CONTINUOUS AND OTHER FINITELY GENERATED CANONICAL COFINAL MAPS ON ULTRAFILTERS

NATASHA DOBRINEN

Abstract. This paper concentrates on the existence of canonical cofinal maps of three types: continuous, generated by finitary monotone end-extension preserving maps, and generated by monotone finitary maps. The main theorems prove that every monotone cofinal map on an ultrafilter from a certain class of ultrafilters is actually canonical when restricted to some filter base. These theorems are then applied to find connections between Tukey, Rudin-Keisler, and Rudin-Blass reducibilities on large classes of ultrafilters.

The main theorems on canonical cofinal maps are the following. Under a mild assumption, basic Tukey reductions are inherited under a Tukey reduction. In particular, every ultrafilter Tukey reducible to a p-point has continuous Tukey reductions. If $\mathcal{U}$ is a Fubini iterate of p-points, then each monotone cofinal map from $\mathcal{U}$ to some other ultrafilter is generated (on a cofinal subset of $\mathcal{U}$) by a finitary map on the base tree for $\mathcal{U}$ which is monotone and end-extension preserving - the analogue of continuous in this context. Further, every ultrafilter which is Tukey reducible to some Fubini iterate of p-points has finitely generated cofinal maps. Similar theorems also hold for some other classes of ultrafilters.

1. Introduction

A map from an ultrafilter $\mathcal{U}$ to another ultrafilter $\mathcal{V}$ is *cofinal* if every image of a filter base for $\mathcal{U}$ is a filter base for $\mathcal{V}$. We say that $\mathcal{V}$ is *Tukey reducible* to $\mathcal{U}$ and write $\mathcal{V} \le_T \mathcal{U}$ if and only if there is a cofinal map from $\mathcal{U}$ to $\mathcal{V}$. When $\mathcal{U} \le_T \mathcal{V}$ and $\mathcal{V} \le_T \mathcal{U}$, then we say that $\mathcal{U}$ is *Tukey equivalent to* $\mathcal{V}$ and write $\mathcal{U} \equiv_T \mathcal{V}$. It is clear that $\equiv_T$ is an equivalence relation, and $\le_T$ on the equivalence classes forms a partial ordering. The equivalence classes are called *Tukey types*. We point out that since $\supseteq$ is a directed partial ordering on an ultrafilter, two ultrafilters are Tukey equivalent if and only if they are *cofinally similar*; that is, there is a partial ordering into which they both embed as cofinal subsets (see [18]). Thus, for ultrafilters, Tukey equivalence is the same as cofinal similarity. An equivalent formulation of Tukey reducibility, noticed by Schmidt in [14], shows that $\mathcal{V} \le_T \mathcal{U}$ if and only if there is a *Tukey map* from $\mathcal{V}$ to $\mathcal{U}$; that is a map $g : \mathcal{V} \to \mathcal{U}$ such

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that every unbounded (with respect to the partial ordering $\supseteq$) subset of $\mathcal{V}$ is unbounded in $\mathcal{U}$.

This paper focuses on the existence of canonical cofinal maps of three types: continuous, approximated by *basic* maps (monotone end-extension and level preserving finitary maps - see Definitions 2.2 and 4.2), and approximated by monotone finitary maps. In each of these cases, the original cofinal map is generated by the approximating finitary maps.

The notion of Tukey reducibility between two directed partial orderings was first introduced by Tukey in [18] to study the Moore-Smith theory of net convergence in topology. This naturally led to investigations of Tukey types of more general partial orderings, directed and later non-directed. These investigations often reveal useful information for the comparison of different partial orderings. For example, Tukey reducibility preserves calibre-like properties, such as the countable chain condition, property K, precalibre $\aleph_1$, σ-linked, and σ-centered (see [16]). For more on classification theories of Tukey types for certain classes of ordered sets, we refer the reader to [18], [3], [11], [15], and [16]. As the focus of this paper is canonical cofinal maps on ultrafilters, and as we have recently written a survey article giving an overview of the motivation and the state of the art of the Tukey theory of ultrafilters (see [6]), we present here only the background and motivations relevant for this work.

For ultrafilters, we may restrict our attention to monotone cofinal maps. A map $f : \mathcal{U} \to \mathcal{V}$ is *monotone* if for any $X, Y \in \mathcal{U}$, $X \supseteq Y$ implies $f(X) \supseteq f(Y)$. It is not hard to show that whenever $\mathcal{U} \geq_T \mathcal{V}$, then there is a *monotone* cofinal map witnessing this (see Fact 6 of [8]).

As cofinal maps between ultrafilters have domain and range of size continuum, a priori, the Tukey type of an ultrafilter may have size $2^{\mathfrak{c}}$. Indeed, this is the case for ultrafilters which have the maximum Tukey type $([\mathfrak{c}]^{<\omega}, \subseteq)$. However, if an ultrafilter has the property that every Tukey reduction from it to another ultrafilter may be witnessed by a continuous map, then it follows that its Tukey type, as well as the Tukey type of each ultrafilter Tukey reducible to it, has size at most continuum. This is the case for p-points.

Definition 1.1. An ultrafilter $\mathcal{U}$ on ω is a *p-point* iff for each decreasing sequence $X_0 \supseteq X_1 \supseteq \dots$ of elements of $\mathcal{U}$, there is an $U \in \mathcal{U}$ such that $U \subseteq^* X_n$, for all $n < \omega$.

Definition 1.2. An ultrafilter $\mathcal{U}$ on ω *has continuous Tukey reductions* if whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ such that $f \restriction \mathcal{C}$ is continuous.

The following Theorem 20 in [8] has provided a fundamental tool for all subsequent research on the classification of Tukey types of p-points.

Theorem 1.3 (Dobrinen/Todorcevic [8]). *Suppose $\mathcal{U}$ is a p-point on ω. Then $\mathcal{U}$ has continuous Tukey reductions.*

Remark 1.4. In fact, the proof of Theorem 20 in [8] shows that p-points have the stronger property of basic Tukey reductions (see Definition 2.2).

It was later proved by Raghavan in [13] that any ultrafilter Tukey reducible to any basically generated ultrafilter has Tukey type of cardinality at most $\mathfrak{c}$.

Definition 1.5 (Definition 15 in [8]). An ultrafilter $\mathcal{U}$ on ω is *basically generated* if it has a filter base (say closed under finite intersections) $\mathcal{B} \subseteq \mathcal{U}$ such that each sequence $\langle A_n : n < \omega \rangle$ of members of $\mathcal{B}$ converging to another member of $\mathcal{B}$ has a subsequence whose intersection is in $\mathcal{U}$.

It was shown in [8] that the class of basically generated ultrafilters contains all p-points and is closed under taking Fubini products. It is still unknown whether the class of all Fubini iterates of p-points is the same as or strictly contained in the class of all basically generated ultrafilters.

Continuous cofinal maps provide one of the main keys to the analysis of the structure of the Tukey types of p-points (see for instance [8], [13] and [12]). Continuous cofinal maps are also crucial to providing a mechanism for applying Ramsey-classification theorems on barriers to classify the initial Tukey structures and Rudin-Keisler structures within these for a large class of p-points: selective ultrafilters in [13]; weakly Ramsey ultrafilters and a large hierarchy of rapid p-points satisfying partition relations in [9] and [10]; and k-arrow ultrafilters, hypercube ultrafilters, and a large class of p-points constructed using products of certain Fraïssé classes in [7].

Continuous cofinal maps are also used in the following theorem, which reveals the surprising fact that the Tukey and Rudin-Blass orders sometimes coincide. Recall that $\mathcal{V} \leq_{RB} \mathcal{U}$ if and only if there is a finite-to-one map $f : \omega \to \omega$ such that $\mathcal{V} = f(\mathcal{U})$. The following is Theorem 10 in [13].

Theorem 1.6 (Raghavan [13]). *Let $\mathcal{U}$ be any ultrafilter and let $\mathcal{V}$ be a q-point. If $\mathcal{V} \leq_T \mathcal{U}$ and this is witnessed by a continuous, monotone cofinal map from $\mathcal{U}$ to $\mathcal{V}$, then $\mathcal{V} \leq_{RB} \mathcal{U}$.*

In Section 2, we prove in Theorem 2.5 that, under a mild assumption, the property of having basic cofinal maps is inherited under Tukey reduction. The proof uses the Extension Lemma 2.4 showing that any basic monotone map on a cofinal subset of an ultrafilter may be extended to a basic monotone map on all of $\mathcal{P}(\omega)$. In particular, p-points satisfy the mild assumption; hence we obtain the following theorem.

Theorem 2.6. *Every ultrafilter Tukey reducible to a p-point has basic, and hence continuous, Tukey reductions.*

Combined, Theorems 1.6 and 2.6 imply the following.

Theorem 2.7. *If $\mathcal{U}$ is Tukey reducible to some p-point, then any q-point Tukey below $\mathcal{U}$ is actually Rudin-Blass below $\mathcal{U}$.*

The rest of the paper involves finding the analogues of Theorems 1.3 and 2.6 for countable iterations of Fubini products of p-points and applying them to connect Tukey reduction with Rudin-Keisler and Rudin-Blass reductions. We now delineate these results.

Section 3 is a primer, explicitly showing how any countable iteration of Fubini products of p-points, which we also simply call a *Fubini iterate of p-points*, can be viewed as an ultrafilter generated by trees on a so-called flat-top front on ω. This precise way of viewing Fubini iterates of p-points sets the stage for finding the analogue of Theorem 1.3 for this more general class of ultrafilters. While it is not possible to show that Fubini iterates of p-points have continuous Tukey reductions (as that is simply not true), we do show that the key properties of continuous maps hold for this class of ultrafilters.

In Section 4 we define the notion of a *basic map* for Fubini iterates, which is in particular an end-extension preserving monotone map from finite subsets of the tree $\hat{B}$ of initial segments of members of a flat-top front B into finite subsets of ω (see Definition 4.2). This is the analogue of continuity for Fubini iterates of p-points. One of the main results of this paper is the following.

Theorem 4.4. *Fubini iterates of p-points have basic Tukey reductions.*

Thus, monotone cofinal maps on Fubini iterates of p-points are continuous, with respect to the product topology on the space $2^{\hat{B}}$. As basic maps on flat-top fronts have the key property (end-extension preserving) of continuous maps used to convert Tukey reduction to Rudin-Keisler reduction in [9], [10], [7], and [5], it seems likely that they will play a crucial role in obtaining similar results for ultra-Ramsey spaces of Chapter 6 of Todorcevic's book [17].

Sections 5 and 6 contain applications of Theorem 4.4 to a broad class of ultrafilters. In Section 5, we directly apply Theorem 4.4 to obtain an analogue of Theorem 10 of Raghavan in [13]. In Theorem 5.1, we prove that if $\mathcal{U}$ is a Fubini iterate of p-points and $\mathcal{V}$ is a q-point Tukey reducible to $\mathcal{U}$, then there is a finite-to-one map on a large subset of $\hat{B}$, where B is the flat-top front base for $\mathcal{U}$, such that its image on $\mathcal{U}$ generates a subfilter of $\mathcal{V}$. One of the consequences of this is the following.

Theorem 5.3. *Suppose $\mathcal{U}$ is a finite iteration of Fubini products of p-points. If $\mathcal{V}$ is a q-point and $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V} \leq_{RK} \mathcal{U}$.*

This improves one aspect of Corollary 56 of Raghavan in [13] as $\mathcal{V}$ is only required to be a q-point, not a selective ultrafilter. The improvement though comes at the expense of limiting $\mathcal{U}$ to a finite Fubini iterate of p-points. It is unknown whether this can be extended to all Fubini iterates of p-points.

In Section 6 we prove the analogue of Theorem 2.5 for ultrafilters Tukey reducible to some Fubini iterate of p-points. Though it is not in general true that the property of having basic cofinal maps is inherited under Tukey reducibility, we do show that a large class of ultrafilters has cofinal maps generated by finitary monotone maps.

Definition 1.7. $\mathcal{U}$ *has finitely generated Tukey reductions* if whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ and a function $\hat{f} : [\omega]^{<\omega} \to [\omega]^{<\omega}$, such that

(a) $\hat{f}$ is monotone: $s \subseteq t \to \hat{f}(s) \subseteq \hat{f}(t)$; and
(b) $\hat{f}$ generates f on $\mathcal{C}$: For each $X \in \mathcal{C}$, $f(X) = \bigcup_{k<\omega} \hat{f}(X \cap k)$.

Analogously to the Extension Lemma 2.4, the Extension Lemma 6.1 shows that basic maps on filter bases on some flat-top front can be extended to the full space. Using this, we prove the following.

Theorem 6.3. *Let $\mathcal{U}$ be any Fubini iterate of p-points. If $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ has finitely generated Tukey reductions.*

These finitary maps are an improvement on the maps ψ_φ used in [13] (see Definition 7 in [13]) in the sense that our finitary maps are shown to generate the original cofinal maps. Theorem 6.3 is used to extend Theorem 17 of Raghavan in [13] to the class of all ultrafilters Tukey reducible to some Fubini iterate of p-points, in contrast to his result where $\mathcal{U}$ is assumed to be basically generated. It is still open whether every ultrafilter Tukey reducible to a p-point is basically generated (see discussion around Problem 7.6), and the class of basically generated ultrafilters and the class of ultrafilters Tukey reducible to a Fubini iterate of p-points may be very different.

Theorem 6.4. *If $\mathcal{U}$ is Tukey reducible to a Fubini iterate of p-points, then for each $\mathcal{V} \le_T \mathcal{U}$, there is a filter $\mathcal{U}(P) \equiv_T \mathcal{U}$ such that $\mathcal{V} \le_{RK} \mathcal{U}(P)$.*

The paper closes with a list of open problems in Section 7.

The results in Sections 2, 3 and 4 were completed in 2010, presented at the Logic Colloquium in Paris that year, and have appeared in the preprint [4]. For various and sundry reasons that article was not published. The present paper includes much revised presentations and proofs of those results, new extensions of them, and additional applications.

2. Basic Tukey reductions inherited under Tukey reducibility

One of the crucial tools used to determine the structure of the Tukey types of p-points is the existence of continuous cofinal maps (see Theorem 20 in [8]). Continuity contributes to the analysis of the structure of the Tukey types of p-points by essentially reducing the number of cofinal maps under consideration from $2^{\mathfrak{c}}$ to $\mathfrak{c}$, with the immediate consequence that there are at most $\mathfrak{c}$ many ultrafilters Tukey reducible to any p-point. Continuity further contributes to finding exact Tukey and Rudin-Keisler structures below certain classes of p-points satisfying partition relations. The fact that each monotone cofinal map on a p-point is approximated by a finitary end-extension preserving function is what allows for application of Ramsey-classification theorems to find the exact Tukey and Rudin-Keisler structures below the p-points forced by certain topological Ramsey spaces (see [13], [9], [10], and [7]). Further applications of cofinal maps represented by finitary end-extension preserving maps to find exact Tukey and Rudin-Keisler structures below non-p-points appear in Dobrinen's contributions in [2] and extensions in [5].

The notion of a basic map is a strengthening of continuity, and is the same as continuity when the domain is a compact subset of 2^ω (see Definition 2.2 below). The Extension Lemma 2.4 shows that all basic Tukey reductions on some cofinal subset of an ultrafilter extend to a basic map on $\mathcal{P}(\omega)$. This will be employed in the proof of the main theorem of this section, Theorem 2.5, which shows that, under mild assumptions, the property of having basic Tukey reductions is inherited under Tukey reducibility. Theorem 2.6 then follows: Every ultrafilter Tukey reducible to a p-point has basic, and hence continuous, Tukey reductions. Combining Theorem 2.6 with Theorem 10 of

Raghavan in [13], we prove that whenever $\mathcal{W}$ is Tukey reducible to a p-point and $\mathcal{V}$ is a q-point, then $\mathcal{W} \geq_T \mathcal{V}$ implies $\mathcal{W} \geq_{RB} \mathcal{V}$ (see Theorem 2.7).

