AUTOMORPHIC LOOPS ARISING FROM MODULE ENDMORPHISMS

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Abstract. A loop is automorphic if all its inner mappings are automorphisms. We construct a large family of automorphic loops as follows. Let $R$ be a commutative ring, $V$ an $R$-module, $E = \text{End}_R(V)$ the ring of $R$-endomorphisms of $V$, and $W$ a subgroup of $(E,+)$ such that $ab = ba$ for every $a, b \in W$ and $1+a$ is invertible for every $a \in W$. Then $Q_{R,V}(W)$ defined on $W \times V$ by

$$(a,u)(b,v) = (a+b, u(1+b) + v(1-a))$$

is an automorphic loop.

A special case occurs when $R = k < K = V$ is a field extension and $W$ is a $k$-subspace of $K$ such that $k1 \cap W = 0$, naturally embedded into $\text{End}_k(K)$ by $a \mapsto M_a$, $bM_a = ba$. In this case we denote the automorphic loop $Q_{R,V}(W)$ by $Q_{k,K}(W)$.

We call the parameters tame if $k$ is a prime field, $W$ generates $K$ as a field over $k$, and $K$ is perfect when $\text{char}(k) = 2$. We describe the automorphism groups of tame automorphic loops $Q_{k,K}(W)$, and we solve the isomorphism problem for tame automorphic loops $Q_{k,K}(W)$.

A special case solves a problem about automorphic loops of order $p^3$ posed by Jedlička, Kinyon and Vojtěchovský.

We conclude the paper with a construction of an infinite 2-generated abelian-by-cyclic automorphic loop of prime exponent.

1. Introduction

A groupoid $Q$ is a quasigroup if for all $x \in Q$ the translations $L_x : Q \to Q$, $R_x : Q \to Q$ defined by $yL_x = xy$, $yR_x = yx$ are bijections of $Q$. A quasigroup $Q$ is a loop if there is $1 \in Q$ such that $1x = x1 = x$ for every $x \in Q$.

Let $Q$ be a loop. The multiplication group of $Q$ is the permutation group $\text{Mlt}(Q) = \langle L_x, R_x : x \in Q \rangle$, and the inner mapping group of $Q$ is the subgroup $\text{Inn}(Q) = \{ \varphi \in \text{Mlt}(Q) : 1\varphi = 1 \}$.

A loop $Q$ is said to be automorphic if $\text{Inn}(Q) \trianglelefteq \text{Aut}(Q)$, that is, if every inner mapping of $Q$ is an automorphism of $Q$. Since, by a result of Bruck [1], $\text{Inn}(Q)$ is generated by the bijections

$$T_x = R_x L_x^{-1}, \quad L_{x,y} = L_x L_y L_{y,x}^{-1}, \quad R_{x,y} = R_x R_y R_{y,x}^{-1},$$

a loop $Q$ is automorphic if and only $T_x$, $L_{x,y}$, $R_{x,y}$ are homomorphisms of $Q$ for every $x, y \in Q$. In fact, by [7, Theorem 7.1], a loop $Q$ is automorphic if and only if every $T_x$ and $R_{x,y}$ are automorphisms of $Q$. The variety of automorphic loops properly contains the variety of groups.

See [1] or [12] for an introduction to loop theory. The first paper on automorphic loops is [2]. It was shown in [2] that automorphic loops are power-associative, that is, every

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element of an automorphic loop generates an associative subloop. Many structural results on automorphic loops were obtained in [9], where an extensive list of references can be found.

1.1. The general construction. In this paper we study the following construction.

**Construction 1.1.** Let \( R \) be a commutative ring, \( V \) an \( R \)-module and \( E = \text{End}_R(V) \) the ring of \( R \)-endomorphisms of \( V \). Let \( W \) be a subgroup of \((E,+)\) such that

\begin{align}
(1) \quad ab = ba \text{ for every } a, b \in W, \text{ and} \\
(2) \quad 1 + a \text{ is invertible for every } a \in W,
\end{align}

where \( 1 \in E \) is the identity endomorphism on \( V \).

Define \( Q_{R,V}(W) \) on \( W \times V \) by

\[
(a,u)(b,v) = (a+b,u(1+b)+v(1-a)).
\]

We show in Theorem 2.2 that \( Q_{R,V}(W) \) is always an automorphic loop.

Two special cases of this construction appeared in the literature. First, in [6], the authors proved that commutative automorphic loops of odd prime power order are centrally nilpotent, and constructed a family of (noncommutative) automorphic loops of order \( p^3 \) with trivial center by using the following construction.

**Construction 1.2.** Let \( k \) be a field and \( M_2(k) \) the vector space of \( 2 \times 2 \) matrices over \( k \) equipped with the determinant norm. Let \( I \) be the identity matrix, and let \( A \in M_2(k) \) be such that \( kI \oplus kA \) is an anisotropic plane in \( M_2(k) \), that is, \( \det(aI+bA) \neq 0 \) for every \( (a,b) \neq (0,0) \). Define \( Q_k(A) \) on \( k \times (k \times k) \) by \( (a,u)(b,v) = (a+b,u(I+bA)+v(I-aA)) \).

We will show in Section 4 that the loops \( Q_k(A) \) are a special case of the construction \( Q_{R,V}(W) \) and hence automorphic. If \( k = \mathbb{F}_p \) then \( Q_k(A) \) has order \( p^3 \), exponent \( p \) and trivial center, by [6, Proposition 5.6].

Second, in [10], Nagy used a construction of automorphic loops based on Lie rings (cf. [8] and [9]) and arrived at the following.

**Construction 1.3.** Let \( V, W \) be vector spaces over \( \mathbb{F}_2 \), and let \( \beta : W \rightarrow \text{End}(V) \) be a linear map such that \( a\beta \beta = b\beta a \beta \) for every \( a, b \in W \), and \( 1 + a\beta \) is invertible for every \( a \in W \). Define a loop \((W \times V, *)\) by \((a,u) \ast (b,v) = (a+b,u(1+b\beta)+v(1+a\beta))\).

When \( \beta \) is injective, Construction 1.3 is a special case of our Construction 1.1, and when \( \beta \) is not injective, it is a slight variation. By [10, Proposition 3.2], \((W \times V, *)\) is an automorphic loop of exponent 2 and, moreover, if \( \beta \) is injective and at least one \( a\beta \) is invertible then \((W \times V, *)\) has trivial center.

1.2. The field extension construction. Most of this paper is devoted to the following special case of Construction 1.1.

**Construction 1.4.** Let \( R = k < K = V \) be a field extension, and let \( W \) be a \( k \)-subspace of \( V \) such that \( k1 \cap W = 0 \). Embed \( W \) into \( \text{End}_K(V) \) via \( a \mapsto M_a, bM_a = ba \). Denote by \( Q_{k<K}(W) \) the loop \( Q_{R,V}(W) \) of Construction 1.1.

