})$ from this. We will leave this to the reader.

3 The Poincaré Group, its Lie Algebra and the Dual of the Lie Algebra

The full Poincaré group $\mathcal{P}$ is a semidirect product; i.e., $\mathcal{P} = \mathbb{R}^4 \rtimes SO(1,3)$, with $\mathbb{R}^4$ given the Minkowski metric. We will write elements of $\mathcal{P}$ by $(\boldsymbol{a}, A)$ with $\boldsymbol{a} = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$, a_0 the "time component" of $\boldsymbol{a}$, and $A \in SO(1,3)$. $\mathcal{P}$ is a Lie group. Therefore its Lie algebra, $\mathfrak{p}$, is a sum of Lie algebras: $\mathfrak{p} = m^4 + so(1,3)$, where m^4 is the Lie algebra for Minkowski space. A basis for elements in $\mathfrak{p}$ are as follows:

$\boldsymbol{P} = (P_1, P_2, P_3)$, generators of "space translations" i.e., translations in a 3-dimensional part of $\mathbb{R}^4$ each of which has signature -1,

P_0, generator for "time translations" i.e., translations in the remaining part of $\mathbb{R}^4$ that has signature 1,

$\boldsymbol{J} = (J_1, J_2, J_3)$, generators of rotations, and

$\boldsymbol{K} = (K_1, K_2, K_3)$, generators of boosts

with the commutation relations

$$\begin{aligned} &[J_j, J_k] = J_l,\ [K_j, K_k] = -J_l,\ [J_j, P_k] = P_l,\ [K_j, P_j] = P_0, \\ &[P_j, P_k] = 0,\ [J_j, K_k] = K_l,\ [P_j, P_0] = 0 \end{aligned} \tag{11}$$

where j, k, l are a cyclic permutation of 1, 2, 3 and all other commutators are zero. The dual basis for $\mathfrak{p}^*$ is the same set of symbols with * after, and we obtain

$$\begin{aligned} &\delta(P_l^*) = J_j^* \wedge P_k^* - P_j^* \wedge J_k^*,\ \delta(P_0^*) = \sum_{j=1,2,3} K_j^* \wedge P_j^*, \\ &\delta(J_l) = J_j^* \wedge J_k^* - K_j^* \wedge K_k^*,\ \delta(K_l^*) = J_j^* \wedge K_k^* - K_j^* \wedge J_k^*. \end{aligned} \tag{12}$$

One may also consider the covering group $\widetilde{\mathcal{P}} = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ where the $\mathbb{R}^4$-part is considered as the set of two-by-two complex matrices in the Cayley representation using standard Pauli spin matrices as a basis: One has, for $\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ and σ_j being the Pauli spin matrices with $\sigma_0 = \mathbf{1}$, $\boldsymbol{r} = \sum_{j=0}^{3} x_j \sigma_j$. Only this covering group is considered, letting $\widetilde{\mathcal{P}} = \mathcal{P}$ in the sequel.

4 Massive Coherent States for Spin Zero for the Poincaré Group

The 2-form in $Z^2(\mathfrak{g}^*)$ corresponding to a massive spinless particle is

$$\omega = m\delta(P_0^*) = m \sum_{j=1,2,3} K_j^* \wedge P_j^*. \tag{13}$$

Therefore, $\mathfrak{h}_\omega$ is the set generated by $\boldsymbol{J}$ and P_0. Next set $H_\omega = \exp(\mathfrak{h}_\omega)$, which is a closed subgroup of $\mathcal{P}$. Then one has the familiar phase space

$$\mathcal{P}/H_\omega = \{\exp(\boldsymbol{q}\cdot\boldsymbol{K} + \boldsymbol{p}\cdot\boldsymbol{P})\circ H_\omega \mid \boldsymbol{q} \text{ and } \boldsymbol{p} \in \mathbb{R}^3\} \tag{14}$$

with an invariant measure given by

$$d\mu = \pi^*(\omega^{\wedge 3}) = m^3 d^3q \wedge d^3p. \tag{15}$$

Now H_ω is also the stability subgroup for $\boldsymbol{t}_0 = (m, 0, 0, 0)$ in $\mathbb{R}^4$:

$$H_\omega = \exp(\mathbb{R}\boldsymbol{t}_0) \rtimes SL(2,\mathbb{C})_{\boldsymbol{t}_0}. \tag{16}$$