We begin with some basic definitions. The following standard notation is used: $2^{<\omega}$ denotes the collection of finite sequences $s : n \to 2$, for $n < \omega$. We use $s, t, u, \dots$ to denote members of $2^{<\omega}$. For $s, t \in 2^{<\omega}$, we write $s \sqsubseteq t$ to denote that s is an initial segment of t; that is, $\mathrm{dom}\,(s) \leq \mathrm{dom}\,(t)$ and $t \restriction \mathrm{dom}\,(s) = s$. We also use $a \sqsubseteq X$ for sets $a, X \subseteq \omega$ to denote that, given their strictly increasing enumerations, a is an initial segment of X. $a \sqsubset X$ denotes that a is a proper initial segment of X.

We would like to identify subsets of ω with their characteristic functions. Of course, since the same finite set determines different characteristic functions on different domains. For $X \subseteq \omega$, we let χ_X denote the characteristic function of X with domain ω; and given $m < \omega$, we let $\chi_X \restriction m$ denote the characteristic function of $X \cap m$ with domain m. We shall often abuse notation and use $X \restriction m$ to denote both the characteristic function $\chi_X \restriction m$ and the set $X \cap m$. No ambiguity will arise from this.

Definition 2.1. Given a subset C of $2^{<\omega}$, we shall call a map $\hat{f} : C \to 2^{<\omega}$ *level preserving* if there is a strictly increasing sequence $(k_m)_{m<\omega}$ such that $\hat{f}$ takes each member of $C \cap 2^{k_m}$ to a member of 2^m. A level preserving map $\hat{f}$ is *end-extension preserving* if whenever $m < m'$, $s \in C \cap 2^{k_m}$, and $s' \in C \cap 2^{k_{m'}}$, then $s \sqsubseteq s'$ implies $\hat{f}(s) \sqsubseteq \hat{f}(s')$. $\hat{f}$ is *monotone* if for each $s, t \in C$, $s \subseteq t$ implies $\hat{f}(s) \subseteq \hat{f}(t)$; more precisely, whenever $\{i \in |s| : s(i) = 1\} \subseteq \{i \in |t| : t(i) = 1\}$, then $\{i \in |\hat{f}(s)| : \hat{f}(s)(i) = 1\} \subseteq \{i \in |\hat{f}(t)| : \hat{f}(t)(i) = 1\}$.

Given a set $C \subseteq \bigcup_{m<\omega} 2^{k_m}$, where $(k_m)_{m<\omega}$ is some strictly increasing sequence, a function $\hat{f} : C \to 2^{<\omega}$ is called *basic* if $\hat{f}$ is level and end-extension preserving and is monotone.

Definition 2.2. We say that a map f on a subset $\mathcal{C} \subseteq 2^\omega$ is *represented by a basic map* $\hat{f}$ if there is a strictly increasing sequence $(k_m)_{m<\omega}$ such that

$$C = \{X \restriction k_m : X \in \mathcal{C},\ m < \omega\} \tag{2.1}$$

and for each $X \in \mathcal{C}$,

$$f(X) = \bigcup_{m<\omega} \hat{f}(X \restriction k_m). \tag{2.2}$$

In this case, we say that $\hat{f}$ *generates* f.

If each monotone cofinal function from an ultrafilter $\mathcal{U}$ to another ultrafilter is represented by a basic map on some cofinal subset of $\mathcal{U}$, then we say that $\mathcal{U}$ *has basic Tukey reductions.*

Note that if $f \restriction \mathcal{C}$ is generated by a basic map $\hat{f}$, then for each $X \in \mathcal{C}$ and $m < \omega$, $f(X) \cap m = \hat{f}(X \restriction k_m)$.

Recall that for a set $C \subseteq 2^{<\omega}$, $[C]$ denotes the set of branches through C. If $C = \{X \restriction k_m : X \in \mathcal{C},\ m < \omega\}$, where $\mathcal{C} \subseteq 2^\omega$ and $(k_m)_{m<\omega}$ is a strictly increasing sequence, then $[C] \subseteq 2^\omega$.

Fact 2.3. *Let $\hat{f}$ be a basic map with domain $C \subseteq 2^{<\omega}$. Then $\hat{f}$ induces a monotone continuous map f^* on $[C]$ by $f^*(X) = \bigcup_{m<\omega} \hat{f}(X \restriction k_m)$, for $X \in [C]$. Further, if $\hat{f}$ generates f on $\mathcal{C}$ and $C \supseteq \{X \restriction k_m : X \in \mathcal{C},\ m < \omega\}$, then $f^* \restriction \mathcal{C} = f \restriction \mathcal{C}$; hence $f \restriction \mathcal{C}$ is continuous.*

Proof. That f^* is continuous on $[C]$ is trivial, since $\hat{f}$ is level and initial segment preserving. Since $\hat{f}$ is monotone, it follows that f^* is monotone.

If $\hat{f}$ generates f on $\mathcal{C}$, then trivially $f^* \restriction \mathcal{C}$ is simply $f \restriction \mathcal{C}$. □

Lemma 2.4 (Extension)**.** *Suppose $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters, $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ such that $f \restriction \mathcal{C}$ is represented by a basic map. Then there is a monotone map $\tilde{f} : 2^\omega \to 2^\omega$ such that*

(1) $\tilde{f}$ is represented by a basic map;
(2) $\tilde{f} \restriction \mathcal{C} = f \restriction \mathcal{C}$; and
(3) $\tilde{f} \restriction \mathcal{U}$ is a cofinal map from $\mathcal{U}$ to $\mathcal{V}$.

Proof. Let $\hat{f}$ be a basic map generating $f \restriction \mathcal{C}$, and let $(k_m)_{m<\omega}$ be the levels on which $\hat{f}$ is defined. Thus, the domain of $\hat{f}$ is $C = \{X \restriction k_m : X \in \mathcal{C},\ m < \omega\}$, and for each $s \in C \cap 2^{k_m}$, $\hat{f}(s) \in 2^m$.

Claim. There is a basic map $\hat{g}$ which generates a function $\tilde{f} : 2^\omega \to 2^\omega$ such that $\tilde{f} \restriction \mathcal{C} = f \restriction \mathcal{C}$.

Proof. Since $\mathcal{C}$ is cofinal in $\mathcal{U}$ and $\mathcal{U}$ is nonprincipal, the finite sequence of zeros of length k_m is in C, for each $m < \omega$. Let $D = \bigcup_{m<\omega} 2^{k_m}$ and define $\hat{g}$ on D as follows: For $t \in 2^{k_m}$, define $\hat{g}(t)$ to be the function from m into 2 such that for $i \in m$,

$$\hat{g}(t)(i) = \max\{\hat{f}(s)(i) : s \in C,\ |s| \le k_m, \text{ and } s \subseteq t\}. \tag{2.3}$$

That is, $\hat{g}(t)(i) = 1$ if and only if there is some $s \in C$ such that $|s| \le k_m$, $s \subseteq t$, and $\hat{f}(s)(i) = 1$. It follows from the definition that $\hat{g}$ is monotone and level preserving. Since $\hat{f}$ is monotone, $\hat{g} \restriction C$ equals $\hat{f} \restriction C$.

To see that $\hat{g}$ is end-extension preserving, suppose $t \sqsubset t'$, where $t \in 2^{k_m}$ and $t' \in 2^{k_{m'}}$ for some $m < m'$. Fix $i < m$. Suppose that $\hat{g}(t')(i) = 1$. Then there is some $s' \in C \cap 2^{k_n}$ such that $i \le n \le m'$, $s' \subseteq t'$, and

$\hat{f}(s')(i) = 1$. Letting $j = \min\{m, n\}$ and $s = s' \restriction k_j$, we see that $s \in C$ and $s \subseteq t' \restriction k_m = t$; moreover, $\hat{f}(s)(i) = 1$, since $\hat{f}(s) = \hat{f}(s') \restriction j$. It follows that $\hat{g}(t)(i) = 1$. On the other hand, if $\hat{g}(t')(i) = 0$, then by the definition of $\hat{g}$, $\hat{f}(s)(i) = 0$ for all $s \in C \cap \bigcup_{n \le m} 2^{k_n}$ such that $s \subseteq t$. Hence, $\hat{g}(t)(i) = 0$. Therefore, $\hat{g}(t') \restriction m = \hat{g}(t)$.

Now define $\tilde{f} : 2^\omega \to 2^\omega$ by

$$\tilde{f}(Z) = \bigcup_{m<\omega} \hat{g}(Z \restriction k_m). \tag{2.4}$$

Then $\tilde{f}$ is generated by the basic map $\hat{g}$. It follows that $\tilde{f}$ is monotone. Since $\hat{g} \restriction C$ equals $\hat{f} \restriction C$, it follows that $\tilde{f} \restriction \mathcal{C} = f \restriction \mathcal{C}$. □

Thus, $\tilde{f}$ is continuous on 2^ω and (1) and (2) of the Lemma hold. To show (3), it suffices to show that $\tilde{f} \restriction \mathcal{U}$ has range inside of $\mathcal{V}$, since $\tilde{f} \restriction \mathcal{C}$ equals $f \restriction \mathcal{C}$ which is monotone and cofinal in $\mathcal{V}$. Let $U \in \mathcal{U}$ be given. Then

$$\begin{aligned}
\tilde{f}(U) &= \bigcup \{\hat{g}(U \restriction k_m) : m < \omega\} && (2.5)\\
&= \bigcup \{\hat{g}(t) : t \in D \text{ and } t \subseteq U\} && (2.6)\\
&= \bigcup \{\hat{f}(s) : s \in C \text{ and } s \subseteq U\} && (2.7)\\
&\supseteq \bigcup \{f(S) : S \in \mathcal{C} \text{ and } S \subseteq U\}, && (2.8)
\end{aligned}$$

where equality (2.5) holds by definition of $\tilde{f}$, (2.6) holds by monotonicity of $\hat{g}$, and (2.7) holds by definition of $\hat{g}$. The containment (2.8) holds since for each $S \subseteq U$ in $\mathcal{C}$, $S \restriction k_m$ is a member of C with $S \cap k_m \subseteq U$. Since $\mathcal{C}$ is cofinal in $\mathcal{U}$, there is at least one $S \in \mathcal{C}$ with $S \subseteq U$, and so $f(S) \subseteq \tilde{f}(U)$. Hence, $\tilde{f}(U)$ is a member of $\mathcal{V}$. □

Now, in what is the main theorem of this section, we show that, assuming the property $(*)$ below, the property of having basic Tukey reductions is inherited under Tukey reducibility.

Theorem 2.5. *Suppose that $\mathcal{U}$ has basic Tukey reductions. Suppose further that for each monotone cofinal map f on $\mathcal{U}$ there is some cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ such that $f \restriction \mathcal{C}$ is represented by a basic function satisfying the following property:*

- $(*)$ *For each $X \in \overline{\mathcal{C}}$ and each $m < \omega$, there is a $Z \in \mathcal{C}$ such that $Z \supseteq X$ and $Z \restriction k_m = X \restriction k_m$, where $(k_m)_{m<\omega}$ is the sequence given by the basic representation of f.*

Then every ultrafilter $\mathcal{V}$ Tukey reducible to $\mathcal{U}$ also has basic Tukey reductions.

Proof. Suppose that $\mathcal{U}$ satisfies the hypotheses and let $\mathcal{V} \le_T \mathcal{U}$. Without loss of generality, we may assume that $\mathcal{U}$ is nonprincipal, for the result holds immediately if $\mathcal{U}$ is principal. By Lemma 2.4, there is a map $\tilde{f} : 2^\omega \to 2^\omega$ generated by a basic map $\hat{f} : \bigcup_{m<\omega} 2^{k_m} \to 2^{<\omega}$, for some increasing sequence $(k_m)_{m<\omega}$, such that $\tilde{f} \restriction \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map. We shall let f denote the restricted map $\tilde{f} \restriction \mathcal{U}$. Suppose $\mathcal{W} \le_T \mathcal{V}$, and let $h : \mathcal{V} \to \mathcal{W}$ be a monotone cofinal map. Extend h to the map $\tilde{h} : 2^\omega \to 2^\omega$ defined as follows: For each $X \in 2^\omega$, let

$$\tilde{h}(X) = \bigcap \{h(V) : V \in \mathcal{V} \text{ and } V \supseteq X\}. \tag{2.9}$$

It follows from h being monotone that $\tilde{h}$ is monotone and $\tilde{h} \restriction \mathcal{V} = h$.

Define $\tilde{g} = \tilde{h} \circ \tilde{f}$. Then $\tilde{g} : 2^\omega \to 2^\omega$ and is monotone. Letting g denote $\tilde{g} \restriction \mathcal{U}$, we see that $g = h \circ f$; hence $g : \mathcal{U} \to \mathcal{W}$ is a monotone cofinal map. By the hypotheses, there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ and a basic map $\hat{g} : C \to 2^{<\omega}$ generating $g \restriction \mathcal{C}$ such that $(*)$ holds, where $(k_m)_{m<\omega}$ is the strictly increasing sequence associated with $\hat{g}$ and $C = \{X \restriction k_m : X \in \mathcal{C} \text{ and } m < \omega\}$.

Without loss of generality, we may assume that $\hat{f}$ and $\hat{g}$ are defined on the same levels $(k_m)_{m<\omega}$: For if $\hat{g}$ is defined on $\{X \restriction j_m : X \in \mathcal{C} \text{ and } m < \omega\}$, we can take $l_m = \max(k_m, j_m)$ and define $\hat{f}'(s) = \hat{f}(s \restriction k_m)$ for $s \in 2^{l_m}$ and $\hat{g}'(X \restriction l_m) = \hat{g}(X \restriction j_m)$ for $X \in \mathcal{C}$ and $m < \omega$. Notice that whenever $s \in C \cap 2^{k_m}$ and $s \sqsubset X \in \mathcal{C}$, then $\hat{f}(s) = f(X) \restriction m$ and $\hat{g}(s) = g(X) \restriction m$.

Define

$$D = \{\hat{f}(s) : s \in C\} \text{ and } \mathcal{D} = f''\mathcal{C}. \tag{2.10}$$

Notice that in fact $D = \{Y \restriction m : Y \in \mathcal{D},\ m < \omega\}$, and $\mathcal{D}$ is cofinal in $\mathcal{V}$ since $f : \mathcal{U} \to \mathcal{V}$ is monotone cofinal and $\mathcal{C}$ is a cofinal subset of $\mathcal{U}$. Let $\overline{\mathcal{C}}$ denote the closure of $\mathcal{C}$ in the topological space 2^ω. Since $\tilde{f}$ is continuous on the compact space 2^ω and $f \restriction \mathcal{C} = \tilde{f} \restriction \mathcal{C}$, it follows that $\overline{\mathcal{D}} = \overline{f''\mathcal{C}} = f''\overline{\mathcal{C}}$.

Claim 1. For each $Y \in \overline{\mathcal{D}}$ and each $m < \omega$, there is an $\tilde{m} \ge m$ satisfying the following: For each $Z \in \overline{\mathcal{C}}$ such that $\tilde{f}(Z) \restriction \tilde{m} = Y \restriction \tilde{m}$, there is an $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$ and $\hat{g}(X \restriction k_m) = \hat{g}(Z \restriction k_m)$.