Assuming the situation of Construction 1.4, the condition (A1) of Construction 1.1 is obviously satisfied because the multiplication in \( K \) is commutative and associative. Moreover, \( k1 \cap W = 0 \) is equivalent to \( 1+a \neq 0 \) for all \( a \in W \), which is equivalent to (A2). Construction 1.1 therefore applies and \( Q_{k<K}(W) \) is an automorphic loop.
For the purposes of this paper, we call the parameters $k$, $K$, $W$ of Construction 1.4 tame if $k$ is a prime field, $W$ generates $K$ as a field over $k$, and $K$ is perfect when $\text{char}(k) = 2$.

In Corollary 3.3 we solve the isomorphism problem for tame automorphic loops $Q_{k<K}(W)$, given a fixed extension $k < K$, and in Theorem 3.5 we describe the automorphism groups of tame automorphic loops $Q_{k<K}(W)$. In particular, we solve the isomorphism problem when $k$ is a finite prime field and $K$ is a quadratic extension of $k$. This answers a problem about automorphic loops of order $p^3$ posed in [6], and it disproves [6, Conjecture 6.5].

Finally, in Section 5 we use the construction $Q_{k<K}(W)$ to obtain an infinite 2-generated abelian-by-cyclic automorphic loop of prime exponent.

2. Automorphic loops from module endomorphisms

Throughout this section, assume that $R$ is a commutative ring, $V$ an $R$-module, $W$ a subgroup of $E = (\text{End}_R(V), +)$ satisfying (A1) and (A2), and $Q_{R,V}(W)$ is defined on $W \times V$ by (1.1) as in Construction 1.1.

It is easy to see that $(0, 0) = (0_E, 0_V)$ is the identity element of $Q_{R,V}(W)$, and that $(a, u) \in Q_{R,V}(W)$ has the two-sided inverse $(-a, -u)$.

Using the notation

$$I_a = 1 + a \quad \text{and} \quad J_a = 1 - a,$$ we can rewrite the multiplication formula (1.1) as

$$(a, u)(b, v) = (a + b, uI_b + vJ_a).$$

A straightforward calculation then shows that the left and right translations $L_{(a,u)}$, $R_{(a,u)}$ in $Q_{R,V}(W)$ are invertible, with their inverses given by

$$(2.1) \quad (a, u)(b, v) = (b, v)L_{(a,u)}^{-1} = (b - a, (v - uI_{b-a})J_{a}^{-1}),$$

$$(2.2) \quad (b, v)/(a, u) = (b, v)R_{(a,u)}^{-1} = (b - a, (v - uJ_{b-a})I_{a}^{-1}),$$

respectively. Hence $Q_{R,V}(W)$ is a loop.

The multiplication formula (1.1) yields $(a, 0)(b, 0) = (a + b, 0)$ and $(0, u)(0, v) = (0, u + v)$, so $W \times 0$ is a subloop of $Q_{R,V}(W)$ isomorphic to the abelian group $(W, +)$ and $0 \times V$ is a subloop of $Q_{R,V}(W)$ isomorphic to the abelian group $(V, +)$. Moreover, the mapping $Q_{R,V}(W) = W \times V \to W$ defined by $(a, u) \mapsto a$ is a homomorphism with kernel $0 \times V$. Thus $0 \times V$ is a normal subgroup of $Q_{R,V}(W)$.

We proceed to show that $Q_{R,V}(W)$ is an automorphic loop.

Let $C_E(W) = \{a \in E : ab = ba \text{ for every } b \in W\}$.

**Lemma 2.1.** For $d \in C_E(W)^*$ and $x \in V$ define $f_{(d,x)} : Q_{R,V}(W) \to Q_{R,V}(W)$ by

$$(a, u)f_{(d,x)} = (a, xa + ud).$$

Then $f_{(d,x)} \in \text{Aut}(Q_{R,V}(W))$.

**Proof.** We have $((a, u)(b, v))f_{(d,x)} = (a + b, uI_b + vJ_a)f_{(d,x)} = (a + b, x(a + b) + (uI_b + vJ_a)d)$, where the second coordinate is equal to $xa + xb + ud + ubd + vd - vad$. On the other hand, $(a, u)f_{(d,x)} \cdot (b, v)f_{(d,x)} = (a, xa + ud)(b, xb + vd) = (a + b, (xa + ud)I_b + (xb + vd)J_a)$, where the second coordinate is equal to $xa + xab + ud + udb + xb - xba + vd - vda$. Note that $ab = ba$ because $a, b \in W$, and $ad = da$, $bd = db$ because $d \in C_E(W)$. The mapping $f_{(d,x)}$ is therefore an endomorphism of $Q_{R,V}(W)$.
Suppose that \((a,u)f_{(d,x)} = (b,v)f_{(d,x)}\). Then \((a, xa + ud) = (b, xb + vd)\) implies \(a = b\) and \(ud = vd\). Since \(d\) is invertible, we have \(u = v\), proving that \(f_{(d,x)}\) is one-to-one.

Given \((b, v) \in Q_{R,V}(W)\), we have \((a,u)f_{(d,x)} = (b,v)\) if and only if \((a, xa + ud) = (b, v)\). We can therefore take \(a = b\) and \(u = (v - xa)d^{-1}\) to see that \(f_{(d,x)}\) is onto. \(\square\)

**Theorem 2.2.** The loops \(Q_{R,V}(W)\) obtained by Construction 1.1 are automorphic.

**Proof.** We have already shown that \(Q = Q_{R,V}(W)\) is a loop. In view of [7, Theorem 7.1], it suffices to show that for every \((a,u), (b,v) \in Q\) the inner mappings \(T_{(a,u)}, L_{(a,u),(b,v)}\) are automorphisms of \(Q\). Using (2.1), we have

\[
(b,v) T_{(a,u)} = (b,v) R_{(a,u)} L_{(a,u)}^{-1} = (b, a + v I_a + u J_b) L_{(a,u)}^{-1}
= (b, (v I_a + u J_b - u J_b) J_a^{-1}) = (b, (u(J_b - I_b) J_a^{-1} + v I_a J_a^{-1})
= (b, -2u b J_a^{-1} + v I_a J_a^{-1}) = (b, (-2u J_a^{-1}) b + v (I_a J_a^{-1}))
\]

where we have also used \(b J_a^{-1} = J_a^{-1} b\). Thus \(T_{(a,u)} = f_{(d,x)}\) with \(d = I_a J_a^{-1}\) and \(x = -2u J_a^{-1} \in V\). Note that \(d \in C_E(W)^*\) by (A1), (A2). By Lemma 2.1, \(T_{(a,u)} \in \text{Aut}(Q)\).