Next, let

$$\boldsymbol{u}_0 = m(0,1,0,0),\ \ \boldsymbol{v}_0 = m(0,0,1,0), \boldsymbol{q}_0 = m(0,0,0,1). \tag{17}$$

Then $\{A\boldsymbol{t}_0, A\boldsymbol{u}_0, A\boldsymbol{v}_0, A\boldsymbol{q}_0\}$ are a Minkowski orthogonal set of vectors with Minkowski "norms" equal to respectively m^2, -m^2, -m^2, -m^2. Thus, for any vector $\boldsymbol{a}'$ in $\mathbb{R}^4$,

$$\boldsymbol{a}' = \beta A\boldsymbol{u}_0 + \gamma A\boldsymbol{v}_0 + \zeta A\boldsymbol{w}_0 + \vartheta A\boldsymbol{t}_0.$$

Here β is the usual Fourier coefficient of $\boldsymbol{a}'$ in the direction $A\boldsymbol{u}_0$, and so forth.

Next work in the Cayley representation for $\mathbb{R}^4$ with the standard Pauli matrices, and obtain the representation of $A_{\boldsymbol{t}}$ by 2×2 matrices so that $A_{\boldsymbol{t}} \cdot \boldsymbol{r} = A_{\boldsymbol{t}}\boldsymbol{r}A_{\boldsymbol{t}}^\dagger$. One has $\boldsymbol{t}_0$ represented by $m\mathbf{1}$. Thus for $A \in SL(2,\mathbb{C})_{\boldsymbol{t}_0}$ we now have $A\cdot(m\ \mathbf{1}) = mA\mathbf{1}A^\dagger = m\mathbf{1}$. Consequently, $A \in \widetilde{SO(3)}$, the covering group of $SO(3)$.

One then obtains

$$H_\omega \simeq \exp(\mathbb{R}\boldsymbol{t}_0) \rtimes \widetilde{SO(3)}. \tag{18}$$

Noting now that $H_\omega = \exp(\mathbb{R}\boldsymbol{t}_0) \rtimes SL(2,\mathbb{C})_{\boldsymbol{t}_0}$, coordinatize $\mathcal{P}/H_\omega$. To do this, observe that

$$\begin{aligned}(\boldsymbol{a}, A) &= (\boldsymbol{a}', A')(\lambda\boldsymbol{t}_0, B),\ \ B \in SL(2,\mathbb{C})_{\boldsymbol{t}_0}\\ \text{iff } (\boldsymbol{a}, A) &= (\boldsymbol{a}' + \lambda A'\boldsymbol{t}_0, A'B)\\ \text{iff } \boldsymbol{a} &= \boldsymbol{a}' + \lambda A'\boldsymbol{t}_0 \text{ and } A = A'B \text{ for some } B.\end{aligned} \tag{19}$$

Now

$$\begin{aligned}\boldsymbol{a}' + \lambda A'\boldsymbol{t}_0 &= \beta A\boldsymbol{u}_0 + \gamma A\boldsymbol{v}_0 + \zeta A\boldsymbol{w}_0 + \vartheta A\boldsymbol{t}_0 + \lambda A'\boldsymbol{t}_0 \\ &= \beta A\boldsymbol{u}_0 + \gamma A\boldsymbol{v}_0 + \zeta A\boldsymbol{w}_0 + (\vartheta + \lambda)A\boldsymbol{t}_0\end{aligned}$$

since $A\boldsymbol{t}_0 = A'B\boldsymbol{t}_0 = A'\boldsymbol{t}_0$. But for $\lambda \in \mathbb{R}$, $\vartheta + \lambda \in \mathbb{R}$. Hence

$$\pi(\boldsymbol{a}, A) = (\beta A\boldsymbol{u}_0 + \gamma A\boldsymbol{v}_0 + \zeta A\boldsymbol{w}_0 + \mathbb{R}\boldsymbol{t}_0, A_{\boldsymbol{t}}SL(2,\mathbb{C})_{\boldsymbol{t}_0}) \tag{20}$$