Proof. Let $Y \in \overline{\mathcal{D}}$ and suppose the claim fails. Then there is an m such that for each $n \ge m$, there is a $Z_n \in \overline{\mathcal{C}}$ such that $\tilde{f}(Z_n) \restriction n = Y \restriction n$, but for each $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$, $\hat{g}(Z_n \restriction k_m) \ne \hat{g}(X \restriction k_m)$. $\overline{\mathcal{C}}$ is compact, so there is a subsequence $(Z_{n_i})_{i<\omega}$ which converges to some $X \in \overline{\mathcal{C}}$. Since $\tilde{f}$ is continuous, $\tilde{f}(Z_{n_i})$ converges to $\tilde{f}(X)$. Since for each $i < \omega$, $\tilde{f}(Z_{n_i}) \restriction n_i = Y \restriction n_i$, it follows that $\tilde{f}(Z_{n_i})$ converges to Y. Therefore, $\tilde{f}(X) = Y$. Further, since $Z_{n_i} \to X$, there is a j such that for all $i \ge j$,

$Z_{n_i} \restriction k_m = X \restriction k_m$, and hence $\hat{g}(Z_{n_i} \restriction k_m) = \hat{g}(X \restriction k_m)$. But this is a contradiction since $X \in \overline{\mathcal{C}}$ and $\tilde{f}(X) = Y$. □

Claim 2. There is a strictly increasing sequence $(j_m)_{m<\omega}$ such that for each $m < \omega$, for all $Y \in \overline{\mathcal{D}}$ and $Z \in \overline{\mathcal{C}}$ with $\tilde{f}(Z) \restriction j_m = Y \restriction j_m$, there is an $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$ and $\hat{g}(X \restriction k_m) = \hat{g}(Z \restriction k_m)$.

Proof. Let $j_0 = 0$ and note that j_0 vacuously satisfies the claim. Now suppose that $m \geq 1$ and suppose we have chosen $j_0 < \cdots < j_{m-1}$ satisfying the claim. For each $Y \in \overline{\mathcal{C}}$, there is an $\tilde{m}(Y) \geq m$ satisfying Claim 1. The finite characteristic functions $Y \restriction \tilde{m}(Y)$ determine basic open sets in 2^ω, and the union of these open sets (over all $Y \in \overline{\mathcal{C}}$) covers $\overline{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is compact, there is a finite subcover, determined by some $Y_0 \restriction \tilde{m}(Y_0), \ldots, Y_l \restriction \tilde{m}(Y_l)$. Take $j_m > \max\{j_{m-1}, \tilde{m}(Y_0), \ldots, \tilde{m}(Y_l)\}$. By this inductive construction, we obtain a sequence $(j_m)_{m<\omega}$ which satisfies the claim. □

Let g^* be the function on $\overline{\mathcal{C}}$ generated by $\hat{g}$; that is, for $X \in \overline{\mathcal{C}}$, define $g^*(X) = \bigcup_{n<\omega} \hat{g}(X \restriction k_n)$. Notice that for each $X \in \overline{\mathcal{C}}$ and each $n < \omega$, $g^*(X) \restriction n = \hat{g}(X \restriction k_n)$, since $\hat{g}$ is initial segment preserving.

Claim 3. $g^* \restriction \mathcal{C} = \tilde{g} \restriction \mathcal{C}$. Moreover, for each $X \in \overline{\mathcal{C}}$,

$$g^*(X) = \bigcap\{\tilde{g}(Z) : Z \in \mathcal{C} \text{ and } Z \supseteq X\} \supseteq \tilde{g}(X). \tag{2.11}$$

Proof. If $X \in \mathcal{C}$, then $g^*(X) = \tilde{g}(X)$, since $\hat{g}$ represents $\tilde{g}$ on $\mathcal{C}$.

Suppose more generally that $X \in \overline{\mathcal{C}}$. Let n be given and let $s = X \restriction k_n$. Then $s \in C$, and for any $Z \in \mathcal{C}$ such that $Z \restriction k_n = s$, $\hat{g}(s) = \hat{g}(Z \restriction k_n) = \tilde{g}(Z) \restriction n$. Since $\tilde{g}$ is monotone and since the property $(*)$ on $\mathcal{C}$ implies there is a $Z \in \mathcal{C}$ such that $Z \supseteq X$ and $Z \restriction k_n = X \restriction k_n$,

$$\hat{g}(X \restriction k_n) = \bigcap\{\tilde{g}(Z) \restriction n : Z \in \mathcal{C} \text{ and } Z \supseteq X\} \supseteq \tilde{g}(X) \restriction n.$$

Unioning over all $n < \omega$, we see that (2.11) holds. □

Claim 4. Let $Y \in \overline{\mathcal{D}}$ and m be given, and let $t = Y \restriction j_m$. Then

$$\tilde{h}(Y) \restriction m \subseteq \hat{g}(s) \restriction m,$$

for each $s \in C \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$.

Proof. Let $Y \in \overline{\mathcal{D}}$ and m be given, and let $t = Y \restriction j_m$. Let s be any member of $C \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$, and fix some $Z \in \mathcal{C}$ such that $Z \restriction k_{j_m} = s$. Then $\tilde{f}(Z) \restriction j_m = \hat{f}(s) = t = Y \restriction j_m$. By Claim 2, there is an $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$ and $\hat{g}(X \restriction k_m) = \hat{g}(Z \restriction k_m)$. To prove the claim, we shall show that the following holds:

$$\tilde{h}(Y) \restriction m = \tilde{g}(X) \restriction m \subseteq \hat{g}(X \restriction k_m) = \hat{g}(s) \restriction m. \tag{2.12}$$

The first equality follows from $\tilde{f}(X) = Y$ and $\tilde{h} \circ \tilde{f}(X) = \tilde{g}(X)$. The last equality holds since $\hat{g}(X \upharpoonright k_m) = \hat{g}(Z \upharpoonright k_m)$, $Z \upharpoonright k_m = s \upharpoonright k_m$, and $\hat{g}(s \upharpoonright k_m) = \hat{g}(s) \upharpoonright m$. To see that the inclusion holds, we recall that by Claim 3, $\tilde{g}(X) \subseteq g^*(X)$, and hence $\tilde{g}(X) \upharpoonright m \subseteq g^*(X) \upharpoonright m$, which by definition of g^* is exactly $\hat{g}(X \upharpoonright k_m)$. □

Finally, we define the finitary function $\hat{h}$ which will represent h on $\mathcal{D}$. Let $D' = \{t \in D : \exists m < \omega\,(|t| = j_m)\}$. For $t \in D' \cap 2^{j_m}$, define $\hat{h}(t)$ to be the function from m into 2 such that for $i \in m$,

$$\hat{h}(t)(i) = \min\{\hat{g}(s)(i) : s \in C \cap 2^{k_{j_m}} \text{ and } \hat{f}(s) = t\}. \tag{2.13}$$

In words, $\hat{h}(t)$ is the characteristic function with domain m of the intersection of the subsets a of m for which there is some $s \in C \cap 2^{k_{j_m}}$ with $\hat{f}(s) = t$ such that $\hat{g}(s \upharpoonright j_m)$ is the characteristic function of a. By definition, $\hat{h}$ is level preserving.

Claim 5. $\hat{h}$ is basic and generates $h \upharpoonright \mathcal{D}$.

Proof. Let $Y \in \mathcal{D}$, Z be a member of $\mathcal{C}$ such that $\tilde{f}(Z) = Y$, and $m < \omega$ be given. Let $t = Y \upharpoonright j_m$ and $u = Z \upharpoonright k_{j_m}$. Then $\hat{f}(u) = t$, so $\hat{g}(u) \upharpoonright m \supseteq \hat{h}(t)$. Since $Z \in \mathcal{C}$, $g(Z) \upharpoonright m = \hat{g}(u) \upharpoonright m$. Thus,

$$h(Y) \upharpoonright m = h \circ f(Z) \upharpoonright m = g(Z) \upharpoonright m = \hat{g}(u) \upharpoonright m \supseteq \hat{h}(t). \tag{2.14}$$

Now suppose s is any member of $C \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$. By Claim 2, there is an $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$ and $\hat{g}(X \upharpoonright k_m) = \hat{g}(s \upharpoonright k_m)$. Then, applying Claim 4, we see that

$$\hat{g}(X \upharpoonright k_m) = g^*(X) \upharpoonright m \supseteq \tilde{g}(X) \upharpoonright m = \tilde{h} \circ \tilde{f}(X) \upharpoonright m = h(Y) \upharpoonright m. \tag{2.15}$$

Hence, $\hat{g}(s \upharpoonright k_m) \supseteq h(Y) \upharpoonright m$. Since s was an arbitrary member of $C \cap 2^{k_{j_m}}$ satisfying that $\hat{f}(s) = t$, it follows that $\hat{h}(t) \supseteq h(Y) \upharpoonright m$. Therefore, $\hat{h}(Y \upharpoonright j_m) = h(Y) \upharpoonright m$.

Thus, $\hat{h}$ generates h on $\mathcal{D}$ and hence, $\hat{h}$ is monotone. It follows that $\hat{h}$ is end-extension preserving: If $t \sqsubset t'$ are members of D' of lengths j_m and $j_{m'}$, respectively, then letting Y be any member of $\mathcal{D}$ such that $t' \sqsubset Y$, we see that $\hat{h}(t) = h(Y) \upharpoonright m = (h(Y) \upharpoonright m') \upharpoonright m = \hat{h}(t') \upharpoonright m$. Therefore, $\hat{h}$ is basic. □

Thus, $h \upharpoonright \mathcal{D}$ is generated by the basic map $\hat{h}$ on D'. Thus, $\mathcal{V}$ has basic Tukey reductions. □

Every p-point has basic Tukey reductions satisfying the additional property $(*)$ of Theorem 2.5, as was shown in the proof of Theorem 20 of [8], the cofinal set $\mathcal{C}$ there being of the simple form $\mathcal{P}(X) \cap \mathcal{U}$ for some $X \in \mathcal{U}$. Hence, the following theorem holds.

Theorem 2.6. *Every ultrafilter Tukey reducible to a p-point has basic, and hence continuous, Tukey reductions.*

Recall that an ultrafilter $\mathcal{V}$ is Rudin-Blass reducible to an ultrafilter $\mathcal{W}$ if there is a finite-to-one map $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{W})$. Thus, Rudin-Blass reducibility implies Rudin-Keisler reducibility. Our Theorem 2.6 combines with Theorem 10 of Raghavan in [13] (see Theorem 1.6 for the statement) to yield the following.

Theorem 2.7. *Suppose $\mathcal{U}$ is Tukey reducible to a p-point. Then for each q-point $\mathcal{V}$, $\mathcal{V} \leq_T \mathcal{U}$ implies $\mathcal{V} \leq_{RB} \mathcal{U}$.*

Remark 2.8. Stable ordered-union ultrafilters are the analogues of p-points on the base set $\mathrm{FIN} = [\omega]^{<\omega} \setminus \{\emptyset\}$ (see [1]). In Theorems 71 and 72 of [8], it was shown that for each stable ordered union ultrafilter $\mathcal{U}$, both $\mathcal{U}$ and its projection $\mathcal{U}_{\mathrm{min,max}}$ have continuous Tukey reductions, with respect to the Ellentuck topology on the Milliken space. It is of interest that the ultrafilter $\mathcal{U}_{\mathrm{min,max}}$ is rapid, but is neither a p-point nor a q-point, and yet, by Theorem 2.5, every ultrafilter Tukey below $\mathcal{U}_{\mathrm{min,max}}$ has continuous Tukey reductions (condition $(*)$ is satisfied). In fact, this was extended to all ultrafilters selective for some topological Ramsey space, under a mild assumption which is satisfied in all known topological Ramsey spaces, by Dobrinen and Trujillo showed in Theorem 56 of in [7]. Many such ultrafilters are not p-points.

It should be the case that by arguments similar to those in Theorem 2.5, one can prove that every ultrafilter Tukey reducible to some stable ordered union ultrafilter, or more generally, any ultrafilter selective for some topological Ramsey space, also has continuous Tukey reductions. We leave this as an open problem in Section 7.

3. Iterated Fubini products of ultrafilters represented as ultrafilters generated by $\vec{\mathcal{U}}$-trees on flat-top fronts

Fubini products of ultrafilters on base set ω are commonly viewed as ultrafilters on base set $\omega \times \omega$. As was pointed out to us by Todorcevic, Fubini products of nonprincipal ultrafilters on base set ω may also be viewed as ultrafilters on base set $[\omega]^2$. This view leads well to precise investigations of ultrafilters constructed by iterating the Fubini product construction. In this section, we review Fubini products of ultrafilters and countable iterations of this construction. After reviewing the notion of front and introducing the new notion of flat-top front, we then show how every ultrafilter obtained by iterating the Fubini product construction can be viewed as an ultrafilter generated by certain subtrees of a base set which is a tree, particularly a flat-top front. This section is a primer for the work in Section 3.

Notation. Let $\mathcal{U}$ and $\mathcal{V}_n$ $(n < \omega)$ be ultrafilters. The *Fubini product* of $\mathcal{V}_n$ over $\mathcal{U}$, denoted $\lim_{n\to\mathcal{U}} \mathcal{V}_n$, is defined as follows:

$$\lim_{n\to\mathcal{U}} \mathcal{V}_n = \{A \subseteq \omega \times \omega : \{n \in \omega : \{j \in \omega : (n,j) \in A\} \in \mathcal{V}_n\} \in \mathcal{U}\}. \tag{3.1}$$

When all $\mathcal{V}_n = \mathcal{V}$, then we let $\mathcal{U} \cdot \mathcal{V}$ denote $\lim_{n\to\mathcal{U}} \mathcal{V}_n$.

The Fubini product construction can be iterated countably many times, each time producing an ultrafilter. For example, given an ultrafilter $\mathcal{V}$, let $\mathcal{V}^1$ denote $\mathcal{V}$, and let $\mathcal{V}^{n+1}$ denote $\mathcal{V} \cdot \mathcal{V}^n$. Naturally, $\mathcal{V}^\omega$ denotes $\lim_{n\to\mathcal{V}} \mathcal{V}^n$. Continuing in this manner, we obtain $\mathcal{V}^\alpha$, for all $2 \le \alpha < \omega \cdot 2$. At this point, it is ambiguous what is meant by $\mathcal{V}^{\omega\cdot 2}$. It is standard practice for countable a limit ordinal α to let $\mathcal{V}^\alpha$ denote any ultrafilter constructed by choosing (arbitrarily) an increasing sequence $(\alpha_n)_{<\omega}$ converging to α and defining $\mathcal{V}^\alpha$ to be $\lim_{n\to\mathcal{V}} \mathcal{V}^{\alpha_n}$, but this is ambiguous, since the choice of the sequence $(\alpha_n)_{n<\omega}$ is completely arbitrary.

However, each countable iteration of Fubini products of ultrafilters (including the choice of sequence at limit stages) can be represented as an ultrafilter generated by $\vec{\mathcal{U}}$-trees (see Definition 3.3) on a base set which is a front. This representation is unambiguous at limit stages. For this reason, Theorem 4.4 in the next section, showing that iterations of Fubini products of p-points have Tukey reductions which are as close to continuous as possible, will be carried out in the setting of $\vec{\mathcal{U}}$-trees.

We now recall the definition of front and define the new notion of *flat-top front*, which is exactly the type of front on which iterated Fubini products of ultrafilters are represented. The reader desiring more background on fronts and $\vec{\mathcal{U}}$-trees than presented here is referred to [17], pages 12 and 190, respectively.

Definition 3.1. A family B of finite subsets of some infinite subset I of ω is called a *front* on I if

(1) $a \not\sqsubset b$ whenever a, b are in B; and
(2) For every infinite $X \subseteq I$ there exists $b \in B$ such that $b \sqsubset X$.

Recall the following standard set-theoretic notation: $[\omega]^k$ denotes the collection of k-element subsets of ω, $[\omega]^{<k}$ denotes the collection of subsets

of ω of size less than k, and $[\omega]^{\le k} = [\omega]^{<k+1}$. It is easy to see that for each $k < \omega$, $[\omega]^k$ is a front.

Every front is lexicographically well-ordered, and hence has a unique lexicographic rank associated with it, namely the ordinal length of its lexicographical well-ordering. For example, $\mathrm{rank}(\{\emptyset\}) = 1$, $\mathrm{rank}([\omega]^1) = \omega$, and $\mathrm{rank}([\omega]^2) = \omega \cdot \omega$. We shall usually drop the adjective 'lexicographic' when talking about ranks of fronts.