Furthermore,

\[
(c,w) L_{(a,u),(b,v)} = ((b,v) \cdot (a,u) (c,w)) L_{(b,v)(a,u)}^{-1}
= ((b,v)(a + c, u I_c + w J_a)) L_{(b+a,v I_a+u J_b)}^{-1}
= (b + a + c, v I_{a+c} + u I_c J_b + w J_a J_b) L_{(b+a,v I_a+u J_b)}^{-1}
= (c, u I_{a+c} + u I_c J_b + w J_a J_b - v I_a - u J_b I_c) J_a^{-1}
= (c, v I_{a+c} - I_a I_c) J_b^{-1} + w J_a J_b J_a^{-1}
= (c, -va J_{b+a} + w J_a J_b J_a^{-1}) = (c, (-va J_{b+a}) c + w (J_a J_b J_a^{-1}))
\]

Thus \(L_{(a,u),(b,v)} = f_{(d,x)}\) with \(d = J_a J_b J_{b+a}^{-1} \in C_E(W)^*\) and \(x = -va J_{b+a} \in V\). By Lemma 2.1, \(L_{(a,u),(b,v)} \in \text{Aut}(Q)\). \(\square\)

For a loop \(Q\), the **associator subloop** \(\text{Asc}(Q)\) is the smallest normal subloop of \(Q\) such that \(Q/\text{Asc}(Q)\) is a group. Given \(x, y, z \in Q\), the **associator** \([x, y, z]\) is the unique element of \(Q\) such that \((xy)z = [x, y, z](xyz)\), so

\[
[x, y, z] = (((xy)z)/(xyz)) = ((xy)z) R_{x(yz)}^{-1}
\]

It is easy to see that \(\text{Asc}(Q)\) is the smallest normal subloop of \(Q\) containing all associators.

**Lemma 2.3.** Let \(Q = Q_{R,V}(W)\). Then

\[
[(a,u), (b,v), (c,w)] = (0, (ubc - wab) I_{a+b+c}^{-1})
\]

for every \((a,u), (b,v), (c,w) \in Q\). In particular, \(\text{Asc}(Q) \leq 0 \times V\).

**Proof.** The associator \([a,u), (b,v), (c,w)]\) is equal to

\[
((a,u)(b,v) \cdot (c,w)) R_{(a,u),(b,v)(c,w)}^{-1}
= (a + b + c, (u I_b + v J_a) I_c + w J_{a+b}) R_{(a+b+c,v I_{b+c}+w I_c+J_b)(a)}^{-1}
= (0, (u J_b I_c + v J_a I_c - w J_{a+b+c} - v I_c J_a - w J_b J_a I_{a+b+c}^{-1})
= (0, (ubc - wab) I_{a+b+c}^{-1})
\]
Corollary 2.4. Let $Q = Q_{R,V}(W)$.

(i) $Q$ is a group if and only if $W^2 = \{ab : a, b \in W\} = 0$.

(ii) If $VW^2 = V$ then $\text{Asc}(Q) = 0 \times V$.

Proof. (i) It is clear that $Q$ is a group if and only if $\text{Asc}(Q) = 0$. Suppose that $Q$ is a group. Taking $w = 0$ and $a = -(b + c)$ in Lemma 2.3, we get $[(a,u), (b,v), (c,w)] = (0, ubc)$, so $W^2 = 0$. Conversely, if $W^2 = 0$ then the formula of Lemma 2.3 shows that every associator vanishes.

(ii) As above, with $w = 0$ and $a = -(b + c)$ we get $[(a,u), (b,v), (c,w)] = (0, ubc)$. Since $VW^2 = V$, we conclude that $0 \times V \leq \text{Asc}(Q)$. The other inclusion follows from Lemma 2.3. □

3. Automorphic loops from field extensions

Throughout this section we will assume that $R = k < K = V$ is a field extension, $k$ embeds into $K$ via $\lambda \mapsto \lambda 1$, and $W$ is a $k$-subspace of $K$ such that $k1 \cap W = 0$, where we identify $a \in W$ with $M_a : K \rightarrow K$, $b \mapsto ba$. We write $M_W = \{M_a : a \in W\}$.

We have already pointed out in the introduction that (A1), (A2) are then satisfied, giving rise to the automorphic loop $Q_{k<K}(W)$ of Construction 1.4. Note that the multiplication formula (1.1) on $W \times K$ makes sense as written even with addition and multiplication from $K$.

Corollary 3.1. Let $Q = Q_{k<K}(W)$ with $W \neq 0$. Then $\text{Asc}(Q) = 0 \times K$.

Proof. Let $0 \neq a \in W$ and note that $M_a$ is a bijection of $V$. Thus $VW^2 \supseteq VM_aM_a = V$, and we are done by Corollary 2.4. □

3.1. Isomorphisms. We proceed to investigate isomorphisms between loops $Q_{k<K}(W)$ for a fixed field extension $k < K$.

Let $W_0$, $W_1$ be two $k$-subspaces of $K$ satisfying $k1 \cap W_0 = 0 = k1 \cap W_1$. Let

$$S(W_0, W_1) = \{A : A \text{ is an additive bijection } K \rightarrow K \text{ and } A^{-1}M_{W_0}A = M_{W_1}\}.$$ 

Any $A \in S(W_0, W_1)$ induces the map $\bar{A} : W_0 \rightarrow W_1$ defined by

$$A^{-1}M_aA = M_{a\bar{A}}, \quad a \in W_0,$$

in fact an additive bijection $W_0 \rightarrow W_1$. Indeed: $\bar{A}$ is onto $W_1$ by definition; if $a, b \in W_0$ are such that $A^{-1}M_aA = A^{-1}M_bA$ then $M_a = M_b$ and $a = a1M_a = 1M_b = b$, so $\bar{A}$ is one-to-one; and $M_{(a+b)\bar{A}} = A^{-1}M_{a+b}A = A^{-1}(M_a + M_b)A = A^{-1}M_aA + A^{-1}M_bA = M_{a\bar{A}} + M_{b\bar{A}}$, so $(a+b)\bar{A} = a\bar{A} + b\bar{A}$.