where $A_{\boldsymbol{t}}$ is any element of $SL(2,\mathbb{C})/SL(2,\mathbb{C})_{\boldsymbol{t}_0}$ with $A_{\boldsymbol{t}}\boldsymbol{t}_0 = \boldsymbol{t}$, and π is the canonical projection from G to G/H_ω. Thus choose

$$\begin{aligned}\sigma \circ \pi(\boldsymbol{a}, A) &= (\beta A_{\boldsymbol{t}}\boldsymbol{u}_0 + \gamma A_{\boldsymbol{t}}\boldsymbol{v}_0 + \zeta A_{\boldsymbol{t}}\boldsymbol{w}_0, A_{\boldsymbol{t}}) \\ &= (\boldsymbol{a} - m^{-2}(\boldsymbol{a}_\mu \boldsymbol{t}^\mu)\boldsymbol{t}, A_{\boldsymbol{t}}).\end{aligned} \tag{21}$$

One may choose [2, p. 465]

$$A_{\boldsymbol{t}} = \begin{pmatrix} (t^0 + t^3)^{1/2} & 0 \\ t^1 + it^2 & (t^0 - t^3)^{1/2} \end{pmatrix}. \tag{22}$$

Hence, one has a homomorphism between $SL(2,\mathbb{C})/SL(2,\mathbb{C})_{\boldsymbol{t}_0}$ and the forward mass shell $V_m^+ = \{\boldsymbol{t} \in \mathbb{R}^4 \mid t_\mu t^\mu = m^2,\ t^0 > 0\}$. Similarly, one has a homomorphism between $\mathcal{P}/H_\omega$ and $\mathbb{R}^3 \rtimes V_m^+$.

G/H_ω also may be taken as isomorphic to $\mathbb{R}^6 = \mathbb{R}^3$ for position $\times$ $\mathbb{R}^3$ for momentum. Properly, one has all z in Minkowski space mod $\mathbb{R}\boldsymbol{t}_0$, and consequently one obtains $\mathbb{R}^3$ for the position in $\mathcal{P}/H_\omega$. The momentum comes from $SL(2,\mathbb{C})$ which is factored by $SL(2,\mathbb{C})_{\boldsymbol{t}_0}$, i.e., by the full rotation group. This leaves another $\mathbb{R}^3$ for the momentum in $\mathcal{P}/H_\omega$. This $\mathcal{P}/H_\omega$ is the phase space.

To obtain an irreducible representation space, $\mathcal{H}$, of $\mathcal{P}$ for a spin zero massive particle with representation U one uses the Mackey machine [4] of induced representations. This $\mathcal{H}$ is just the usual representation space of quantum mechanics for zero spin and mass $m > 0$.

Now, take $\eta \in L^2(V_m^+)$ satisfying the square-integrability condition

$$\int_{\mathbb{R}^3 \rtimes V_m^+} |\langle U(\sigma(\boldsymbol{x}))\eta, \eta\rangle_{\mathcal{H}}|^2 d\mu(\boldsymbol{x}) < \infty$$

and the extra (α-admissibility) condition for H_ω at $(\mathbf{0}, 1)$

$$U(h)\eta = \alpha(h)\eta,\ \ h \in H_\omega.$$

But any h is of the form $(\lambda\boldsymbol{t}_0, R)$ where $R \in \widetilde{SO(3)}$, and α is a one dimensional representation of H_ω. The only one dimensional representation of $\widetilde{SO(3)}$ is $\mathbf{1}$. Thus, η may have dependence on $p_1^2 + p_2^2 + p_3^2$

only, $(p_0, p_1, p_2, p_3) \in V_m^+$. The remaining factor of $\lambda \boldsymbol{t}_0$ one can eliminate in favor of $\mathbf{1}$ by simply requiring η to be time independent, or more properly, requiring that the η which describes the measuring instrument is regenerated by turning on the instrument at any time and then performing the measurement with $A^\eta(f)$. In this way one avoids the wave function spreading of η with time.