Given a front B, for each $n \in \omega$, we define $B_n = \{b \in B : n = \min(b)\}$ and $B_{\{n\}} = \{b \setminus \{n\} : b \in B_n\}$. Then $B = \bigcup_{n\in\omega} B_n$, and each $B_n = \{\{n\} \cup a : a \in B_{\{n\}}\}$. Note that for each $n \in \omega$, $B_{\{n\}}$ is a front on $\omega \setminus (n+1)$ with rank strictly less than the rank of B. Conversely, given any collection of fronts $B_{\{n\}}$ on $\omega \setminus (n+1)$, the union $\bigcup_{n\in\omega} B_n$ is a front on ω, where B_n is defined as above to be $\{\{n\} \cup a : a \in B_{\{n\}}\}$.

Definition 3.2. We call a set $B \subseteq [\omega]^{<\omega}$ a *flat-top front* if B is a front on ω, $B \ne \{\emptyset\}$, and

(1) Either $B = [\omega]^1$; or
(2) $B \subseteq [\omega]^{\ge 2}$ and for each $b \in B$, letting $a = b \setminus \{\max(b)\}$, $\{c \setminus a : c \in B,\ c \sqsupset a\}$ is equal to $[\omega \setminus (\max(a) + 1)]^1$.

Flat-top fronts are exactly the fronts on which iterated Fubini products of ultrafilters are represented, as will be seen in Facts 3.4 and 3.5. For example, $[\omega]^2$ is the flat-top front on which a Fubini product of the form $\lim_{n\to\mathcal{U}} \mathcal{V}_n$ is represented. For each $k < \omega$, $[\omega]^k$ is a flat-top front. Moreover, flat-top fronts are preserved under the following recursive construction: Given flat-top fronts $B_{\{n\}}$ on $\omega \setminus (n+1)$, $n < \omega$, the union $\bigcup_{n\in\omega} B_n$ is a flat-top front on ω.

Given any front B, we let $\hat{B}$ denote the collection of all initial segments of members of B. Let $\hat{B}^-$ denote the collection of all proper initial segments of members of B; that is, $\hat{B}^- = \hat{B} \setminus B$. Both $\hat{B}$ and $\hat{B}^-$ form trees under the partial ordering $\sqsubseteq$.

Definition 3.3. Given a flat-top front B and a sequence $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$ of nonprincipal ultrafilters $\mathcal{U}_c$ on ω, a *$\vec{\mathcal{U}}$-tree* is a tree $T \subseteq \hat{B}$ such that $\emptyset \in T$ and for each $c \in T \cap \hat{B}^-$, $\{n \in \omega : c \cup \{n\} \in T\} \in \mathcal{U}_c$.

Notation. Given a flat-top front B and a sequence $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$ of nonprincipal ultrafilters on ω, let $\mathfrak{T} = \mathfrak{T}(\vec{\mathcal{U}})$ denote the collection of all $\vec{\mathcal{U}}$-trees. For any $c \in \hat{B}^-$ and $T \in \mathfrak{T}$, let $T_c = \{t \in T : t \sqsubseteq c \text{ or } t \sqsupset c\}$, the tree with stem c consisting of all nodes in T comparable with c. For any tree T, let $[T]$ denote the collection of maximal branches through T.

Note that if T is a $\mathcal{U}$-tree, then the fact that $\emptyset \in T$ implies that the set $[T]$ of maximal branches through T is contained in B.

The following Facts 3.4 and 3.5 were pointed out to us by Todorcevic.

Fact 3.4. *The Fubini product* $\lim_{n\to\mathcal{U}} \mathcal{V}_n$ *of nonprincipal ultrafilters on* ω *is isomorphic to the ultrafilter on* $B = [\omega]^2$ *generated by* $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in [\omega]^{\le 1})$*-trees, where* $\mathcal{U}_\emptyset = \mathcal{U}$ *and for each* $n \in \omega$, $\mathcal{U}_{\{n\}} = \mathcal{V}_n$.

Proof. Suppose that $\mathcal{W} = \lim_{n\to\mathcal{U}} \mathcal{V}_n$. Define $\mathcal{U}_\emptyset = \mathcal{U}$, and $\mathcal{U}_{\{n\}} = \mathcal{V}_n$ for each $n < \omega$. Let $B = [\omega]^2$. Then $\hat{B} = [\omega]^{\le 2}$ and $\hat{B}^- = [\omega]^{\le 1}$. Let $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$. Let Δ denote the upper triangle $\{(m,n) : m < n < \omega\}$ on $\omega \times \omega$. Since $\mathcal{U}$ and $\mathcal{V}_n$ are all nonprincipal, $\Delta \in \lim_{n\to\mathcal{U}} \mathcal{V}_n$. Let $\theta : \Delta \to B$ by $\theta((m,n)) = \{m,n\}$. Then θ witnesses that the ultrafilter $\mathcal{W} \restriction \Delta := \{W \in \mathcal{W} : W \subseteq \Delta\}$ on base set Δ is isomorphic to the ultrafilter $\{[S] : S \subseteq \hat{B}$ and $\exists T \in \mathfrak{T}(\vec{\mathcal{U}})\,(T \subseteq S)\}$ on base set B. Since $\mathcal{W} \restriction \Delta$ is isomorphic to the original $\mathcal{W}$, we have that the ultrafilter on B generated by the set $\{[T] : T \in \mathfrak{T}(\vec{\mathcal{U}})\}$ is isomorphic to $\mathcal{W}$. In particular, $\{[T] : T \in \mathfrak{T}(\vec{\mathcal{U}})\}$ is isomorphic to a base for $\mathcal{W}$. □

Next we shall generalize Fact 3.4 to all iterates of Fubini products of nonprincipal ultrafilters on ω. Let $\mathcal{P}_0$ denote the collection of all nonprincipal ultrafilters on ω. Given $\alpha < \omega_1$, define $\mathcal{P}_{\alpha+1} = \{\lim_{n\to\mathcal{U}} \mathcal{V}_n : \mathcal{U} \in \mathcal{P}_0$ and $\mathcal{V}_n \in \mathcal{P}_\alpha\}$. For each limit ordinal α, define $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_\beta$. Then $\mathcal{P}_{<\omega_1} := \bigcup\{\mathcal{P}_\alpha : \alpha < \omega_1\}$ is the collection of all iterated Fubini products of nonprincipal ultrafilters on ω. Each $\mathcal{W} \in \mathcal{P}_{<\omega_1}$ has a well-defined notion of rank, namely $\mathrm{rank}(\mathcal{W})$ is the least $\alpha < \omega_1$ for which it is a member of $\mathcal{P}_\alpha$.

Fact 3.5. *If* $\mathcal{W}$ *is a countable iteration of Fubini products of nonprincipal ultrafilters, then there is a flat-top front* B *and p-points* $\mathcal{U}_c$, $c \in \hat{B}^-$ *such that* $\mathcal{W}$ *is isomorphic to the ultrafilter on* B *generated by the* $(\mathcal{U}_c : c \in \hat{B}^-)$*-trees.*

Proof. We prove by induction on $\alpha < \omega_1$ that the fact holds for every ultrafilter in $\mathcal{P}_\alpha$. If $\mathcal{W} \in \mathcal{P}_0$, then $\mathcal{W}$ is a nonprincipal ultrafilter and is represented on the flat-top front $B = [\omega]^1$ via the obvious isomorphism $n \mapsto \{n\}$. If $\mathcal{W} \in \mathcal{P}_1$, then Fact 3.4 proves our claim.

Let $2 \le \alpha < \omega_1$ and assume the fact holds for each ultrafilter in $\bigcup_{\gamma<\alpha} \mathcal{P}_\gamma$. If α is a limit ordinal, then there is nothing to prove, so assume $\alpha = \beta + 1$ for some $1 \le \beta < \omega_1$. Suppose that $\mathcal{W} \in \mathcal{P}_\alpha$. Then $\mathcal{W} = \lim_{n\to\mathcal{U}} \mathcal{W}_n$, where $\mathcal{U}$ is a nonprincipal ultrafilter and for each n, $\mathcal{W}_n \in \mathcal{P}_\beta$. By the induction hypothesis, for each $n < \omega$ there is a flat-top front $C(n)$ on ω and there are nonprincipal ultrafilters $\mathcal{U}_c(n)$, $c \in \widehat{C(n)}^-$, such that $\mathcal{W}_n$ is isomorphic to the ultrafilter generated by $(\mathcal{U}_c : c \in \widehat{C(n)}^-)$-trees on $C(n)$. In the standard way, we glue the fronts together to obtain a new flat-top front: Let $B_{\{n\}}$ be the front on $\omega \setminus (n+1)$ which is the isomorphic image of $C(n)$, via the isomorphism $\varphi_n : \omega \to \omega \setminus (n+1)$ by $\varphi_n(m) = n+1+m$. That is, for each $c \in C(n)$, let $\varphi_n(c) = \{n+1+m : m \in c\}$, and let $B_{\{n\}} = \{\varphi_n(c) : c \in C(n)\}$. Then $B = \bigcup_{n<\omega}\{\{n\} \cup b : b \in B_{\{n\}}\}$ is a flat-top front.

Given $n < \omega$, for each $c \in \widehat{C(n)}^-$, $\mathcal{U}_c(n)$ is isomorphic to $\varphi_n(\mathcal{U}_c(n))$. Therefore, the ultrafilter generated by $(\mathcal{U}_c(n) : c \in \widehat{C(n)}^-)$-trees on (n) is isomorphic to the ultrafilter generated by $(\varphi_n(\mathcal{U}_{\varphi_n^{-1}(a)}(n)) : a \in \hat{B}^-_{\{n\}})$. For each $n < \omega$ and $a \in \hat{B}^-_{\{n\}}$, let $\mathcal{V}_{\{n\}\cup a}$ denote $\mathcal{U}_{\varphi_n^{-1}(a)}(n)$. Finally, let $\mathcal{V}_\emptyset = \mathcal{U}$. Then the ultrafilter on B generated by the $(\mathcal{V}_a : a \in \hat{B}^-)$-trees is isomorphic to $\lim_{n\to\mathcal{U}} \mathcal{W}_n$. $\square$

4. Basic cofinal maps on iterated Fubini products of p-points

Fubini products of p-points do not in general have continuous Tukey reductions. However, we will show that they do have canonical cofinal maps satisfying many of the properties of continuous maps, which we call *basic* (see Definition 4.2 below). Making use of the natural representation of Fubini iterates of p-points as ultrafilters generated by $\vec{\mathcal{U}}$-trees on some flat-top front B (recall Fact 3.5), we show in Theorem 4.4 that countable iterates of Fubini products of p-points have *basic Tukey reductions*. Such Tukey reductions are represented by finite end-extension preserving maps and hence are continuous on the space $2^{\hat{B}}$ with the Cantor topology, where $\hat{B}$ is the tree consisting of all initial segments of members of the front B. This extends a key property of p-points (recall Theorem 1.3) to a large class of ultrafilters. Theorem 4.4 will be applied in Sections 5 and 6.

Definition 4.1. Let $\prec$ denote the following well-ordering on $[\omega]^{<\omega}$. Given any $a, b \in [\omega]^{<\omega}$ with $a \neq b$, enumerate their elements in increasing order as $a = \{a_1, \dots, a_m\}$ and $b = \{b_1, \dots, b_n\}$. Here m equals the cardinality of a and n equals the cardinality of b, and no comparison between m and n is assumed. Define $a \prec b$ iff

(1) $a = \emptyset$; or
(2) $\max(a) < \max(b)$; or
(3) $\max(a) = \max(b)$ and $a_i < b_i$, where i is the least such that $a_i \neq b_i$.

Thus, $\prec$ well-orders $[\omega]^{<\omega}$ in order type ω as follows: $\emptyset \prec \{0\} \prec \{0,1\} \prec \{1\} \prec \{0,1,2\} \prec \{0,2\} \prec \{1,2\} \prec \{2\} \prec \{0,1,2,3\} \prec \dots$. Moreover, for each $k < \omega$, the set $\{c \in [\omega]^{<\omega} : \max(c) = k\}$ forms a finite interval in $([\omega]^{<\omega}, \prec)$.

The following example illustrates why it is impossible for a Fubini product of p-points to have continuous Tukey reductions, with respect to the Cantor topology on 2^B, where B is the base for the ultrafilter. Let $\mathcal{U}$ and $\mathcal{V}$ be any nonprincipal ultrafilters, p-points or otherwise, and let $f : \omega \times \omega \to \omega$ be given by $f((n, j)) = n$. Then $f : \mathcal{U} \cdot \mathcal{V} \to \mathcal{U}$ is a monotone cofinal map, and there is no cofinal $\mathcal{X} \subseteq \mathcal{U} \cdot \mathcal{V}$ for which $f \restriction \mathcal{X}$ is basic on the topological space $2^{\omega\times\omega}$. However, we will soon show that each ultrafilter $\mathcal{W}$ which is an iterated Fubini product of p-points has finitely generated Tukey reductions which, moreover, are basic, and hence continuous, on the appropriate tree space. Toward this end, we proceed to give the definition of basic for this context, and then prove the main results of this section.

Notation. For any subset $A \subseteq [\omega]^{<\omega}$, recall that $\hat{A}$ denotes the set of all initial segments of members of A. For any front B, we let $\hat{B}^-$ denote $\hat{B} \setminus B$. For any subset $A \subseteq [\omega]^{<\omega}$ and $k < \omega$, let $A \restriction k$ denote $\{a \in A : \max(a) < k\}$. For $A \subseteq \hat{B}$ and $k < \omega$, let $\chi_A \restriction k$ denote the characteristic function of $A \restriction k$ on domain $\hat{B} \restriction k$. For each $k < \omega$, let $2^{\hat{B}\restriction k}$ denote the collection of characteristic functions of subsets of $\hat{B} \restriction k$ on domain $\hat{B} \restriction k$.

Definition 4.2. Let B be a flat-top front on ω, $\tilde{T} \subseteq \hat{B}$ be a tree, and $(n_k)_{k<\omega}$ be an increasing sequence. We say that a function $\hat{f} : \bigcup_{k<\omega} 2^{\tilde{T}\restriction n_k} \to 2^{<\omega}$ is *level preserving* if $\hat{f} : 2^{\tilde{T}\restriction n_k} \to 2^k$, for each $k < \omega$. $\hat{f}$ is *end-extension preserving* if for all $k < m$, $A \subseteq \tilde{T} \restriction n_k$ and $A' \subseteq \tilde{T} \restriction n_m$, if $A = A' \restriction n_k$ then $\hat{f}(\chi_A) = \hat{f}(\chi_{A'}) \restriction k$. $\hat{f}$ is *monotone* if whenever $A \subseteq A' \subseteq \tilde{T}$ are finite, then $\hat{f}(\chi_A) \subseteq \hat{f}(\chi_{A'})$. $\hat{f}$ is *basic* if it is level and end-extensions preserving and is monotone.

Let $\mathcal{U}$ be an ultrafilter on B generated by $(\mathcal{U}_c : c \in \hat{B}^-)$-trees, let $f : \mathcal{U} \to \mathcal{V}$ be a monotone cofinal map, where $\mathcal{V}$ is an ultrafilter on base ω, and let $\tilde{T} \in \mathfrak{T}(\vec{\mathcal{U}})$. Let $\mathfrak{T} \restriction \tilde{T}$ denote the set of all $\vec{\mathcal{U}}$-trees contained in $\tilde{T}$. We say that $\hat{f} : \bigcup_{k<\omega} 2^{\tilde{T}\restriction n_k} \to 2^{<\omega}$ *generates* f on $\mathfrak{T} \restriction \tilde{T}$ if for each $T \in \mathfrak{T} \restriction \tilde{T}$,

$$f([T]) = \bigcup_{k<\omega} \hat{f}(\chi_T \restriction n_k). \tag{4.1}$$

We say that $\mathcal{U}$ has *basic Tukey reductions* if whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, then there is a $\tilde{T} \in \mathfrak{T}(\vec{\mathcal{U}})$ and a basic map $\hat{f}$ which generates f on $\mathfrak{T} \restriction \tilde{T}$.