Proposition 3.2. For $i \in \{0,1\}$, let $Q_i = Q_{k<K}(W_i)$ with $W_i \neq 0$. Suppose that $K$ is perfect if $\text{char}(k) = 2$. Then there is a one-to-one correspondence between the set $\text{Iso}(Q_0, Q_1)$ of all isomorphisms $Q_0 \rightarrow Q_1$ and the set $S(W_0, W_1) \times K$. The correspondence is given by

$$\Phi : \text{Iso}(Q_0, Q_1) \rightarrow S(W_0, W_1) \times K, \quad f\Phi = (A, c),$$

where $(A, c)$ are defined by

$$(0, u)f = (0, uA) \quad \text{and} \quad (a, 0)f = (a\bar{A}, c \cdot a\bar{A}),$$

for $a \in W_0$, $c \in K$. □
and by the converse map
\[ \Psi : S(W_0, W_1) \times K \to \text{Iso}(Q_0, Q_1), \quad (A, c) \Psi = f, \]
where \( f \) is defined by
\[ (a, u)f = (a\bar{A}, c \cdot a\bar{A} + uA). \]

Proof. Given \( A \in S(W_0, W_1) \) and \( c \in K \), let \( f : Q_0 \to Q_1 \) be defined by (3.1). It is not difficult to see that \( f \) is a bijection. We claim that \( f \) is a homomorphism. Indeed, \( \bar{A} \) is additive, we have
\[ (a, u)f \cdot (b, v) = (a\bar{A}, c \cdot a\bar{A} + uA)(b\bar{A}, c \cdot b\bar{A} + vA) \]
\[ = (a\bar{A} + b\bar{A}, (c \cdot a\bar{A} + uA)I_{b\bar{A}} + (c \cdot b\bar{A} + vA)J_{a\bar{A}}) \]
and
\[ ((a, u)(b, v))f = (a + b, uI_b + vJ_a)f = ((a + b)\bar{A}, c \cdot (a + b)\bar{A} + (uI_b + vJ_a)A), \]
so it remains to show \( AI_{b\bar{A}} = I_bA \) and \( AJ_{a\bar{A}} = J_aA \) for every \( a, b \in W_0 \). This follows from \( A^{-1}M_aA = M_{a\bar{A}} \), and we conclude that \( \Psi \) is well-defined.

Conversely, let \( f : Q_0 \to Q_1 \) be an isomorphism. Corollary 3.1 gives \( \text{Asc}(Q_0) = 0 \times K = \text{Asc}(Q_1) \), and so \((0 \times K)f = 0 \times K\). Hence there is a bijection \( A : K \to K \) such that \((0, u)f = (0, uA)\) for every \( u \in K \). Then \((0, uA + vA) = (0, uA)(0, vA) = (0, u)f(0, v)f = ((0, u)(0, v))f = (0, u + v)f = (0, (u + v)A)\) shows that \( A \) is additive.

Let \( B : W_0 \to W_1, C : W_0 \to K \) be such that \((a, 0)f = (aB, aC)\) for every \( a \in W_0 \). Note that \((0, 0)f = (0, 0)\) implies \( 0B = 0 = 0C \). Because \((a, u) = (0, 0)(0, uJ_\alpha^{-1})\), we must have
\[ (a, u)f = (a, 0)f \cdot (0, uJ_\alpha^{-1})f = (aB, aC)(0, uJ_\alpha^{-1})A = (aB, aC + uJ_\alpha^{-1}A)J_{aB}. \]
This proves that \( B \) is onto \( W_1 \). Since
\[ ((a + b)B, (a + b)C) = (a + b, 0)f = ((a, 0)(b, 0))f = (a, 0)f \cdot (b, 0)f \]
\[ = (aB, aC)(bB, bC) = (aB + bB, aCJ_{bB} + bCJ_{aB}), \]
\( B \) is additive. To show that \( B \) is one-to-one, suppose that \( aB = bB \). Then \((a - b)B = 0\) by additivity, and \( a = b \) follows from the fact that \((0, K)f = (0, K)\).

We also deduce from the above equality that
\[ (a + b)C = aC + aC \cdot bB + bC - bC \cdot aB. \]
Using (3.3) and \((a + b)C = (b + a)C\), we obtain \( 2(aC \cdot bB) = 2(bC \cdot aB) \). If \( \text{char}(k) \neq 2 \), we deduce
\[ aC \cdot bB = bC \cdot aB. \]
If \( \text{char}(k) = 2 \), we can use (3.3) repeatedly to get
\[ bC = ((a + b) + a)C = (a + b)C + aC + (a + b)C \cdot aB + aC \cdot (a + b)B \]
\[ = (aC + bC + aC \cdot bB + bC \cdot aB) + aC + (aC + bC + aC \cdot bB + bC \cdot aB) \cdot aB \]
\[ + aC \cdot aB + aC \cdot bB \]
\[ = bC + aC \cdot bB \cdot aB + bC \cdot aB \cdot aB. \]
Hence \( aC \cdot bB \cdot aB = bC \cdot aB \cdot aB \). When \( a \neq 0 \), we can cancel \( ab \neq 0 \) and deduce (3.4). When \( a = 0, (3.4) \) holds thanks to \( 0B = 0 = 0C \).
Therefore, in either characteristic, we can fix an arbitrary \( 0 \neq b \in W_0 \) and obtain from (3.4) the equality \( aC = ((bB)^{-1} \cdot bC) \cdot aB \) for every \( a \in W_0 \). Hence \( aC = c \cdot aB \) for some (unique) \( c \in K \).

We proceed to show that

\[
(3.5) \quad A^{-1} MaA = M_aB
\]

for every \( a \in W_0 \). By (3.2),

\[
(a, u)f \cdot (b, v)f = (aB, aC + uJ_a^{-1}AJ_aB)(bB, bC + vJ_b^{-1}AJ_bB)
\]

\[
= (aB + bB, (aC + uJ_a^{-1}AJ_aB)I_{bb} + (bC + vJ_b^{-1}AJ_bB)J_{ab})
\]

is equal to

\[
((a, u)(b, v))f = (a + b, uI_b + vJ_a)f = ((a + b)B, (a + b)C + (uI_b + vJ_a)J_a^{-1}bA\phi_{a + b}B).
\]

Thus

\[
(a + b)C + (uI_b + vJ_a)J_a^{-1}bA\phi_{a + b}B = (aC + uJ_a^{-1}AJ_aB)I_{bb} + (bC + vJ_b^{-1}AJ_bB)J_{ab}B.
\]

Since \( (a + b)C = aCI_{bb} + bCJ_{ab}B \) by (3.3), the last equality simplifies to

\[
(uI_b + vJ_a)J_a^{-1}bA\phi_{a + b}B = uJ_a^{-1}AJ_aB I_{bb} + vJ_b^{-1}AJ_bBJ_{ab}B.
\]

With \( v = 0 \) we obtain the equality of maps \( K \to K \)

\[
(3.6) \quad I_bJ_a^{-1}bA\phi_{a + b}B = J_a^{-1}AJ_aB I_{bb}B.
\]

Similarly, with \( u = 0 \) we deduce another equality of maps \( K \to K \), namely

\[
(3.7) \quad J_a^{-1}AJ_aB = J_b^{-1}AJ_bBJ_{ab}B.
\]

Using both (3.6) and (3.7), we see that

\[
I_bJ_a^{-1}A J_aB I_{bb}B = J_aI_{bb}A\phi_{a + b}B = J_a^{-1}bA\phi_{a + b}B = J_a^{-1}bAJ_bBJ_{ab}B.
\]

and upon commuting certain maps and canceling we get \( I_b^{-1}A I_{bb}B = J_b^{-1}AJ_bBJ_{ab}B \), and therefore also \( J_bA I_{bb}B = I_aA J_aB \). Upon expanding and canceling like terms, we get \( 2M_aA = 2M_BB \).