When one moves to $(\boldsymbol{a}, A)$ by means of the group, then $U(\boldsymbol{a}, A)\eta$ satisfies the same conditions at $(\boldsymbol{a}, A)$; i.e., $A\boldsymbol{t}_0 = \boldsymbol{t}$ is not a variable in $U(\boldsymbol{a}, A)\eta$ and there is no spin/angular momentum as a variable in $U(\boldsymbol{a}, A)\eta$.

One now is in a position in which any coherent state based on η as any α-admissible square-integrable vector is defined, with or without the condition of informational completeness. [5] As an example without the condition of informational completeness on η, take $\eta = \chi_\Delta$ for Δ a sphere in all the variables in p_1, p_2, p_3, the square-integrability being trivial and the α-admissibilty following from the condition of no spherical dependence and time independence. To obtain a coherent state with the extra condition, take $\eta =$ a Gaussian in the variables p_1, p_2, p_3 convolved with χ_Δ or even the Gaussian alone. One stresses that there are many coherent states with various properties, such as analyticity, for example.

5 Massive Coherent States with Spin for the Poincaré Group

The 2-form for a massive, spinning particle in the Poincaré group is

$$\omega = \delta(mP_0^* + SJ_3^*) \tag{23}$$

where m is the mass and S is the spin. (One may take $\boldsymbol{S}$ to be in $\mathbb{R}^3$, $||\boldsymbol{S}|| = S$, and replace SJ_3^* with $\boldsymbol{S} \cdot \boldsymbol{J}^*$ if one chooses to have a general expression.) Now from (10),

$$\omega = m \sum_{j=1}^{3} K_j^* \wedge P_j^* + S(J_1^* \wedge J_2^* - K_1^* \wedge K_2^*). \tag{24}$$

Thus $\mathfrak{h}_\omega$ is just the subalgebra generated by P_0 and J_3. Take $H_\omega = \exp\{\mathfrak{h}_\omega\}$, which is a closed subgroup of $\mathcal{P}$ again. Then the phase space is

$$\mathcal{P}/H_\omega = \{\exp(\boldsymbol{q} \cdot \boldsymbol{K} + \boldsymbol{p} \cdot \boldsymbol{P} + \theta_1 J_1 + \theta_2 J_2) \circ H_\omega \mid \boldsymbol{q}, \boldsymbol{p} \in \mathbb{R}^3, \theta_1, \theta_2 \in \mathbb{R}\} \tag{25}$$

with the invariant measure

$$d\mu = \pi^*(\omega^{\wedge 4}) = Sm^3 d^3k d^3p d\Omega \tag{26}$$

where $d\Omega$ is the invariant measure on the surface of the sphere of radius S.

Now H_ω is also the stability group of $\boldsymbol{t}_0 = m(1,0,0,0)$ and $\boldsymbol{s}_0 = S(0,0,0,1)$:

$$H_\omega = \exp(\mathbb{R}\boldsymbol{t}_0) \rtimes SL(2,\mathbb{C})_{\boldsymbol{t}_0,\boldsymbol{s}_0}. \tag{27}$$

To coordinatize $\mathcal{P}/H_\omega$, one notes again that

$$\begin{aligned} (\boldsymbol{a}, A) &= (\boldsymbol{a}', A')(\lambda \boldsymbol{t}_0, B), \;\; B \in SL(2,\mathbb{C})_{\boldsymbol{t}_0\boldsymbol{s}_0} \\ \text{iff } (\boldsymbol{a}, A) &= (\boldsymbol{a}' + \lambda A'\boldsymbol{t}_0, A'B) \\ \text{iff } \boldsymbol{a} &= \boldsymbol{a}' + \lambda A'\boldsymbol{t}_0 \text{ and } A = A'B \text{ for some } B. \end{aligned} \tag{28}$$