Remark 4.3. Note that if $\hat{f}$ witnesses that f is basic on $\mathfrak{T} \restriction \tilde{T}$, then $\hat{f}$ generates a continuous map on the collection of *trees* in $\mathfrak{T} \restriction \tilde{T}$, continuity being with respect to the Cantor topology on $2^{\hat{B}}$. Moreover, we may define a map $\hat{g}$ on B as follows: For each finite subset $A \subseteq B$, define $\hat{g}(A) = \hat{f}(\hat{A})$, where $\hat{A}$ is the collection of all initial segments of members of A. Then $\hat{g}$ is finitary, but not necessarily continuous on 2^B, and $\hat{g}$ generates f on $\{[T] : T \in \mathfrak{T} \restriction \tilde{T}\}$ which is a base for the ultrafilter. Thus, for ultrafilters generated by $\mathcal{U}$-trees, basic Tukey reductions imply finitely represented Tukey reductions on the original base set B.

Now we prove the main theorem of this section.

Theorem 4.4. *Let B be any flat-top front and $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$ be a sequence of p-points. Then the ultrafilter $\mathcal{U}$ on base B generated by the $\vec{\mathcal{U}}$-trees has basic Tukey reductions. Therefore, every countable iteration of Fubini products of p-points has basic Tukey reductions.*

Proof. Let $\mathcal{V}$ be some ultrafilter Tukey reducible to $\mathcal{U}$, and let $f : \mathcal{U} \to \mathcal{V}$ be a monotone cofinal map. We let $\mathfrak{T}$ denote $\mathfrak{T}(\mathcal{U}_c : c \in \hat{B}^-)$, the set of all $\vec{\mathcal{U}}$-trees. Recall that $\mathfrak{T}$ is a base for the ultrafilter $\mathcal{U}$. For each $k < \omega$, let $\hat{B} \restriction k$ denote the collection of all $b \in \hat{B}$ with $\max b < k$. Thus, $\hat{B} \restriction 0 = \{\emptyset\}$, $\hat{B} \restriction 1 = \{\emptyset, \{0\}\}$, and so forth. Fix an enumeration of the finite, non-empty $\sqsubseteq$-closed subsets of $\hat{B}$ as $\langle A_i : i < \omega\rangle$ so that for each $i < j$, $\max \bigcup A_i \le \max \bigcup A_j$. Let $(p_k)_{k<\omega}$ denote the strictly increasing sequence so that for each k, the sequence $\langle A_i : i < p_k\rangle$ lists all $\sqsubseteq$-closed subsets of $\hat{B} \restriction k$. (For example, if $B = [\omega]^2$, then $\hat{B} = [\omega]^{\le 2}$ and $\hat{B}^- = [\omega]^{\le 1}$, and we may let $A_0 = \{\emptyset\}$, $A_1 = \{\emptyset, \{0\}\}$, $A_2 = \{\emptyset, \{0\}, \{1\}\}$, $A_3 = \{\emptyset, \{0\}, \{0, 1\}\}$, $A_4 = \{\emptyset, \{0\}, \{0, 1\}, \{1\}\}$, $A_5 = \{\emptyset, \{1\}\}$. Note that $p_0 = 1$, $p_1 = 2$, and $p_2 = 6$.)

For $k < \omega$ and $i < p_k$, define

$$\hat{B}^k_i = A_i \cup \{b \in \hat{B} : \exists a \in A_i (b \sqsupset a \text{ and } \min(b \setminus a) > k)\}. \tag{4.2}$$

Thus, $\hat{B}^k_i$ is the maximal tree in $\mathfrak{T}$ for which $T \restriction k = A_i$. For a tree $T \subseteq \hat{B}$ and $c \in T \cap \hat{B}^-$, define the notation

$$U_c(T) = \{l > \max(c) : c \cup \{l\} \in T\}. \tag{4.3}$$

We refer to $U_c(T)$ as *the set of immediate extensions of c in T*. Note that if $T \in \mathfrak{T}$, then for each $c \in T \cap \hat{B}^-$, $U_c(T)$ is a member of $\mathcal{U}_c$. For $c \in \hat{B}^-$, recall that $\hat{B}_c$ denotes the tree of all $a \in \hat{B}$ such that either $a \sqsubseteq c$ or else $a \sqsupset c$.

Our goal is to construct a tree $\tilde{T} \in \mathfrak{T}$ and find a sequence $(n_k)_{k<\omega}$ of good cut-off points such that the following (⊛) holds.

($\circledast$) For each $T \subseteq \tilde{T}$ in $\mathfrak{T}$, $k < \omega$, and $i < p_{n_k}$ such that $A_i = T \restriction n_k$, for every $j \le k$,

$$j \in f([T]) \iff j \in f([\tilde{T} \cap \hat{B}^{n_k}_i]).$$

Claim 1. The property ($\circledast$) implies that f is basic on $\mathfrak{T} \restriction \tilde{T}$.

Proof. For $k < \omega$ and $T \in \mathfrak{T} \restriction \tilde{T}$, define

$$\hat{f}(T \restriction n_k) = f([\tilde{T} \cap \hat{B}^{n_k}_i]) \cap (k+1), \tag{4.4}$$

where i is *the* integer below p_{n_k} such that $T \restriction n_k = A_i$. By definition, $\hat{f}$ is level preserving. Let $l > k$ and let m be such that $T \restriction n_l = A_m$. Then $A_m \restriction n_k = A_i$. For $j \le k$, ($\circledast$) implies that $j \in f([\tilde{T} \cap \hat{B}^{n_k}_i])$ if and only if $j \in f([T])$ if and only if $j \in f([\tilde{T} \cap \hat{B}^{n_l}_m])$. Thus, $j \in \hat{f}(T \restriction n_k)$ if and only if $j \in \hat{f}(T \restriction n_l)$. Therefore, $\hat{f}$ is end-extension preserving; that is, $\hat{f}(T \restriction n_l) \restriction (k+1) = \hat{f}(T \restriction n_k)$. Furthermore, $f(T) = \bigcup_{k<\omega} \hat{f}(T \restriction n_k)$; thus, $\hat{f}$ generates f on $\mathfrak{T} \restriction \tilde{T}$. Since f is monotone and $\hat{f}$ is end-extension preserving and generates f, it follows that $\hat{f}$ is also monotone. Thus, $\hat{f}$ witnesses that f is basic on $\mathfrak{T} \restriction \tilde{T}$. $\square$

The construction of $\tilde{T}$ and $(n_k)_{k<\omega}$ takes place in three stages.

Stage 1. In the first stage toward the construction of $\tilde{T}$, we will choose some $R^k_i \in \mathfrak{T}$ such that for all $k < \omega$, the following holds:

$(*)_k$ For all $i < p_k$ and $T \subseteq R^k_i$ in $\mathfrak{T}$ with $T \restriction k = A_i$, for each $j \le k$, $j \in f([T]) \iff j \in f([R^k_i])$.

We begin the construction by first choosing R^0_0. If there is an $R \in \mathfrak{T}$ such that and $0 \notin f([R])$, then let R^0_0 be such an R. If there is no such R, then let $R^0_0 = \hat{B}$. (We remark that for any flat-top front, A_0 is always $\{\emptyset\}$ and $p_0 = 1$. Moreover for each $R \in \mathfrak{T}$, $R \restriction 0 = \{\emptyset\}$, which is exactly A_0. So if there is no $R \in \mathfrak{T}$ such that and $0 \notin f([R])$, then $\mathcal{V}$ must be the principal ultrafilter generated by $\{0\}$.)

Now let $k > 0$, and suppose we have chosen R^l_j for all $l < k$ and $j < p_l$. Let $i < p_k$. For $i < p_{k-1}$, if there is an $R \subseteq R^{k-1}_i$ in $\mathfrak{T}$ such that $R \restriction k = A_i$ and $k \notin f([R])$, then let R^k_i be such an R; if not, let $R^k_i = R^{k-1}_i \cap \hat{B}^k_i$. Now suppose that $p_{k-1} \le i < p_k$. If there is an $R \in \mathfrak{T}$ such that $R \restriction k = A_i$ and $0 \notin f([R])$, let $R^k_{i,0}$ be such an R; if not, let $R^k_{i,0} = \hat{B}^k_i$. Given $R^k_{i,j}$ for $j < k$, if there is an $R \in \mathfrak{T}$ such that $R \subseteq R^k_{i,j}$, $R \restriction k = A_i$, and $j+1 \notin f([R])$, then let $R^k_{i,j+1}$ be such an R; if not, then let $R^k_{i,j+1} = R^k_{i,j}$. Finally, let $R^k_i = R^k_{i,k}$.

It follows from the construction that for all $1 \le l \le k$ and $p_{l-1} \le i < p_l$,

$$R^l_{i,0} \supseteq R^l_{i,1} \supseteq \ldots \supseteq R^l_{i,l} = R^l_i \supseteq R^k_i,$$

and moreover, $R^l_{i,j} \restriction l$ for any $j \le l$, $R^l_i \restriction l$ and $R^k_i \restriction k$ are all equal A_i.

Fix $k < \omega$: we check that $(*)_k$ holds. Let $i < p_k$, $T \subseteq R^k_i$ in $\mathfrak{T}$ with $T \restriction k = A_i$, and $j \le k$ be given. If $j \in f([T])$, then j must be in $f([R^k_i])$, since $T \subseteq R^k_i$ and f is monotone. Now suppose that $j \notin f([T])$; we will show that $j \notin f([R^k_i])$. Let $p_{-1} = 0$, and let $l \le k$ be the integer satisfying $p_{l-1} \le i < p_l$. If $l = 0$, then $i = 0$ in which case $A_i = \{\emptyset\}$; if $l \ge 1$, then $\max \bigcup A_i = l - 1$. Either way, we see that $T \restriction k = A_i = T \restriction l$. We now have two cases to check.

Case 1: $j \le l$. Notice that $T \subseteq R^k_i \subseteq R^l_i \subseteq R^l_{i,j}$. If $j = 0$, then $R^l_{i,j} \subseteq \hat{B}^l_i$ and we let R' denote $\hat{B}^l_i$; if $j > 0$, then $R^l_{i,j} \subseteq R^l_{i,j-1}$ and we let R' denote $R^l_{i,j-1}$. Since j is not in $f([T])$ and $T \restriction l = A_i$, T is a witness that there is an $R \subseteq R'$ with $R \restriction l = A_i$ such that $j \notin f([R])$. Thus, $R^l_{i,j}$ was chosen so that $j \notin f([R^l_{i,j}])$. It follows that $j \notin f([R^k_i])$, since $R^k_i \subseteq R^l_{i,j}$ and f is monotone.

Case 2: $l < j \le k$. In this case, $T \subseteq R^k_i \subseteq R^j_i \subseteq R^{j-1}_i$. Since T is a witness that there is an $R \subseteq R^{j-1}_i$ with $R \restriction l = A_i$ and $j \notin f([R])$, R^j_i was chosen so that $j \notin f([R^j_i])$. Thus, $j \notin f([R^k_i])$, since $R^k_i \subseteq R^j_i$ and f is monotone.

Therefore, $j \in f([T])$ if and only if $j \in f([R^k_i])$; hence $(*)_k$ holds. This concludes Stage 1 of our construction.

Given $k < \omega$ and $c \in \hat{B}^- \restriction k$, define

$$S^k_c = \bigcap \{R^l_i : l \le k,\ i < p_l,\ \text{and } c \in R^l_i\}. \tag{4.5}$$

S^k_c is well-defined and $c \in S^k_c$. To see this, notice that for $c \in \hat{B}^- \restriction k$, letting $l \le k$ be least such that $l > \max c$, then c is in A_i for at least one $i < p_l$. Since $A_i \subseteq R^l_i$, the set $\{(l,i) : l \le k,\ i < p_l,\ \text{and } c \in R^l_i\}$ is nonempty. Moreover, S^k_c is a member of $\mathfrak{T}$, since it is a finite intersection of members of $\mathfrak{T}$. Thus, for each $a \in S^k_c \cap \hat{B}^-$, the set of $\{l > \max(a) : a \cup \{l\} \in S^k_c\}$ is a member of the ultrafilter $\mathcal{U}_c$. Define

$$U^k_c := U_c(S^k_c) = \{l > \max(c) : c \cup \{l\} \in S^k_c\}. \tag{4.6}$$

Thus, for each $c \in \hat{B}^-$ and $j = \max(c) + 1$, we have $S^j_c \supseteq S^{j+1}_c \supseteq \dots$, each of which is a member of $\mathfrak{T}$; and $U^j_c \supseteq U^{j+1}_c \supseteq \dots$, each of which is a member of the p-point $\mathcal{U}_c$.

Stage 2. In this stage we construct a tree T^* in $\mathfrak{T}$ which will be thinned down one more time in Stage 3 to obtain a subtree $\tilde{T} \subseteq T^*$ in $\mathfrak{T}$ such that that $f \restriction \mathfrak{T} \restriction \tilde{T}$ is basic. The tree T^* which we construct in this stage will have sets of immediate successors $U_c := \{l \ge \max(c) + 1 : c \cup \{l\} \in T^*\}$,

$c \in T^* \cap \hat{B}^-$. The sets U_c will have interval gaps which have right endpoints which line up often and in a useful way (meshing), aiding us in finding the good cut-off points n_k needed in Stage 3 to thin T^* down to $\tilde{T}$. Toward obtaining these interval gaps, we will construct a family of functions which we call *meshing functions* $m(c, \cdot) : \omega \to \omega$ satisfying the following 'meshing property':

(†) Given $k < \omega$, for each $c \in \hat{B}^- \restriction k$ there corresponds an i_c such that for all $a, c \in \hat{B}^- \restriction k$, $m(a, 2i_a) = m(c, 2i_c)$.

The meshing functions of (†) will then aid in obtaining a tree $T^* \in \mathfrak{T}$ with the following properties:

(‡) For all $c \in T^* \cap \hat{B}^-$,
  (a) $U_c \subseteq U_c^{\max(c)+1}$; and
  (b) For all $i < \omega$, $U_c \setminus m(c, 2i) = U_c \setminus m(c, 2i+1) \subseteq U_c^{m(c,2i)}$.

We now begin the construction of the meshing functions $m(c, \cdot)$ and the sets U_c, proceeding by recursion on the well-ordering $(\hat{B}^-, \prec)$. Since $\emptyset$ is $\prec$-minimal in $\hat{B}^-$, we start by choosing $g_\emptyset$, $m(\emptyset, \cdot)$, and $Y_\emptyset$ as follows. Since $\mathcal{U}_\emptyset$ is a p-point, we may choose a $U_\emptyset^* \in \mathcal{U}_\emptyset$ such that $U_\emptyset^* \subseteq^* U_\emptyset^k$ for each k. (Recall the definition of U_c^k from equation (4.6).) Let $g_\emptyset : \omega \to \omega$ be a strictly increasing function such that for each k, $U_\emptyset^* \setminus g_\emptyset(k+1) \subseteq U_\emptyset^{g_\emptyset(k)}$, and $g_\emptyset(0) > 0$. If $\bigcup_{i\in\omega}[g_\emptyset(2i), g_\emptyset(2i+1)) \in \mathcal{U}_\emptyset$, then define $m(\emptyset, k) = g_\emptyset(k+1)$; otherwise, $\bigcup_{i\in\omega}[g_\emptyset(2i+1), g_\emptyset(2i+2)) \in \mathcal{U}_\emptyset$, and we define $m(\emptyset, k) = g_\emptyset(k)$. Let $Y_\emptyset = \bigcup_{i\in\omega}[m(\emptyset, 2i+1), m(\emptyset, 2i+2))$ and define

$$U_\emptyset = U_\emptyset^0 \cap U_\emptyset^* \cap Y_\emptyset. \tag{4.7}$$

Note that for each k, $U_\emptyset \setminus m(\emptyset, k+1) \subseteq U_\emptyset^{m(\emptyset,k)}$.