If \( \text{char}(k) \neq 2 \), we deduce \( M_aA = AM_BB \) and (3.5). Suppose that \( \text{char}(k) = 2 \). Then (3.6) with \( a = b \) yields \( I_bA = I_b^{-1}A I_{bb}B B \), so \( I_b^2A = I_b^{-1}A I_{bb}B B \). Since \( M_a^2 = M_a^2 \) and \( I_b^2 = I_b^{-1}A I_{bb}B B \), we get \( I_b^2A = A I_{bb}B B \), \( M_aA = M_BB \), and \( A^{-1}M_BB \).

Since \( K \) is perfect (this is the only time we use this assumption), the last equality shows that every \( A^{-1}M_BA \) is of the form \( M_c \), so, in particular, \( A^{-1}M_BA = M_c \) for some \( c \). Then \( M_c^2 = (A^{-1}M_BA)^2 = A^{-1}M_BB = M_BB \), and evaluating this equality at \( e \) yields \( e^2 = (bB)^2 \) and \( e = bB \). We have again established (3.5).

Since \( A : K \to K \) is an additive bijection, (3.5) holds and \( \text{Im}(B) = W_1 \), it follows that \( A \in S(W_0, W_1) \) and \( B : A : W_0 \to W_1 \). We therefore have \( (a, 0)f = (aA, c \cdot aA) \), and \( \Phi \) is well-defined by \( (A, c) = \Phi \).

It remains to show that \( \Phi \) and \( \Psi \) are mutual inverses. If \( f \in \text{Iso}(Q_0, Q_1) \) and \( f \Phi = (A, c) \), then (3.5) yields \( J_a^{-1}AJ_aB = A \). This means that (3.2) can be rewritten as (3.1), and thus \( f \Phi \Psi = f \). Conversely, suppose that \( (A, c) \in S(W_0, W_1) \times K \) and let \( f = (A, c) \Psi \) and \( (D, d) = f \Phi = (A, c) \Psi \Phi \). Then \( (0, u)f = (0, uA) \) by (3.1) and \( (0, u)f = (0, uD) \) by definition of \( \Phi \), so \( A = D \). Finally, \( (a, 0)f = (aA, c \cdot aA) \) by (3.1) and \( (a, 0)f = (aD, d \cdot aD) = (aA, d \cdot aA) \) by definition of \( \Psi \), so \( c = d \).

\[ \square \]
3.2. **Isomorphisms and automorphisms in the tame case.** For the rest of this section suppose that the triple $k, K, W_i$ is tame, that is, $k$ is a prime field, $(W_i)_k = K$, and $K$ is perfect if $\text{char}(k) = 2$. In particular, $W_i \neq 0$. Let $\text{GL}_k(K)$ be the group of all $k$-linear transformations of $K$, and let $\text{Aut}(K)$ be the group of all field automorphisms of $K$.

Since $k$ is prime, any additive bijection $K \to K$ is $k$-linear, and so $S(W_0, W_1) = \{ A \in \text{GL}_k(K) : A^{-1}M_a A = M_{W_1} \}$. We have shown that $A \in S(W_0, W_1)$ gives rise to an additive bijection $\tilde{A} : W_0 \to W_1$. This map extends uniquely into a field automorphism $\tilde{A}$ of $K$ such that $A^{-1}M_a A = M_{\tilde{A}}$ for every $a \in K$. To see this, first note that $A \in \text{GL}_k(K)$ implies $A^{-1}M_a A = A^{-1}M_a a B A = A^{-1}M_a A A^{-1} B A$, $A^{-1}M_a B A = A^{-1}M_a A A^{-1} B A$ and $A^{-1}M_a A = M_\lambda$ for every $a, b \in K$ and $\lambda \in k$. If $\tilde{A}$ is already defined on $a, b$, let $(a + b)\tilde{A} = a \tilde{A} + b \tilde{A}$, $(ab)\tilde{A} = a \tilde{A} \cdot b \tilde{A}$, and $(\lambda a)\tilde{A} = \lambda \cdot a \tilde{A}$, where $\lambda \in k$. This procedure defines $\tilde{A}$ well. For instance, if $ab = c + d$, we have $a \tilde{A} \cdot b \tilde{A} = M_{a \tilde{A} b \tilde{A}} = M_{a \tilde{A} A b} = 1A^{-1}M_a A A^{-1} M_b A = 1A^{-1}M_a A A^{-1} M_b A = 1A^{-1}M_\lambda + M_{d \tilde{A}} = c \tilde{A} + d \tilde{A}$, and so on.

Here is a solution to the isomorphism problem for a fixed extension $k < K$:

**Corollary 3.3.** For $i \in \{0, 1\}$, let $k, K, W_i$ be a tame triple and $Q_i = Q_{k < K}(W_i)$. Then $Q_0$ is isomorphic to $Q_1$ if and only if there is $\varphi \in \text{Aut}(K)$ such that $W_0 \varphi = W_1$.

**Proof.** Suppose that $f : Q_0 \to Q_1$ is an isomorphism. By Proposition 3.2, $f$ induces a map $A \in S(W_0, W_1)$, which gives rise to $\tilde{A} : W_0 \to W_1$, which extends into $\tilde{A} \in \text{Aut}(K)$ such that $W_0 \tilde{A} = W_1$.

Conversely, suppose that $\varphi \in \text{Aut}(K)$ satisfies $W_0 \varphi = W_1$. Then for every $a \in W_0$ and $b \in K$ we have $b \varphi^{-1}M_a \varphi = ((b \varphi^{-1}) 
 a) \varphi = b \varphi^{-1} \varphi \cdot a \varphi = b \cdot a \varphi = b M_a \varphi$, so $\varphi \in S(W_0, W_1)$. The set $S(W_0, W_1) \times K$ is therefore nonempty, and we are done by Proposition 3.2.

We proceed to describe the automorphism groups of tame loops $Q_{k < K}(W)$. Let $S(W) = S(W, W) = \{ A \in \text{GL}_k(K) : A^{-1}M_a A = M_{W} \}$.