Now $A' \in SL(2,\mathbb{C})/SL(2,\mathbb{C})_{\boldsymbol{t}_0\boldsymbol{s}_0}$, $A' : \boldsymbol{t}_0 \mapsto \boldsymbol{t}$, $A' : \boldsymbol{s}_0 \mapsto \boldsymbol{s}$; so, $\boldsymbol{a} = \boldsymbol{a}' + \lambda\boldsymbol{t}$. However, using the Cayley representation of $\mathbb{R}^4$ by the standard Pauli matrices gives $\boldsymbol{t}_0$ represented by $m\mathbf{1}$ and $\boldsymbol{s}_0$ represented by $S\sigma_3$. Thus $SL(2,\mathbb{C})_{\boldsymbol{t}_0,\boldsymbol{s}_0} = \{diag(a,\overline{a}) \mid a \in \mathbb{C},\ |a| = 1\} \simeq O(2)\tilde{}$. Consequently, $A \in SL(2,\mathbb{C})/SL(2,\mathbb{C})_{\boldsymbol{t}_0\boldsymbol{s}_0}$, $A : \boldsymbol{t}_0 \mapsto \boldsymbol{t}$, $A : \boldsymbol{s}_0 \mapsto \boldsymbol{s}$, implies $A \in \{A_{\boldsymbol{ts}} SL(2,\mathbb{C})_{\boldsymbol{t}_0\boldsymbol{s}_0}\}$, where [2, p. 468]

$$A_{\boldsymbol{ps}} = \begin{bmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{bmatrix} \tag{29}$$

and

$$\mathfrak{a} = N[S(t^0 + t^3) + m(s^0 + s^3)]^{1/2},$$
$$\mathfrak{b} = N[(t^1 - it^2)(s^0 + s^3) - (s^1 + is^2)(t^0 + t^3)][S(t^0 + t^3) + m(s^0 + s^3)]^{-1/2},$$
$$\mathfrak{c} = N[(m(s^1 + is^2) + S(t^1 + it^2)][S(t^0 + t^3) + m(s^0 + s^3)]^{-1/2},$$
$$\mathfrak{d} = N[(t^0 - t^3)(s^0 + s^3) - (t^1 + it^2)(s^1 - is^2) + mS][S(t^0 + t^3) + m(s^0 + s^3)]^{-1/2},$$

where $N = (2mS)^{-1/2}$. Consequently, one has a homomorphism between $SL(2,\mathbb{C})/SL(2,\mathbb{C})_{\boldsymbol{t}_0\boldsymbol{s}_0}$ and the set

$$\begin{aligned} X = \{(\boldsymbol{t}, \boldsymbol{s}) &\in \mathbb{R}^4 \times \mathbb{R}^4 \\ &\mid t_\mu t^\mu = m^2,\ t_\mu s^\mu = 0,\ s_\mu s^\mu = -S^2,\ t^0 > 0\}. \end{aligned} \tag{30}$$

One may also write $\boldsymbol{a} \in \mathbb{R}^4$ in the basis

$$\{A\boldsymbol{t}_0, A\boldsymbol{u}_0, A\boldsymbol{v}_0, A\boldsymbol{w}_0\} = \{\boldsymbol{t}, A\boldsymbol{u}_0, A\boldsymbol{v}_0, A\boldsymbol{w}_0\};$$

so, one may choose the λ such that $\boldsymbol{a} = \boldsymbol{a}' + \mathbb{R}\boldsymbol{t}$ with

$$\begin{aligned} \boldsymbol{a}' = m^{-2}&(\boldsymbol{a}_\mu[A_{\boldsymbol{ts}}\boldsymbol{u}_0]^\mu)A_{\boldsymbol{ts}}\boldsymbol{u}_0 + m^{-2}(\boldsymbol{a}_\mu[A_{\boldsymbol{ts}}\boldsymbol{v}_0]^\mu)A_{\boldsymbol{ts}}\boldsymbol{v}_0 \\ &+ m^{-2}(\boldsymbol{a}_\mu[A_{\boldsymbol{ts}}\boldsymbol{w}_0]^\mu)A_{\boldsymbol{ts}}\boldsymbol{w}_0. \end{aligned} \tag{31}$$

Note that the last term may be shortened, as $\boldsymbol{w}_0$ is a multiple of s_3, and as such is invariant under $A_{\boldsymbol{ts}}$. We will abbreviate $\boldsymbol{a}'$ as $\boldsymbol{a} - m^{-2}(\boldsymbol{a}_\mu \boldsymbol{t}^\mu)\boldsymbol{t}$. Thus $[(\boldsymbol{a}, A)] \in \mathcal{P}/H_\omega$ implies that $[(\boldsymbol{a}, A)] = (\boldsymbol{a}', A_{\boldsymbol{ps}})H_\omega$. This establishes a homomorphism from $\mathcal{P}/H_\omega$ to $\mathbb{R}^3 \rtimes X$.