Now suppose $c \in \hat{B}^-$ and for all $b \prec c$ in $\hat{B}^-$, g_b and $m(b, \cdot)$ have been defined. Since $\mathcal{U}_c$ is a p-point, there is a $U_c^* \in \mathcal{U}_c^*$ for which $U_c^* \subseteq^* U_c^k$, for all $k > \max(c)$. Let a denote the immediate $\prec$-predecessor of c in $\hat{B}^-$. Let $g_c : \omega \to \omega$ be a strictly increasing function such that $g_c(0) > \max(c)$ and

(1_{g_c}) For each $i < \omega$, $U_c^* \setminus g_c(i+1) \subseteq U_c^{g_c(i)}$; and
(2_{g_c}) For each $j < \omega$, there is an i such that $g_c(j) = m(a, 2i)$.

Let Y_c denote the one of the two sets $\bigcup_{i\in\omega}[g_c(2i+1), g_c(2i+2))$ or $\bigcup_{i\in\omega}[g_c(2i+2), g_c(2i+3))$ which is in $\mathcal{U}_c$. In the first case define $m(c, i) = g_c(i)$; in the second case define $m(c, i) = g_c(i+1)$. Then

$$Y_c = \bigcup_{i<\omega}[m(c, 2i+1), m(c, 2i+2)) \tag{4.8}$$

and is in $\mathcal{U}_c$. Let

$$U_c = U_c^{\max(c)+1} \cap U_c^* \cap Y_c. \tag{4.9}$$

This concludes the recursive definition.

We check that (†) holds. Let $c \in \hat{B}^-$ and let $a_0 \prec \cdots \prec a_l \prec c$ be the enumeration of all $\prec$-predecessors of c in $\hat{B}^-$. Let $j < \omega$ be given. Either $m(c, 2j) = g_c(2j)$ or $m(c, 2j) = g_c(2j+1)$. By (2_{g_c}), $g_c(2j) = m(a_l, 2i)$ for some i, and $g_c(2j+1) = m(a_l, 2i)$ for some i. Thus, there is an i such that $m(a_l, 2i) = m(c, 2j)$. Let i_l denote this i. Likewise, either $m(a_l, 2i_l) = g_{a_l}(2i_a)$ or $g_{a_l}(2i_a+1)$. By $(2_{g_{a_l}})$, $g_{a_l}(2i_a) = m(a_{l-1}, 2i)$ for some i, and $g_{a_l}(2i_a+1) = m(a_{l-1}, 2i)$ for some i. Let i_{l-1} denote the i such that $m(a_{l-1}, 2i_{l-1}) = m(a_l, 2i_l)$. Continuing in this manner, we obtain numbers i_k, $k \le l$, such that

$$m(c, 2j) = m(a_l, 2i_l) = m(a_{l-1}, 2i_{l-1}) = \cdots = m(a_0, 2i_0). \tag{4.10}$$

Hence, (†) holds.

Let T^* be the tree in $\mathfrak{T}$ defined by declaring for each $c \in \hat{B}^- \cap T^*$, $U_c(T^*) = U_c$. If the reader is not satisfied with this top-down construction (which *is* precise as $\emptyset$ is in every member of $\mathfrak{T}$ and this completely determines the rest of T^*), we point out that T^* can also be seen as being constructed level by level as follows. Let $\emptyset \in T^*$, and for each $l \in U_\emptyset$, put $\{l\}$ in T^*, so that the first level of T^* is exactly $\{\{l\} : l \in U_\emptyset\}$. Suppose we have constructed the tree T^* up to level k, meaning that we know exactly what $T^* \cap \hat{B} \cap [\omega]^{\le k}$ is. For each $c \in \hat{B}^- \cap T^* \cap [\omega]^k$, let the immediate successors of c in T^* be exactly the set U_c; in other words, for each $l > \max(c)$, put $c \cup \{l\} \in T^*$ if and only if $l \in U_c$. Recalling that $\max(c) < g_c(0) \le m(c, 0)$ and $U_c \subseteq Y_c = Y_c \setminus m(c, 0)$, we see that each element of U_c is strictly greater than $\max(c)$. Hence, by constructing T^* in this manner, we obtain a member of $\mathfrak{T}$ such that for each $c \in T^* \cap \hat{B}^-$, $U_c(T^*)$ is exactly U_c.

We now check that (‡) holds. Let $c \in T^* \cap \hat{B}^-$ be given. By (4.9), $U_c \subseteq U_c^{\max(c)+1}$, so (‡) (a) holds. By equation (4.8), we see that $Y_c \cap [m(c, 2i), m(c, 2i+1)) = \emptyset$ for each i. Thus,

$$U_c \cap [m(c, 2i), m(c, 2i+1)) = \emptyset, \tag{4.11}$$

since $U_c \subseteq Y_c$ by (4.9). Recall that U_c^* diagonalizes the collection of sets U_c^k for all $k > \max(c)$, and the function g_c was chosen to witness this diagonalization so that (1_{g_c}) holds. Either $m(c, i) = g_c(i)$ and $m(c, i+1) = g_c(i+1)$, or else $m(c, i) = g_c(i+1)$ and $m(c, i+1) = g_c(i+2)$. In either case, (1_{g_c}) implies that $U_c^* \setminus m(c, i+1) \subseteq U_c^{m(c,i)}$. Thus

$$U_c \setminus m(c, i+1) \subseteq U_c^{m(c,i)}, \tag{4.12}$$

since $U_c \subseteq U_c^*$ by (4.9). (‡) (b) follows from (4.11) and (4.12). This finishes Stage 2 of our construction.

We point out that the constructions in Stages 1 and 2 were handled directly for any flat-top front, without any reference to its rank. Stage 3, however, proceeds by induction on the rank of the flat-top front. Though by Theorem 1.3, $B = [\omega]^1$ could be used as the basis for the induction scheme for Theorem 4.4, it neither informs intuition nor does it prime the reader for the more complex construction needed for flat-top fronts in general. Thus, we shall first finish the proof of Theorem 4.4 for the case $B = [\omega]^2$. This includes Stage 3 and the construction of a $\tilde{T} \in \mathfrak{T}$ witnessing that the function f is basic for the case of $B = [\omega]^2$. After that, we shall finish the proof of Theorem 4.4 for flat-top fronts in general, by induction on rank.

Completion of the proof of Theorem 4.4 for $B = [\omega]^2$:

Stage 3 for $B = [\omega]^2$. We will show there is a strictly increasing sequence $(n_k)_{k<\omega}$ and a subtree $\tilde{T} \subseteq T^*$ in $\mathfrak{T}$ so that the following holds:

(4.13)
$$\text{For all } k < \omega \text{ and } c \in \tilde{T} \cap [n_k]^{\le 1}, \text{ there is an } r_c \text{ such that } m(c, 2r_c) = n_k.$$

For the case of $B = [\omega]^2$, we will obtain $\tilde{T}$ by thinning the first level of T^* to a set $Z_\emptyset \in \mathcal{U}_\emptyset$ and then taking all extensions of this set into T^*. The set $Z_\emptyset$ will be chosen so that for each $c \in [n_k]^{\le 1} \cap \tilde{T}$, the interval $[m(c, 2r_c - 1), m(c, 2r_c))$ has empty intersection with $Z_\emptyset$. Toward finding such a sequence $(n_k)_{k<\omega}$ and such a $Z_\emptyset$, we do the following construction.

Define a strictly increasing function $h : \omega \to \omega$ as follows. Let $h(0) = m(\emptyset, 0)$. Note that $h(0) > 0$, since $m(\emptyset, 0) > 0$. Given $h(k)$, let $l_k = h(k) - 1$. It follows from (†) that there are infinitely many integers q with the property that for each $c \in [h(k)]^{\le 1}$, there is an r_c such that $m(c, 2r_c) = m(\{l_k\}, 2q)$. Thus, for each $c \in [h(k)]^{\le 1}$, we may choose some r'_c such that each $m(c, 2r'_c) > h(k)$. Now choose q_k such that $m(\{l_k\}, 2q_k) > m(c, 2r'_c + 1)$, for each $c \in [h(k)]^{\le 1}$. Define $h(k + 1) = m(\{l_k\}, 2q_k)$. Since the $m(c, \cdot)$ functions are increasing, it follows from (†) that

($*_h$) For all $c \in [h(k)]^{\le 1}$, there is an $r_c \ge 1$ such that
$$h(k) < m(c, 2r_c - 1) < m(c, 2r_c) = h(k + 1).$$

Let $Z^0 = \bigcup_{i<\omega}[h(2i), h(2i+1))$ and $Z^1 = \bigcup_{i<\omega}[h(2i+1), h(2i+2))$. If Z^0 is in $\mathcal{U}_\emptyset$, then define $n(\emptyset, i) = h(i)$; if Z^1 is in $\mathcal{U}_\emptyset$, then define $n(\emptyset, i) = h(i+1)$. Define $Z_\emptyset := \bigcup_{i<\omega}[n(\emptyset, 2i), n(\emptyset, 2i+1))$. Then $Z_\emptyset$ is in $\mathcal{U}_\emptyset$. Let $\tilde{T}$ be obtained from T^* simply by thinning the first level of T^* through $Z_\emptyset$. That is, define level 1 of $\tilde{T}$ to be $\{\{l\} : l \in U_\emptyset(T^*) \cap Z_\emptyset\}$. Then for each $l \in Z_\emptyset$, for $l' > l$ define $\{l, l'\}$ to be in $\tilde{T}$ if and only if $\{l, l'\} \in T^*$. Define $n_k = n(\emptyset, 2k + 2)$.

Note that $n_k > k$, for every $k < \omega$, since $n(\emptyset, \cdot)$ is a strictly increasing function. This completes the construction of $\tilde{T}$ and $(n_k)_{k<\omega}$.

By the definition of n_k and $(*_h)$, it follows that for each k there is an $r_\emptyset$ such that $m(\emptyset, 2r_\emptyset) = n_k$. Further, note that

$$Z_\emptyset \setminus n_k = Z_\emptyset \setminus n(\emptyset, 2k+2) = Z_\emptyset \setminus n(\emptyset, 2k+1). \tag{4.14}$$

Thus, if $l \in U_\emptyset(\tilde{T}) \cap n_k$, then $l < n(\emptyset, 2k+1)$. If $Z_\emptyset = Z^0$, then $n(\emptyset, 2k+1)$ is equal to $h(2k+1)$ and $n_k = n(\emptyset, 2k+2) = h(2k+2)$. By $(*_h)$, there is some i such that $m(\{l\}, 2i) = h(2k+2)$, which is exactly n_k. Similarly, if $Z_\emptyset = Z^1$, then $n(\emptyset, 2k+1)$ is equal to $h(2k+2)$ and $n_k = n(\emptyset, 2k+2) = h(2k+3)$. By $(*_h)$, there is some i such that $m(\{l\}, 2i) = h(2k+3)$, which is the same as n_k. Thus, (4.13) holds. This finishes Stage 3 of the construction for the case of $B = [\omega]^2$.

Now we check that $(\circledast)$ holds for the case of $B = [\omega]^2$.

Claim 2. Let $k < \omega$ and $i < p_{n_k}$ be given, and suppose that $A_i \subseteq \tilde{T}$. Then $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq R_i^{n_k}$.

Proof. Let Q denote $\tilde{T} \cap \hat{B}_i^{n_k}$. Since $A_i \subseteq \tilde{T}$, we see that $Q \restriction n_k = A_i$ which equals $R_i^{n_k} \restriction n_k$. Thus, to prove the claim it is enough to show that for each $c \in Q \cap [\omega]^{\le 1}$, $U_c(Q) \setminus n_k \subseteq U_c(R_i^{n_k})$. Since $Q \subseteq \tilde{T} \subseteq T^*$, we see that for all $c \in Q \cap [\omega]^{\le 1}$, $U_c(Q) \setminus n_k \subseteq U_c(\tilde{T}) \setminus n_k \subseteq U_c(T^*) \setminus n_k$. We have two cases for c.

Case 1: $c \in Q \cap [n_k]^{\le 1}$. Then by (4.13), there is an r_c such that $m(c, 2r_c) = n_k$. By (‡) (b), we have that $U_c(T^*) \setminus n_k = U_c(T^*) \setminus m(c, 2r_c+1) \subseteq U_c^{m(c,2r_c)}$, which is exactly $U_c^{n_k}$. Note that $c \in Q \cap [n_k]^{\le 1}$ implies $c \in A_i$, which is a subset of $R_i^{n_k}$. Thus, $c \in R_i^{n_k}$ so, recalling the definition from (4.5), $S_c^{n_k} \subseteq R_i^{n_k}$. Therefore, recalling the definition from (4.6), $U_c^{n_k} = U_c(S_c^{n_k}) \subseteq U_c(R_i^{n_k})$. Hence,

$$U_c(Q) \setminus n_k \subseteq U_c(T^*) \setminus n_k \subseteq U_c^{n_k} \subseteq U_c(R_i^{n_k}). \tag{4.15}$$

Case 2: $c = \{l\} \in Q$ and $l \ge n_k$. Then $U_c(T^*) \setminus n_k = U_c(T^*)$. By (‡) (a), $U_c(T^*) \subseteq U_c^{l+1}$, which by definition is exactly $U_c(S_c^{l+1})$. Since $l \in U_\emptyset(Q) \setminus n_k$, which by Case 1 is contained in $U_\emptyset(R_i^{n_k})$, we have that $c \in R_i^{n_k}$. Therefore, $U_c^{l+1} \subseteq R_i^{n_k}$. Hence, $U_c(Q) \subseteq U_c(T^*) \subseteq U_c(R_i^{n_k})$.

By Cases 1 and 2, $Q \subseteq R_i^{n_k}$; hence Claim 2 holds. □

Now suppose $T \in \mathfrak{T} \restriction \tilde{T}$, let $k < \omega$ be given, and let $i < p_{n_k}$ be such that $T \restriction n_k = A_i$. Again, let Q denote $\tilde{T} \cap \hat{B}_i^{n_k}$. Then $T \subseteq Q$; so by $(*)_{n_k}$, for each $j \le k$, $j \in f([T]) \iff j \in f([R_i^{n_k}])$. Likewise, $Q \restriction n_k = A_i$: Since A_i equals $T \restriction n_k$, then it must be the case that $A_i \subseteq Q$; on the other hand,

$\hat{B}_i^{n_k} \restriction n_k$ equals A_i, so $Q \restriction n_k$ must be a subset of A_i. Thus, by $(*)_{n_k}$, for each $j \le k$, $j \in f([R_i^{n_k}]) \Longleftrightarrow k \in f([Q])$. Therefore, $(\circledast)$ holds.

As was shown in Claim 1, $(\circledast)$ implies that f is basic on $\mathfrak{T} \restriction \tilde{T}$. This completes the proof of Theorem 4.4 for the case $B = [\omega]^2$. $\square$

Now we prove Theorem 4.4 for all flat-top fronts. Stage 3 is more intricate for general flat-top fronts than when B is simply $[\omega]^2$. It involves the following lemma proved by induction on the rank of the front.

Stage 3. Given a front B, we let $\hat{B}^{--}$ denote the set of c in $\hat{B}^-$ which are not $\sqsubseteq$-maximal in $\hat{B}^-$. That is, $\hat{B}^{--} = \{c \setminus \{\max(c)\} : c \in \hat{B}^-\}$. To get one's bearings, note that for $B = [\omega]^2$, $\hat{B}^{--} = \{\emptyset\}$. In that case we only constructed one n-function, namely $n(\emptyset, \cdot)$. If $B = [\omega]^3$ then $\hat{B}^{--} = [\omega]^{\le 1}$; if $B = [\omega]^4$ then $\hat{B}_* = [\omega]^2$. To find good cut-off points n_k, we first construct functions $n(c, \cdot)$, $c \in \hat{B}^{--}$, as in the following lemma.