**Lemma 3.4.** Suppose that $k, K, W$ is a tame triple. Then the mapping $S(W) \to \text{Aut}(K)$, $A \mapsto \tilde{A}$ is a homomorphism with kernel $N(W) = M_{K^*}$ and image $I(W) = \{ C \in \text{Aut}(K) : WC = W \}$. Moreover, $S(W) = I(W) N(W)$ is isomorphic to the semidirect product $I(W) \ltimes K^*$ with multiplication $(A, c)(B, d) = (A, cB \cdot d)$.

**Proof.** With $A, B \in S(W)$ and $a \in K$ we have $M_{a \tilde{A} B} = (AB)^{-1}M_a (AB) = B^{-1}A^{-1}M_a AB = B^{-1}M_a B = M_a \tilde{A}$, so $\tilde{AB} = \tilde{A} B$. The kernel of this homomorphism is equal to $N(W) = \{ A \in S(W) : M_a A = AM_a \text{ for every } a \in K \}$. If $A \in N(W)$, we can apply the defining equality to 1 and deduce $aA = (1A)a$, so $A = M_1 A \in M_{K^*}$. Conversely, if $M_b \in M_{K^*}$ then obviously $M_b \in N(W)$.

For the image, note that $\tilde{A}$ satisfies $W \tilde{A} = W$. We have seen above that $\tilde{A} \in \text{Aut}(K)$. Conversely, if $C \in \text{Aut}(K)$ satisfies WC = W then $C \in S(W)$, and $C^{-1}M_a C = M_{ac}$ for every $a \in K$ because $C$ is multiplicative. Thus $C = \tilde{C} \in I(W)$.

Since $I(W), N(W)$ are subsets of $S(W)$, we have $I(W) N(W) \subseteq S(W)$. To show that $S(W) \subseteq I(W) N(W)$, let $A \in S(W)$ and consider $D = (\tilde{A})^{-1}A \in S(W)$. Then $D^{-1}M_{a \tilde{A}} D = A^{-1}M_{a \tilde{A}} (\tilde{A})^{-1} A = A^{-1}M_a A = M_{a \tilde{A}}$ shows that $D \in N(W)$. Then $A = \tilde{A} D$ is the desired decomposition.

Let $A, B \in S(W) = I(W) N(W) = I(W) M_{K^*}$, where $A = \tilde{A} M_c, B = \tilde{B} M_d$ for some $c, d \in K^*$. Then $AB = \tilde{A} M_c \tilde{B} M_d = \tilde{A} \tilde{B} M_{c \tilde{B} d} = \tilde{A} \tilde{B} M_{c \tilde{B} d}$.


Theorem 3.5. Let $Q = Q_{k < K}(W)$, where $k$ is a prime field, $k < K$ is a field extension, $W$ is a $k$-subspace of $K$ such that $k1 \cap W = 0$, and $(W)_k = K$. If $\text{char}(k) = 2$, suppose also that $K$ is a perfect field. Then the group $\text{Aut}(Q)$ is isomorphic to the semidirect product $S(W) \rtimes K$ with multiplication $(A,c)(B,d) = (AB,cB + d)$.

Proof. By Proposition 3.2, there is a one-to-one correspondence between the sets $\text{Aut}(Q)$ and $S(W) \times K$. Suppose that $f\Phi = (A,c)$, $g\Phi = (B,d)$, so that $(a,u)f = (aA,c \cdot aA + uA)$ and $(a,u)g = (aB,d \cdot aB + uB)$ for every $(a,u) \in W \times K$. Then

$$(a,u)fg = (aA,c \cdot aA + uA)g = (aA\bar{B},d \cdot a\bar{A}B + (c \cdot a\bar{A} + uA)B).$$

We want to prove that $(fg)\Phi = (AB,cB + d)$, which is equivalent to proving

$$(a,u)fg = (a\bar{A}B, (cB + d) \cdot a\bar{A}B + uAB).$$

Keeping $\bar{A}B = \bar{AB}$ of Lemma 3.4 in mind, it remains to show that $(c \cdot a\bar{A})B = cB \cdot a\bar{AB}$, but this follows from $B^{-1}M_{a\bar{A}}B = M_{a\bar{A}B}$.

A finer structure of $\text{Aut}(Q_{k < K}(W))$ is obtained by combining Theorem 3.5 with Lemma 3.4.

4. Automorphic loops of order $p^3$

The following facts are known about automorphic loops of odd order and prime power order.

Automorphic loops of odd order are solvable [9, Theorem 6.6]. Every automorphic loop of prime order $p$ is a group [9, Corollary 4.12]. More generally, every automorphic loop of order $p^2$ is a group, by [3] or [9, Theorem 6.1]. For every prime $p$ there are examples of automorphic loops of order $p^3$ that are not centrally nilpotent [9], and hence certainly not groups.

There is a commutative automorphic loop of order $2^3$ that is not centrally nilpotent [5]. By [6, Theorem 1.1], every commutative automorphic loop of odd order $p^k$ is centrally nilpotent. For any prime $p$ there are precisely 7 commutative automorphic loops of order $p^3$ up to isomorphism [4, Theorem 6.4].

We will use a special case of Corollary 3.3 to construct a class of pairwise non-isomorphic automorphic loops of odd order $p^3$, for $p$ odd.

Suppose that $p$ is odd. The field $\mathbb{F}_{p^2}$ can be represented as $\{x + y\sqrt{d} : x, y \in \mathbb{F}_p\}$, where $d \in \mathbb{F}_p$ is not a square. Let $\mathbb{F}_p = k < K = \mathbb{F}_{p^2}$, and let

$$W_0 = k\sqrt{d} \text{ and } W_a = k(1 + a\sqrt{d}) \text{ for } 0 \neq a \in \mathbb{F}_p.$$ 

We see that every $W_a$ is a 1-dimensional $k$-subspace of $K$ such that $k1 \cap W_a = 0$. Conversely, if $W$ is a 1-dimensional $k$-subspace of $K$ such that $k1 \cap W = 0$, there is $a + b\sqrt{d}$ in $W$ with $a, b \in k$, $b \neq 0$. If $a = 0$ then $W = W_0$. Otherwise $a^{-1}(a + b\sqrt{d}) = 1 + a^{-1}b\sqrt{d} \in W$, and $W = W_{a^{-1}b}$. Hence there is a one-to-one correspondence between the elements of $k$ and 1-dimensional $k$-subspaces $W$ of $K$ satisfying $k1 \cap W = 0$, given by $a \mapsto W_a$.

Theorem 4.1. Let $p$ be a prime and $\mathbb{F}_p = k < K = \mathbb{F}_{p^2}$.

(i) Suppose that $p$ is odd. If $a, b \in k$, then the automorphic loops $Q_{k < K}(W_a), Q_{k < K}(W_b)$ of order $p^3$ are isomorphic if and only if $a = \pm b$. In particular, there are $(p + 1)/2$
pairwise non-isomorphic automorphic loops of order $p^3$ of the form $Q_{k<K}(W)$, where we can take $W \in \{W_a : 0 \leq a \leq (p - 1)/2\}$.