The canonical projection is

$$\pi(\boldsymbol{a}, A) = (\boldsymbol{a} - m^{-2}(\boldsymbol{a}_\mu \boldsymbol{t}^\mu)\ \boldsymbol{t} + \mathbb{R}\boldsymbol{t}, A_{\boldsymbol{ts}} SL(2,\mathbb{C})_{\boldsymbol{t}_0 \boldsymbol{s}_0}) \tag{32}$$

where $A\boldsymbol{t}_0 = A_{\boldsymbol{ts}}\boldsymbol{t}_0 = \boldsymbol{t}$, and $A\boldsymbol{s}_0 = A_{\boldsymbol{ps}}\boldsymbol{s}_0 = \boldsymbol{s}$. The (continuous) Borel section is

$$\sigma \circ \pi(\boldsymbol{a}, A) = (\boldsymbol{a} - m^{-2}(\boldsymbol{a}_\mu \boldsymbol{t}^\mu)\boldsymbol{t}, A_{\boldsymbol{ts}}). \tag{33}$$

Coherent states based on $\eta \in L^2(X) = \mathcal{H}$ (or any other representation like the familiar representation with the spin being represented by vectors in $\mathbb{C}^n$) are defined as any vector satisfying the square-integrability condition and the α-admissibility condition which now reads

$$U(\exp\{\lambda P_0 + \theta J_3\})\eta = \alpha(\exp\{\lambda P_0 + \theta J_3\})\eta. \tag{34}$$

Thus, having η as an eigenfunction of J_3 and "time independent" as in the spin zero case, one obtains the α-admissibility condition. This conforms to the usual decomposition of η as being time independent, having a Y_l^m dependence on the J's, and having a dependence on the momenta p_1 and p_2 only through $p_1^2 + p_2^2$. This holds for all massive particles of any non-zero spin and in particular for spin 1/2 and spin 1 particles.

6 Fuzzy Quantization of Space-time

There are two ways to discuss this.

The first way: Having the coherent states in hand, one now turns to the quantum measurement of the position of any massive quantum particle. Take f equal to any function that has a dependence on position of, say, any characteristic function of a Borel set, and is unconstrained otherwise. Take η to be a vector in $\mathcal{H}$ that has quantum expectation of position to be zero in all components other than the nonzero spin in which case it is N, the North pole of the sphere. Then $U(g)\eta$, $g \in G$, will have quantum expectation of having the momentum $= \boldsymbol{p}$, the configuration position $= \boldsymbol{q}$, and the spin direction $\boldsymbol{s}$ for $g = \sigma(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})$. η must also be square-integrable and α-admissible (and perhaps even informationally complete). One uses $\eta(\boldsymbol{x})$, or really $T^\eta(\boldsymbol{x})$, to mark the point $\boldsymbol{x} \in \Gamma = G/H_\omega$. Then any state ρ has the probability $Tr(\rho T^\eta(\boldsymbol{x}))$ of being observed at $\boldsymbol{x}$. If we take the marginals over the spin and the momentum, we will obtain the probability of ρ being at the resulting position when measuring with η. This is a fuzzy position. If η has more than one position at which

$$"|\eta(\boldsymbol{z})|^2" \equiv \int_{\boldsymbol{y}\in\Gamma(mom,spin)} |\eta(\boldsymbol{x})|^2 d\mu(\boldsymbol{y},\boldsymbol{z}),$$

$\boldsymbol{x} = (\boldsymbol{y},\boldsymbol{z})$, with $\Gamma(mom, spin)$ = the part of Γ involving momentum and spin only, is a maximum, then the position will be effectively not well defined but rather having a multiplicity equal to the number of positions at which "$|\eta(\boldsymbol{z})|^2$" is a maximum. One assumes a single maximum. Then the position is well-defined albeit fuzzily.