Lemma 4.5. *Let B be a flat-top front with lexicographic rank at least $\omega \cdot \omega$ (so $[\omega]^2 \subseteq \hat{B}$). Let $(\mathcal{U}_c : c \in \hat{B}^-)$ be a sequence of p-points. Suppose that for $c \in \hat{B}^-$, we have strictly increasing functions $m(c, \cdot) : \omega \to \omega$ satisfying (†) and (‡). Let $(j_i)_{i<\omega}$ be any strictly increasing sequence such that*

$$\forall c \in \hat{B}^- \restriction j_i \ \ \exists r_c < \omega \, (j_i < m(c, 2r_c - 1) < m(c, 2r_c) = j_{i+1}). \tag{4.16}$$

Then there are strictly increasing functions $n(c, \cdot) : \omega \to \omega$ and sets $Z_c \in \mathcal{U}_c$, $c \in \hat{B}^{--}$, which satisfy the following:

(i) For each $l < \omega$ and $c \in \hat{B}^{--}$, there is an $i < \omega$ such that $n(c, l) = j_i$.

(ii) If c is not $\sqsubseteq$-maximal in $\hat{B}^{--}$, then for each $q \ge 1$ and each $l \in U_c(T^*) \cap n(c, q-1)$, there is a q' such that

$$n(c, q-1) < n(c \cup \{l\}, 2q' - 1) < n(c \cup \{l\}, 2q') = n(c, q). \tag{4.17}$$

(iii) For each $c \in \hat{B}^{--}$, $Z_c := \bigcup_{i<\omega} [n(c, 2i), n(c, 2i+1)) \in \mathcal{U}_c$.

Proof. The proof is by induction on the rank of the flat-top front B. Stage 3 in the proof of Theorem 4.4 for $B = [\omega]^2$ gives the lemma for $[\omega]^2$: In this case, $\hat{B}^{--} = \{\emptyset\}$. Letting $j_i = h(i)$ provides a sequence satisfying (4.16) and for which the sequence $(n_k)_{k<\omega}$ satisfies this lemma. On the other hand, given a sequence $(j_i)_{i<\omega}$ satisfying (4.16), we may take $h(i) = j_i$ and then the rest of the previous construction satisfies the lemma.

Now suppose that B is a flat-top front with rank $\alpha > \omega \cdot \omega$ and that the lemma holds for all flat-top fronts of smaller rank. By Stages 1 and 2 applied to B, we obtain functions $m(c, \cdot)$, $c \in \hat{B}^-$, satisfying $(*)_k$ for all $k < \omega$, (†), and (‡). Using $(*)_k$, let $j_0 = m(\emptyset, 0)$, and choose $j_{i+1} > j_i$ such that for each $c \in \hat{B}^- \restriction j_i$, there is an r such that $j_i < m(c, 2r-1)$ and $m(c, 2r) = j_{i+1}$. Then the sequence $(j_i)_{i<\omega}$ satisfies (i).

For $l < \omega$, let $B_l = \{b \in B : \min(b) = l\}$. Note that B_l is isomorphic to $B_{\{l\}} := \{b \setminus \{l\} : b \in B_l\}$, which is a flat-top front on $\omega \setminus (l+1)$ of rank less than α. Thus, the induction hypothesis applies to each B_l. Define C_l to be $\{c \in \hat{B}^- : c \sqsupseteq \{l\}\}$. Use the induction hypothesis on B_0 with the sequence $(j_i)_{i<\omega}$ to find meshing functions $n(c, \cdot) : \omega \to \omega$ and Z_c, $c \in C_0 \cap \hat{B}^{--}$, which satisfy (i) - (iii). Define $j^1_i = n(\{0\}, 2i)$, for each $i < \omega$. In general, for $l \geq 1$, given the sequence $(j^l_i)_{i<\omega}$, use the induction hypothesis on B_l to find meshing functions $n(c, \cdot) : \omega \to \omega$ and Z_c, $c \in C_l \cap \hat{B}^{--}$, which satisfy (i) - (iii) with regard to $(j^l_i)_{i<\omega}$. Define

$$j^{l+1}_i = n(\{l\}, 2i), \text{ for each } i < \omega. \tag{4.18}$$

Continuing in this manner, we obtain for all $l < \omega$ functions $n(\{l\}, \cdot)$, sequences $(j^l_i)_{i<\omega}$, and $Z_{\{l\}} \in \mathcal{U}_{\{l\}}$ satisfying (i) - (iii) for the flat-top front B_l. Moreover, the functions mesh:

$$\forall l < l',\ \forall r',\ \exists r\ \ (n(c \cup \{l'\}, r') = n(c \cup \{l\}, 2r)). \tag{4.19}$$

This will be important in the construction of $n(\emptyset, \cdot)$.

Finally, we construct $n(\emptyset, \cdot) : \omega \to \omega$ to satisfy (i) - (iii) for B.

Claim 3. For each number $h < \omega$, there are infinitely many $i < \omega$ with the property that for each $l < h$, there is an $r_l \geq 1$ such that

$$h < n(\{l\}, 2r_l - 1) < n(\{l\}, 2r_l) = j^h_i. \tag{4.20}$$

Proof. Let $h < \omega$ and $r < \omega$ be given. By definition j^h_r is equal to $n(\{h-1\}, 2r)$, and this value $n(\{h-1\}, 2r)$ was chosen to equal $j^{h-1}_{i_1}$ for some $i_1 < \omega$. $j^{h-1}_{i_1}$ is by definition equal to $n(\{h-2\}, 2i_1)$, which was chosen to be $j^{h-2}_{i_2}$ for some i_2. Continuing in this manner, we see that there are integers $i_1, \dots, i_h$ such that

$$j^h_r = n(\{h-1\}, 2r) = j^{h-1}_{i_1} = n(\{h-2\}, 2i_1) = \cdots = j^1_{i_{h-1}} = n(\{0\}, 2i_{h-1}) = j_{i_h}. \tag{4.21}$$

Now for each $l < h$, take r'_l minimal such that $h < n(\{l\}, 2r'_l - 1)$. Let k be any integer such that j^h_k is greater than all $n(\{l\}, 2r'_l)$, $l < h$. Then choosing r_l, $l < h$, so that each $n(\{l\}, 2r_l) = j^h_k$ will automatically satisfy (4.20). $\square$

Define a strictly increasing function $h : \omega \to \omega$ as follows. Let $h(0) = n(\{0\}, 2)$. Given $h(i)$, by Claim 3, there is a p_i such that $j^{h(i)}_{p_i} > h(i)$, and for each $l < h(i)$, there is an r_l such that $h(i) < n(\{l\}, 2r_l - 1) < n(\{l\}, 2r_l) = j^{h(i)}_{p_i}$. Define $h(i+1) = j^{h(i)}_{p_i}$. If $\bigcup_{i<\omega}[h(2i), h_(2i+1)) \in \mathcal{U}_\emptyset$, then let $n(\emptyset, i) =$

$h(i+2)$. If $\bigcup_{i<\omega}[h(2i+1), h(2i+2)) \in \mathcal{U}_\emptyset$, then let $n(\emptyset, i) = h(i+1)$. Then $Z_\emptyset := \bigcup_{i<\omega}[n(\emptyset, 2i), n(\emptyset, 2i+1))$ is in $\mathcal{U}_\emptyset$. Note that (iii) is satisfied.

Then each $n(\emptyset, r)$ equals $h(r+1)$, which is j_i for some i, so (i) holds. Given $r \geq 1$, and letting k be the one of i or $i+1$ such that $n(\emptyset, r) = h(k+1)$, we see that for each $l < n(\emptyset, r-1)$, there is an r_l such that $n(\emptyset, r-1) = h_\emptyset(k) < n(\{l\}, 2r_l - 1) < n(\{l\}, 2r_l) = h(k+1) = n(\emptyset, r)$. Thus, (ii) holds. This completes the proof of the Lemma. $\square$

With Lemma 4.5, we are prepared to construct $\tilde{T}$. Define $\tilde{T}$ to be T^* thinned through the Z_c, $c \in \hat{B}^{--}$, from Lemma 4.5. That is, $\emptyset \in \tilde{T}$; $U_\emptyset(\tilde{T}) = U_\emptyset \cap Z_\emptyset$; and in general if $c \in \hat{B}^{--} \cap \tilde{T}$, then $U_c(\tilde{T}) = U_c$; and if $c \in (\hat{B}^- \setminus \hat{B}^{--}) \cap \tilde{T}$, then $U_c(\tilde{T}) = U_c(T^*)$. For each $c \in \hat{B}^- \cap \tilde{T}$, let $\tilde{U}_c$ denote $U_c(\tilde{T})$. For each $k < \omega$, define

$$n_k = n(\emptyset, 2k+2). \tag{4.22}$$

This finishes Stage 3 of the construction.

Finally, we check that ($\circledast$) holds. Toward this, we first show that for all $k < \omega$ and $i < p_{n_k}$, $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq R_i^{n_k}$. It will follow that for each $T \in \mathfrak{T} \restriction \tilde{T}$ with $T \restriction n_k = A_i$, we in fact have $T \subseteq R_i^{n_k}$. This along with $(*)_{n_k}$ for all $k < \omega$ will yield ($\circledast$).

Claim 4. Let $k < \omega$ and $i < p_{n_k}$ be given such that $A_i \subseteq \tilde{T}$. Then $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq R_i^{n_k}$.

Proof. Recall the definition of $\hat{B}_i^{n_k}$ from (4.2) and the definition of $R_i^{n_k}$ from Stage 1. Let Q denote $\tilde{T} \cap \hat{B}_i^{n_k}$. Since $Q \restriction n_k \subseteq A_i = R_i^{n_k} \restriction n_k$, it suffices to show that for each $c \in Q \cap \hat{B}^-$, $U_c(Q) \setminus n_k \subseteq U_c(R_i^{n_k})$. Since $Q \subseteq \tilde{T} \subseteq T^*$, it follows that for all $c \in Q \cap \hat{B}^-$, $U_c(Q) \setminus n_k \subseteq U_c(\tilde{T}) \setminus n_k \subseteq U_c(T^*) \setminus n_k$.

We handle the two cases, $c \in Q \cap \hat{B}^- \restriction n_k$ and $c \in (Q \cap \hat{B}^-) \setminus (\hat{B}^- \restriction n_k)$, in the following two subclaims.

Subclaim (i). For all $c \in Q \cap \hat{B}^- \restriction n_k$, we have $U_c(Q) \setminus n_k \subseteq U_c(R_i^{n_k})$.

Proof. The proof makes full use of the properties (i) - (iii) of Lemma 4.5. Let $c \in Q \cap \hat{B}^- \restriction n_k$. By (i), there is some $i^* < \omega$ such that $n_k = j_{i^*}$, since $n_k = n(\emptyset, 2k+2)$ which is equal to some j_{i^*}. Let $c = \{l_0, \dots, l_r\} \in Q \cap \hat{B}^- \restriction n_k$. For each $i \leq r+1$, let a_i denote $\{l_j : j < i\}$; in particular, $a_0 = \emptyset$ and $a_{r+1} = c$. Note that $a_r \in \hat{B}^{--}$.

We proceed by induction on $i \leq r$. Now $\{l_0\}$ is a member of $\tilde{T}$, and $U_\emptyset(\tilde{T}) \subseteq Z_\emptyset$, by our construction of $\tilde{T}$. By (iii), $Z_\emptyset \cap [n(\emptyset, 2k+1), n(\emptyset, 2k+2))$ and we defined n_k to equal $n(\emptyset, 2k+2)$. Thus, l_0 must be less than $n(\emptyset, 2k+$

1). In particular, l_0 is in $U_\emptyset(T^*) \cap n(\emptyset, 2k+1)$. Now $a_1 = \{l_0\} = \emptyset \cup \{l_0\}$, so by (ii) (letting r=2k+2), there is a $r_1 \geq 1$ (that is, an r_{a_1}) such that

$$n(\emptyset, 2k+1) < n(a_1, 2r_1 - 1) < n(a_1, 2r_1) = n(\emptyset, 2k+2) = n_k.$$

For the induction step, first suppose that $1 \leq i \leq q$ and we have found $r_i \geq 1$ such that $n(a_i, 2r_i) = n_k$. Now $a_{i+1} \in \tilde{T}$ and $U_{a_i}(\tilde{T}) \subseteq Z_{a_i}$. Since $a_i \in \hat{B}^{--}$, (iii) implies that $Z_{a_i} \cap [n(a_i, 2r_i - 1), n(a_i, 2r_i)) = \emptyset$. Since $l_i < n_k$ it follows that $l_i < n(a_i, 2r_i - 1)$. Thus, $\max(a_{i+1}) < n(a_i, 2r_i - 1)$.

Note that if $i < q$, then a_i is not maximal in $\hat{B}^{--}$. Thus, (ii) implies that there is a $r_{i+1} \geq 1$ such that

$$n(a_i, 2r_i - 1) < n(a_{i+1}, 2r_{i+1} - 1) < n(a_{i+1}, 2r_{i+1}) = n(a_i, 2r_i) = n_k.$$

Thus, by induction, we have that for all $1 \leq i \leq q$, there is a $r_i \geq 1$ such that $n(a_i, 2r_i) = n_k$. In particular, there is a r_q such that $n(a_q, 2r_q) = n_k$, which is equal to j_{i^*}. Since $a_q \in \hat{B}^{--}$, by (iii) we have that $Z_{a_q} \cap [n(a_q, 2r_q - 1), n(a_q, 2r_q)) = \emptyset$. Therefore, $l_q < n(a_q, 2r_q - 1) < n(a_q, 2r_q) = j_{i^*}$. Since $n(a_q, \cdot)$ is a strictly increasing function, with each of its values being equal to j_i for some i, it follows that $n(a_q, 2r_q - 1) \leq j_{i^*-1}$. Thus, $l_q < j_{i^*-1}$, and therefore $a \in \hat{B}^- \restriction j_{i^*-1}$. By our assumption (4.16), there is an r_c such that

$$j_{i^*-1} < m(c, 2r_c - 1) < m(c, 2r_c) = j_{i^*} = n_k.$$

By (‡)(b), $U_c(T^*) \setminus n_k \subseteq U_c(S_c^{n_k})$. Therefore, $U_c(\tilde{T}) \setminus n_k \subseteq U_c(S_c^{n_k})$. Now since $c \in A_i \subseteq R_i^{n_k}$, we have that $S_c^{n_k} \subseteq R_i^{n_k}$. Hence, $U_c(\tilde{T}) \setminus n_k \subseteq U_c(R_i^{n_k})$. Since $Q \subseteq \tilde{T}$, we have that $U_c(Q) \subseteq U_c(R_i^{n_k})$. □

Subclaim (ii). For all $c \in Q \cap \hat{B}^-$ such that $\max(c) \geq n_k$, we have $c \in R_i^{n_k}$ and $U_c(Q) \subseteq U_c(R_i^{n_k})$.

Proof. Let $c \in Q \cap \hat{B}^-$ such that $\max(c) \geq n_k$, and let l denote $\max(c)$. By (‡) (a), $U_c(T^*) \subseteq U_c(S_c^{l+1})$. The proof will proceed by induction on the cardinality of $c \setminus n_k$.