(ii) Suppose that $p = 2$. Then there is a unique automorphic loop of order $p^3$ of the form $Q_{k<K}(W)$ up to isomorphism.

Proof. (i) By Theorem 2.2, the loops $Q_a = Q_{k<K}(W_a)$ and $Q_b = Q_{k<K}(W_b)$ are automorphic loops of order $p^3$. By Corollary 3.3, the loops $Q_a$, $Q_b$ are isomorphic if and only if there is an automorphism $\varphi$ of $K$ such that $W_a\varphi = W_b$. Let $\sigma$ be the unique nontrivial automorphism of $K$, given by $(a + b\sqrt{d})\sigma = a - b\sqrt{d}$. Then $W_a\sigma = W_{-a}$ for every $a \in k$. Therefore $Q_a$ is isomorphic to $Q_b$ if and only if $a = \pm b$. The rest follows.

Part (ii) is similar, and follows from Corollary 3.3 by a direct inspection of subspaces and automorphisms of $\mathbb{F}_4$.

We will now show how to obtain the loops of Construction 1.2 as a special case of Construction 1.4.

Lemma 4.2. Let $k$ be a field and $A \in M_2(k) \setminus kI$. Then $kI + kA$ is an anisotropic plane if and only if $kI + kA$ is a field with respect to the operations induced from $M_2(k)$.

Proof. Certainly $kI + kA$ is an abelian group. It is well known and easy to verify directly that every $A \in M_2(k)$ satisfies the characteristic equation

$$A^2 = \text{tr}(A)A - \det(A)I.$$ 

This implies that $kI + kA$ is closed under multiplication, and it is therefore a subring of $M_2(k)$.

If $kI + kA$ is a field then every nonzero element $B \in kI + kA$ has an inverse in $kI + kA$, so $B$ is an invertible matrix and $kI + kA$ is an anisotropic plane. Conversely, suppose that $kI + kA$ is an anisotropic plane, so that every nonzero element $B \in kI + kA$ is an invertible matrix. The characteristic equation for $B$ then implies that $B^{-1} = (\det(B)^{-1})(\text{tr}(B)I - B)$, certainly an element of $kI + kA$, so $kI + kA$ is a field.

Proposition 4.3. Let $k$ be a field. Let

$$S = \{Q_{k<K}(W) : k < K \text{ is a quadratic field extension} \ , \dim_k(W) = 1, k1 \cap W = 0\},$$

$$T = \{Q_k(A) : A \in M_2(k), kI + kA \text{ is an anisotropic plane}\}.$$

Then, up to isomorphism, the loops of $S$ are precisely the loops of $T$.

Proof. Let $Q_{k<K}(W) \in S$. Then there is $\theta \in K$ such that $W = k\theta$, $K = k(\theta)$, and $\theta^2 = e + f \theta$ for some $e, f \in k$. The multiplication in $K$ is determined by $(a + b\theta)(c + d\theta) = (ac + bd\theta^2) + (ad + bc)\theta$ and $\theta^2 = e + f \theta$. With respect to the basis $\{1, \theta\}$ of $K$ over $k$, the multiplication by $\theta$ is given by the matrix $A = M_\theta = \begin{pmatrix} 0 & 1 \\ e & f \end{pmatrix}$. The multiplication on $kI + kA$ is then determined by $(aI + bA)(cI + dA) = (acI + bdA^2) + (ad + bc)A$ and $A^2 = -\det(A)I + \text{tr}(A)A = eI + fA$, so $kI + kA$ is a field isomorphic to $K$. By Lemma 4.2, $kI + kA$ is an anisotropic plane, and the loop $Q_k(A)$ is defined.

The multiplication in $Q_{k<K}(W)$ on $W \times V = k\theta \times (k1 + k\theta)$ is given by $(a\theta, u)(b\theta, v) = (a\theta + b\theta, u(1 + b\theta) + v(1 - a\theta))$, while the multiplication in $Q_k(A) = Q_k(M_\theta)$ on $k \times (k \times k)$ is given by $(a, u)(b, v) = (a + b, u(1 + b\theta) + v(1 - a\theta))$. This shows that $Q_{k,K}(W)$ is isomorphic to $Q_k(A)$, and $S \subseteq T$. 

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Conversely, if $Q_k(A) \in T$ then the anisotropic plane $K = kI + kA$ is a field by Lemma 4.2, clearly a quadratic extension of $k$. Moreover, $W = kA$ is a 1-dimensional $k$-subspace of $K$ such that $k1 \cap W = 0$, so $Q_{k < K}(W) \in S$. We can again show that $Q_{k < K}(W)$ is isomorphic to $Q_k(A)$. \hfill \Box

Conjecture 6.5 of [6] stated that there is precisely one isomorphism type of loops $Q_{\mathbb{F}_2}(A)$, two isomorphism types of loops $Q_{\mathbb{F}_3}(A)$, and three isomorphism types of loops $Q_{\mathbb{F}_p}(A)$ for $p \geq 5$. The conjecture was verified computationally in [6] for $p \leq 5$, using the GAP package LOOPS [11]. Since $\mathbb{F}_p^2$ is the unique quadratic extension of $\mathbb{F}_p$, Theorem 4.1 and Proposition 4.3 now imply that the conjecture is actually false for every $p > 5$. (But note that $(p + 1)/2$ gives the calculated answer for $p = 3$ and $p = 5$, and the case $p = 2$ is also in agreement.)

The full classification of automorphic loops of order $p^3$ remains open.

5. Infinite examples

We conclude the paper by constructing an infinite 2-generated abelian-by-cyclic automorphic loop of exponent $p$ for every prime $p$.

**Lemma 5.1.** Let $p$ be an odd prime, $k = \mathbb{F}_p$, $K = \mathbb{F}_p((t))$ the field of formal Laurent series over $\mathbb{F}_p$, $W = \mathbb{F}_p[t]$, and $Q = Q_{k < K}(W)$ the automorphic loop from Construction 1.4 defined by (1.1) on $W \times K = \mathbb{F}_p t \times \mathbb{F}_p((t))$. Let $L = \langle (t,0),(0,1) \rangle$ be the subloop of $Q$ generated by $(t,0)$ and $(0,1)$. Then $L = W \times U$, where $U$ is the localization of $\mathbb{F}_p[t]$ with respect to \{1 + a : a \in W\}. Moreover, $L$ is an infinite nonassociative 2-generated abelian-by-cyclic automorphic loop of exponent $p$.