Notice that then the position is defined independent of spin and momentum in a G-invariant fashion! Now, one may take a collection η_s of square-integrable and α-admissible vectors, one for each spin s, and compute the fuzzy positions accordingly in $\mathcal{H} = \oplus_s \mathcal{H}^s$ with $\mathcal{H}^s$ equal to the Hilbert space for particles with spin s (and $m > 0$). The variances of each will depend on s, but the centers will not.

This will be the fuzzy quantization of space-time. Notice that the fuzziness is necessary if one takes the resonable requirement that one must measure a quantum particle with another quantum particle (η) that is chosen beforehand.

The second way: One may consider the measurement operator $A^\eta(f)$ and its spectrum. First take f = the characteristic function of a Borel set

$$\Delta = \{(q,p) \mid a_{1,j} \le q_j < a_{2,j}, b_{1,j} \le p_j < b_{2,j}, j \in \{1, \cdots, n\}\}$$

for some $a_{1,j} < a_{2,j}$ and $b_{1,j} < b_{2,j}$ for all components j. If one has the usual interpretations that $\boldsymbol{q}$ is the position operator and $\boldsymbol{p}$ is the momentum operator, then one would expect that $\sum_j (a_{2,j} - a_{1,j})(b_{2,j} - b_{1,j}) \lesssim \hbar$ would give that $A^\eta(\chi_\Delta)$ has one eigenvalue equal to 1 and the rest 0. But in general, there is also the spin to consider. One could take for spin variables that they have variances equal to the entire spin space, or for spin J that they could have variances equal to $1/(2J+1)$th the entire spin space because all one is interested in are the $\boldsymbol{q}$'s and $\boldsymbol{p}$'s. Ignore this for the present.

A problem arises at once. Having only eigenvalues in the set $\{0, 1\}$ turns out to be impossible for the $A^\eta(\chi_\Delta)$ as it is not a non-trivial projection. [6] What one may say is that $A^\eta(\chi_\Delta)$ is a positive operator bounded by the $\min\{1, \mu(\Delta)\}$, is compact so has discrete spectrum only, and has its spectrum in l_p, $1 \le p \le \infty$. [2, pp. 121-133, 243-247] Plotting its spectrum in decreasing order one discovers that initially the eigenvalues are close to but less than 1 and eventually are close to but strictly larger than 0. What is surprising is that, for our choice of Δ, the drop is quite dramatic. (That is also the fact if Δ is a Borel set with for example a non-fractal boundary.) One now asks what the "channel capacity theorem" is for $A^\eta(\chi_\Delta)$, practically speaking.

Definition 1 *A Hilbert space vector ψ is said to be η-localized in Borel region Δ with attenuation factor $|\gamma|^2$ iff $A^\eta(\chi_\Delta)\psi = \gamma\psi$.*

For the spectrum as described above, the eigenvalues have the first n of them about equal to 1 and thus their eigenvectors are η-localized, the eigenvectors of the middle eigenvalues have a questionable localization, and the remaining ones are practically annihilated by $A^\eta(\chi_\Delta)$. With our particular choice of Δ, one may arrange it so that $n = 1$ and the remaining eigenvectors are effectively annihilated by $A^\eta(\chi_\Delta)$. This is what we describe as Δ having the minimum uncertainty property.

Next take a group element that describes translation by integral multiples of $a_{2,j} - a_{1,j}$ and $b_{2,j} - b_{1,j}$. Repeated application of these translations to χ_Δ gives a tiling of the space G/H_ω, modulo the spin. Because the operator $A^\eta(\chi_\Delta)$ is covariant under the group [2, p. 312], and the operators of translation are unitary, one finds that each of these displaced $A^\eta(\chi_\Delta)$'s has exactly the same spectrum (and with the first eigenvector) as the translation of $A^\eta(\chi_\Delta)$ (and with the first eigenvector of it). Thus, for all practical purposes, one has discretized the space. One may include the spin in a similar fashion.