**Proof.** First we observe that $W \times U$ is a subloop of $Q$. Indeed, $W \times U$ is clearly closed under multiplication. Since $(1 \pm a)^{-1} \in U$ for every $a \in W$ by definition, the formulas (2.1), (2.2) show that $W \times U$ is closed under left and right divisions, respectively. To prove that $L = W \times U$, it therefore suffices to show that $W \times U \subseteq L$.

We claim that $0 \times \mathbb{F}_p[t] \subseteq L$, or, equivalently, that $(0, t^n) \in L$ for every $n \geq 0$. First note that for any integer $m$ we have

$$ (0, t^m)(t,0) \cdot (t,0)^{-1}(0,t^m) = (t,t^m(1+t)) \cdot (-t,t^m(1+t)) = (0, 2(t^m - t^{m+2})). $$

We have $(0,t^0) = (0,1) \in L$ by definition. The identity (5.1) with $m = 0$ then yields $(0, 2(1 - t^2)) \in L$, so $(0,t^2) \in L$. Since also

$$ (-t,0) \cdot (0,1)(t,0) = (-t,0)(t,1+t) = (0,1 + 2t + t^2) $$

belongs to $L$, we conclude that $(0,t) \in L$. The identity (5.1) can then be used inductively to show that $(0,t^n) \in L$ for every $n \geq 0$.

We now establish $0 \times U \subseteq L$ by proving that $(0, (1 + a)^n) \in L$ for every $n \in \mathbb{Z}$ and every $a \in W = \mathbb{F}_p$. We have already seen this for $n \geq 0$. The identity

$$ ((a,0) \cdot (0, (1 - a)^m))/(-a,0) = (-a, (1 - a)^{m-1})/(-a,0) = (0, (1 - a)^{m-2}) $$

then proves the claim by descending induction on $m$, starting with $m = 1$.

Given $(a,0) \in W \times 0 \subseteq L$ and $(0,u) \in 0 \times U \subseteq L$, we note that $(0,u(a(1 - a)^{-1}) \in L$, and thus

$$ (a,0)(0,u) \cdot (0,u(a(1 - a)^{-1})) = (a,u(1 - a))(0,u(a(1 - a)^{-1})) = (a,u) $$

is also in $L$, concluding the proof that $W \times U \subseteq L$. 

\hfill \Box
The loop \( L \) is certainly infinite and 2-generated, and it is automorphic by Theorem 2.2. The homomorphism \( W \times U \rightarrow \mathbb{F}_p, (it, u) \mapsto i \) has the abelian group \((U, +)\) as its kernel and the cyclic group \((\mathbb{F}_p, +)\) as its image, so \( L \) is abelian-by-cyclic. An easy induction yields \((a, u)^m = (ma, mu)\) for every \((a, u) \in Q\) and \(m \geq 0\), proving that \( L \) has exponent \(p\).

Finally, \((t, 0)(t, 0) \cdot (0, 1) = (2t, 1 - 2t) \neq (2t, 1 - 2t + t^2) = (t, 0) \cdot (t, 0)(0, 1)\) shows that \( L \) is nonassociative. \( \square \)

**Lemma 5.2.** Let \( k = \mathbb{F}_2, K = \mathbb{F}_2((t))\) the field of formal Laurent series over \( \mathbb{F}_2\), \( W = \mathbb{F}_2t\), and \( Q = Q_{k < k}(W)\) the automorphic loop from Construction 1.4 defined by (1.1) on \( W \times K = \mathbb{F}_2t \times \mathbb{F}_2((t))\). Let \( L = \langle (t, 0), (0, 1) \rangle\) be the subloop of \( Q\) generated by \((t, 0)\) and \((0, 1)\). Then \( L = \{(it, f(1 + t)^i) : f \in U, i \in \{0, 1\}\}\), where \( U\) is the localization of \( \mathbb{F}_2[t^2]\) with respect to \( \{1 + t^2\}\). Moreover, \( L\) is an infinite nonassociative 2-generated abelian-by-cyclic commutative automorphic loop of exponent 2.

**Proof.** In our situation the multiplication formula (1.1) becomes

\[(a, u)(b, v) = (a + b, u(1 + b) + v(1 + a)),\]

so \( Q\) is commutative and of exponent 2. Note that (2.1) becomes

\[(a, u)(b, v) = (a + b, (v + u(1 + a + b))(1 + a)^{-1}).\]

Let us first show that \( S = \{(it, f(1 + t)^i) : f \in U, i \in \{0, 1\}\} = (0 \times U) \cup (t, 0)(0 \times U)\) is a subloop of \( Q\). Indeed, \( 0 \times U \subseteq S\) is a subloop, and with \( f, g \in U\), we have

\[(t, f(1 + t))(t, g(1 + t)) = (0, f(1 + t)^2 + g(1 + t)^2) = (0, (f + g)(1 + t^2)),\]

\[(0, f)(t, g(1 + t)) = (t, g(1 + t) + f(1 + t)) = (t, (g + f)(1 + t)),\]

\[(t, f(1 + t)) \cdot (0, g) = (t, g + f(1 + t)^2)(1 + t)^{-1} = (t, (g(1 + t^2)^{-1} + f)(1 + t)),\]

\[(t, f(1 + t))\cdot (t, g(1 + t)) = (0, (g(1 + t) + f(1 + t))(1 + t)^{-1}) = (0, g + f),\]

always obtaining an element of \( S\).

To prove that \( S = L\), it suffices to show that \((0, t^{2m}), (0, t^{2m}(1 + t^2)^{-1}) \in L\) for every \( m \geq 0\), since this implies \( 0 \times U \subseteq L\) and thus \( S = (0 \times U) \cup (t, 0)(0 \times U) \subseteq L\). We have \((0, 1) \in L\) by definition, \((t, 1 + t) = (t, 0)(0, 1) \in L\), \((t, (1 + t^2)^{-1}(1 + t)) = (t, 0)(0, 1) \in L\), and \((0, 1 + (1 + t^2)^{-1}) = (t, 1 + t) \cdot (t, (1 + t^2)^{-1}(1 + t)) \in L\), so also \((0, (1 + t^2)^{-1}) \in L\). The inductive step follows upon observing the identity

\[(t, 0) \cdot (0, u)(t, 0) = (t, 0)(t, u(1 + t)) = (0, u(1 + t^2)).\]

The loop \( L \) is certainly infinite, 2-generated, commutative, automorphic and of exponent 2. It is abelian-by-cyclic because the map \( L \rightarrow \mathbb{F}_2, (it, f(1 + t)^i) \mapsto i \) is a homomorphism with the abelian group \((U, +)\) as its kernel and the cyclic group \((\mathbb{F}_2, +)\) as its image. Finally, \((t, 0)(t, 0) \cdot (0, 1) = (0, 1) \neq (0, 1 + t^2) = (t, 0) \cdot (t, 0)(0, 1)\) shows that \( L \) is nonassociative. \( \square \)

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References


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