Because the operators $A^\eta(\chi_\Delta)$ and $A^\eta(\chi_{\Delta'})$ are fuzzy, one doesn't have their first eigenvectors orthogonal for $\Delta \cap \Delta' = \emptyset$. But one may make the overlap quite small; it is a matter of fiddling with the expression for η.

7 An Example - Planck Length

Suppose one has two particles of the same mass and spin impinging on each other with some initial relative momentum $\boldsymbol{p}$. Let the two particles have wave functions equal to ψ and φ. Take η to be the square-integrable and α-admissible vector of the same mass and spin generating a well-defined position distribution. Then one has the appropriate marginals of the $prob_\varphi(\boldsymbol{x}) = Tr(P_\varphi T^\eta(\boldsymbol{x}))$ and $prob_\psi(\boldsymbol{y}) = Tr(P_\psi T^\eta(\boldsymbol{y}))$ describing the positions of both. For whatever reason, the trajectories of both are such that there is a closest "distance" each gets to the other. What is that "distance"? It will be the Euclidean distance from the $\mathbb{R}^4$ part of $\boldsymbol{x}$ to the $\mathbb{R}^4$ part of $\boldsymbol{y}$ at the closest distance since we have defined η to have a unique maximum of $|\eta(\boldsymbol{z})|^2$. (And note that the time component of both is the same.) It seems that this is independent of the other particulars of η, since the $prob_\varphi(\boldsymbol{x})$ depends on η only on $\boldsymbol{x}$ and this is the location of the *classical* center of $U(\sigma(\boldsymbol{x}))\eta$. Also note that at the closest approach point one obtains the Planck length squared which is reflected in $\sigma(\boldsymbol{z})$ and may be realized in

$$
\begin{aligned}
&\langle \eta, C^2\eta\rangle_{\mathcal{H}}\langle\psi, U(\sigma(\boldsymbol{z}))\varphi\rangle_{\mathcal{H}} \\
&= \int_{G/H_\omega} \langle\psi, T^\eta(\boldsymbol{x})U(\sigma(\boldsymbol{z}))\varphi\rangle_{\mathcal{H}} d\mu(\boldsymbol{x}) \\
&= \int_{G/H_\omega} \langle\psi, U(\sigma(\boldsymbol{x}))\eta\rangle_{\mathcal{H}}\langle U(\sigma(\boldsymbol{x}))\eta, U(\sigma(\boldsymbol{z}))\varphi\rangle_{\mathcal{H}} d\mu(\boldsymbol{x}) \\
&\leq \int_{G/H_\omega} \left[Tr(P_\psi T^\eta(\boldsymbol{x}))Tr(P_{U(\sigma(\boldsymbol{z}))\varphi}T^\eta(\boldsymbol{x}))\right]^{1/2} d\mu(\boldsymbol{x}).
\end{aligned}
$$

This Euclidean-distance-at-closest-approach is what may be described as the Planck length. But what about the dependance on the momentum or spin? Here there may be a closest approach with relative momentum $\boldsymbol{p}$ and/or spin $\boldsymbol{s}$ that depends on $\boldsymbol{p}$ and/or $\boldsymbol{s}$. That can be handled with our phase space approach. Then the "closest" they get is the minimum over $\boldsymbol{p}$ and $\boldsymbol{s}$ of all the closest distances that depend on $\boldsymbol{p}$ and $\boldsymbol{s}$.

Having the φ describe the electron in orbit around a nucleus and the ψ a free electron is a special case, and won't be further discussed except to note that for that, one would have to use the particular case of spin equal to 1/2.

8 Conclusion

We have shown that the phase spaces for massive representations of the Poincaré group lead to general formulas for the coherent states which we have made explicite here for the first time. This is interpreted as measurement theory based on a quantum particle. This, in turn, leads to fuzzy quantization of space-time (1) by taking the marginals of $\sigma(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})$ over $\boldsymbol{p}$ and $\boldsymbol{s}$ or (2) by taking space-time-momentum and tiling it with tiles Δ such that $A^\eta(\chi_\Delta)$ has one eigenvector with eigenvalue equal to $1-\varepsilon$ and all the rest of the eigenvalues being less than $\varepsilon \approx 0$.

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