Suppose $|c \setminus n_k| = 1$. Let a denote $c \setminus \{l\}$. Then a is in $Q \cap \hat{B}^{--} \restriction n_k$ and $l \in U_a(Q) \setminus n_k$. Since $Q \restriction n_k \subseteq A_i \subseteq \tilde{T} \restriction n_k \subseteq T^* \restriction n_k$, it follows that a is a member of $R_i^{n_k}$. Since by Subclaim (i), $U_a(Q) \setminus n_k$ is contained in $U_a(R_i^{n_k})$, we have that $c \in R_i^{n_k}$. Further,

$$U_c(Q) \subseteq U_c(T^*) \subseteq U_c(S_c^{l+1}) \subseteq U_c(R_i^{n_k}). \tag{4.23}$$

since $l+1 > n_k$, $i < p_{n_k}$ and $c \in R_i^{n_k}$ imply that $S_c^{l+1} \subseteq R_i^{n_k}$, by the definition (4.5) of S_c^{l+1}. Thus, Subclaim (ii) holds for the basis of our induction scheme.

Now assume that for all $c \in Q \cap \hat{B}^-$ with $1 \leq |c \setminus n_k| \leq m$, Subclaim (ii) holds. Suppose $c \in Q \cap \hat{B}^-$ with $|c \setminus n_k| = m+1$. Letting $l = \max(c)$

and $a = c \setminus \{l\}$, the induction hypothesis applied to a yields that $a \in R_i^{n_k}$ and $U_a(Q) \subseteq U_a(R_i^{n_k})$. Thus, $c \in R_i^{n_k}$. Again, as in (4.23), we find that $U_c(Q) \subseteq U_c(R_i^{n_k})$, which finishes the proof of Subclaim (ii). $\square$

Since $Q \restriction n_k \subseteq R_i^{n_k}$ and since Subclaims (i) and (ii) imply that for all $c \in Q \cap \hat{B}^-$, we have that $c \in R_i^{n_k}$ and $U_c(Q) \subseteq U_c(R_i^{n_k})$, we conclude that $Q \subseteq R_i^{n_k}$. This finishes the proof of Claim 4. $\square$

To finish, we prove that ($\circledast$) holds. Let $T \in \mathfrak{T} \restriction \tilde{T}$, $k < \omega$, and suppose $i < p_{n_k}$ is the integer such that $T \restriction n_k = A_i$. Letting Q denote $\tilde{T} \cap \hat{B}_i^{n_k}$, we see that $T \subseteq Q$. By Claim 4, $Q \subseteq R_i^{n_k}$; so for all $j \leq k$,

$$j \in f([T]) \Longleftrightarrow j \in f([R_i^{n_k}]) \tag{4.24}$$

by $(*)_{n_k}$. Since $T \subseteq Q$, we have $A_i = T \restriction n_k \subseteq Q \restriction n_k$. On the other hand, $A \subseteq \hat{B}_i^{n_k}$, so $Q \restriction n_k \subseteq \hat{B}_i^{n_k} \restriction n_k = A_i$. Thus, $Q \restriction n_k = A_i$, so for all $j \leq k$,

$$j \in f([Q]) \Longleftrightarrow j \in f([R_i^{n_k}]) \tag{4.25}$$

by $(*)_{n_k}$. Equations (4.24) and (4.25) complete the proof of ($\circledast$). By Claim 1, f is basic on $\mathfrak{T} \restriction \tilde{T}$. The concludes the proof of the theorem. $\square$

Given a set $\mathcal{C} \subseteq 2^B$, let $\hat{\mathcal{C}}$ denote $\{\hat{X} : X \in \mathcal{X}\}$, a subset of $2^{\hat{B}}$. Letting C denote $\{\hat{X} \restriction k_m : X \in \mathcal{C}$ and $m < \omega\}$, we point out that any finitary function $\hat{f} : \{X \restriction k_m : X \in \mathcal{C},\ m < \omega\} \to 2^{<\omega}$ determines functions $f' : \mathcal{C} \to 2^\omega$ and $f^* : \hat{\mathcal{C}} \to 2^\omega$ by setting $f'(X) = f^*(\hat{X}) = \bigcup_{m<\omega} \hat{f}(\hat{X} \restriction k_m)$. In particular, $f'(X) = f^*(\hat{X})$ for each $X \in \mathcal{C}$.

Fact 4.6. *Suppose $\mathcal{C}$ is a subset of 2^B and $C = \{\hat{X} \restriction k_m : X \in \mathcal{C},\ m < \omega\}$. If $\hat{f} : C \to 2^{<\omega}$ is a basic map, then f^* is continuous on $\hat{\mathcal{C}}$ as a subspace of $2^{\hat{B}}$.*

In particular, in the setting of Theorem 4.4, the map $f^* : \mathfrak{T} \restriction \tilde{T} \to \mathcal{V}$ defined by $f^*(T) = \bigcup_{m<\omega} \hat{f}(T \restriction k_m)$ is a continuous map on its domain $\mathfrak{T} \restriction \tilde{T}$, which is a compact subspace of $2^{\hat{B}}$. This map f^* is equivalent to f in the following sense: For each $T \in \mathfrak{T} \restriction \tilde{T}$, $f^*(T) = f([T])$. In contrast, the map $f : \mathcal{C} \to \mathcal{V}$ is in general not continuous. However, $f \restriction \mathcal{C}$ is still represented by a monotone finitary map: Defining $\hat{g} : \bigcup_{m<\omega} 2^{B\restriction k_m} \to 2^{<\omega}$ by $\hat{g}(s) = \hat{f}(\hat{s})$, (where $\hat{s}$ denotes $\{a \in \hat{B} : \exists b \in s\, (a \sqsubseteq b)\}$), we see that $\hat{g}$ is a monotone function and that for each $X \in \mathcal{C}$, $f(X) = \bigcup_{m<\omega} \hat{g}(X \restriction k_m)$. The existence of such a finitary map which generates the original map f is the focus of the next section.

5. Further connections between Tukey, Rudin-Blass, and Rudin-Keisler reductions

In Lemma 9 of [13], Raghavan distilled properties of cofinal maps which, when satisfied, yield that Tukey reducibility above a q-point implies Rudin-Blass reducibility. He then showed that continuous cofinal maps satisfy these properties, thus yielding his Theorem 10 in [13], (see Theorem 1.6 in Section 1). The proof of the following theorem follows the general structure of Raghavan's proofs. The key differences are that we start with a weaker assumption, basic maps on Fubini iterates of p-points, and obtain a finite-to-one map on the base tree $\hat{B}$ for the ultrafilter rather than the base B itself. While the following theorem may be of interest in itself, we apply it to prove Theorems 5.3 and 5.4 showing that for finite Fubini iterates of p-points, and for generic ultrafilters forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, Tukey reducibility above a q-point is equivalent to Rudin-Keisler reducibility.

Theorem 5.1. *Suppose $\mathcal{U}$ is a Fubini iterate of p-points and $\mathcal{V}$ is a q-point. If $\mathcal{V} \le_T \mathcal{U}$, then there is a finite-to-one function $\tau : \tilde{T} \to \omega$, for some $\tilde{T} \in \mathfrak{T}$, such that $\{\tau(T) : T \in \mathfrak{T} \restriction \tilde{T}\} \subseteq \mathcal{V}$.*

Proof. Let B be the flat-top front which is a base for $\mathcal{U}$, and as usual, let $\mathfrak{T}$ denote the set of all $\vec{\mathcal{U}}$-trees on $\hat{B}$. We begin by establishing some useful notation: Given $m < n$ and $T \in \mathfrak{T}$, let $T \restriction [m, n)$ denote $T \restriction n \setminus T \restriction m$. Thus $T \restriction [m, n)$ is the set of all $a \in T$ such that $m \le \max(a) < n$.

Let $f : \mathcal{U} \to \mathcal{V}$ be a monotone cofinal map. Let $\tilde{T} \in \mathfrak{T}$ be given by Theorem 4.4, so that $f \restriction (\mathfrak{T} \restriction \tilde{T})$ is generated by some basic map $\hat{f} : \bigcup_{m<\omega} 2^{\tilde{T} \restriction k_m} \to 2^{<\omega}$. Let $\psi : \mathcal{P}(\tilde{T}) \to \mathcal{P}(\omega)$ be defined by

$$\psi(s) = \{k \in \omega : \forall T \in \mathfrak{T} \restriction \tilde{T}\, [s \subseteq t \to k \in f([T])]\}, \tag{5.1}$$

for each $s \subseteq \tilde{T}$. Note that ψ is monotone and further that for each $T \in \mathfrak{T} \restriction \tilde{T}$ and each $m < \omega$, $\hat{f}(T \restriction k_m) = \psi(T \restriction k_m) \cap (m+1)$. By the same argument as in Lemma 8 in [13], we may assume that for each finite $s \subseteq \tilde{T}$, $\psi(s)$ is finite.

Note that for all $m, j < \omega$, if $j \notin \psi(\tilde{T} \restriction m)$, then there is a $T \in \mathfrak{T} \restriction \tilde{T}$ such that $T \sqsupset \tilde{T} \restriction m$ and $j \notin f(T)$. Therefore, $j \notin \hat{f}(T \restriction k_j)$. It follows that $j \notin f(S)$ for all $S \sqsupseteq T \restriction k_j$ and hence also $j \notin \psi(S)$ for all $S \sqsupseteq T \restriction k_j$. Without loss of generality, assume that $T \restriction [m, k_j) = \emptyset$. (This means that each $a \in T$ has $\max(a) \notin [m, k_j)$.) Now if t is a finite $\sqsubset$-closed subset of $\tilde{T}$ and $t \restriction [m, k_j) = \emptyset$, then $\tilde{T} \restriction m \cup t$ can be extended to an $S \in \mathfrak{T} \restriction \tilde{T}$ such that $\tilde{T} \restriction m \cup t \subseteq S$, $S \restriction [m, k_j) = \emptyset$, and $S \sqsupset T \restriction k_j$. Then $j \notin f(S)$, so $j \notin \psi(\tilde{T} \restriction m \cup t)$.

Define $g : \omega \to \omega$ by $g(0) = 0$; and given $g(n)$, choose $g(n+1) > g(n)$ so that

$$g(n+1) > \max\{k_{g(n)}, \max(\psi(\tilde{T} \restriction k_{g(n)}))\}. \tag{5.2}$$

Since $\mathcal{V}$ is a q-point, there is a $V_0 \in \mathcal{V}$ such that for each $n < \omega$, $|V_0 \cap [g(n), g(n+1))| = 1$. We may, without loss of generality, assume that $V_1 = \bigcup_{n\in\omega}[g(2n), g(2n+1))$ is in $\mathcal{V}$, and let $V = V_0 \cap V_1$. Enumerate V as $\{v_i : i < \omega\}$. Notice that for each $i < \omega$,

$$g(2i) \leq v_i < g(2i+1). \tag{5.3}$$

Without loss of generality, assume that $v_0 > 0$. Then $v_0 \notin \psi(\emptyset)$, since assuming $\mathcal{V}$ is nonprincipal, $\psi(\emptyset)$ must be empty.

Our construction ensures the following properties: For all $i < \omega$,

(1) $g(i+1) > \max(k_{g(i)}, \max(\psi(\tilde{T} \restriction k_{g(i)})))$;
(2) $g(2i) \leq v_i < g(2i+1)$;
(3) $k_{v_i} < g(2i+2)$;
(4) $\psi(\tilde{T} \restriction k_{v_i}) \subseteq g(2i+2)$.

We will now define a strictly increasing function $h : \omega \to \omega$ so that the following hold:

(a) $h(i) < k_{v_i}$;
(b) $v_i \notin \psi(\tilde{T} \restriction h(i))$;
(c) For each finite, $\sqsubset$-closed set $s \subseteq \tilde{T}$, if $v_i \in s$ then $s \restriction [h(i), h(i+1)) \neq \emptyset$.

Define $h(0) = 0$. Then (a) - (c) trivially hold. Suppose $h(i)$ has been defined so that (a) - (c) hold. Define $h(i+1) = k_{v_i}$. Then $h(i+1) > h(i)$, since $k_{v_i} > h(i)$ by (a) of the induction hypothesis. (a) holds, since $k_{v_i} < k_{v_{i+1}}$.

To see that (b) holds, note that

$$\psi(\tilde{T} \restriction h(i+1)) = \psi(\tilde{T} \restriction k_{v_i}) \subseteq \psi(\tilde{T} \restriction k_{g(2i+1)}) \subseteq g(2i+1) \leq v_{i+1}, \tag{5.4}$$

where the inclusions hold by (3) and (1), and the inequality holds by (2). Thus, $v_{i+1} \notin \psi(\tilde{T} \restriction h(i+1))$.

To check (c), fix a finite $\sqsubset$-closed set $s \subseteq \tilde{T}$ such that $s \restriction [h(i), h(i+1)) = \emptyset$; that is, for all $a \in s$, $s \cap [h(i), h(i+1)) = \emptyset$. Let $t = s \setminus \tilde{T} \restriction h(i+1)$. We claim that $v_i \notin \psi(\tilde{T} \restriction h(i) \cup t)$. By (b), $v_i \notin \psi(\tilde{T} \restriction h(i))$. Therefore, $v_i \notin \hat{f}(\tilde{T} \restriction h(i))$. Now $(\tilde{T} \restriction h(i) \cup t) \restriction h(i+1) = \tilde{T} \restriction h(i)$, since $t \restriction [h(i), h(i+1)) = \emptyset$. So $v_i \notin \hat{f}(\tilde{T} \restriction h(i) \cup t) \restriction h(i+1)$. Therefore, for each $S \in \mathfrak{T} \restriction \tilde{T}$ such that $S \restriction h(i+1) = (\tilde{T} \restriction h(i) \cup t) \restriction h(i+1)$, which we point out is the same as $\tilde{T} \restriction h(i)$, we have $v_i \notin f([S])$. Thus, if $S \sqsupseteq \tilde{T} \restriction h(i) \cup t$ and satisfies $S \restriction [h(i), h(i+1)) = \emptyset$, then $v_i \notin f([S]$, since this gets decided by height k_{v_i} which equals $h(i+1)$. Therefore, $v_i \notin \psi(\tilde{T} \upharpoonright h(i) \cup t)$, which proves (c).

Now we define a function $\tau : \tilde{T} \to \omega$ as follows: For $i < \omega$ and $a \in \tilde{T}$, if $\max(a) \in [h(i), h(i+1))$, then $\tau(a) = v_i$.

Claim. For each $T \in \mathfrak{T} \upharpoonright \tilde{T}$, $\tau(T) \in \mathcal{V}$.

Proof. Suppose not. Then there is a $T \in \mathfrak{T} \upharpoonright \tilde{T}$ such that $\tau(T) \notin \mathcal{V}$; so $\tau(T) \in \mathcal{V}^*$. Then there is an $S \in \mathfrak{T} \upharpoonright T$ such that $f([S]) \subseteq \omega \setminus \tau(T) \cap V \in \mathcal{V}$. Let j be least such that $\hat{f}(S \upharpoonright k_j) \neq \emptyset$ and let $s = S \upharpoonright k_j$. Then $\emptyset \neq \hat{f}(s) \subseteq \psi(s)$. Fix some $v_i \in \psi(s)$. Then $v_i \notin \tau(T)$ since $v_i \in \psi(s) \subseteq f([S]) \subseteq \omega \setminus \tau(T)$. However, $v_i \in \psi(s)$ implies $s \cap [h(i), h(i+1)) \neq \emptyset$, by (c). For each $a \in s \cap [h(i), h(i+1))$, $\tau(a) = v_i$. Since $s \subseteq T$, we have $v_i \in \tau(T)$, a contradiction. Thus, $\tau(T) \in \mathcal{V}$. □

Therefore, τ is a finite-to-one map from $\tilde{T}$ into ω, and the set of τ-images of members of $\mathfrak{T} \upharpoonright \tilde{T}$ generate a filter contained inside $\mathcal{V}$. □

The previous theorem does not necessarily imply that the τ-image of $\mathfrak{T} \upharpoonright \tilde{T}$ generates $\mathcal{V}$. However, under certain conditions, it does. In the case that $\mathfrak{T} \upharpoonright \tilde{T}$ generates an ultrafilter on base set the *tree* $\tilde{T}$, as is the case when all the p-points are the same selective ultrafilter, then the τ-image of $\mathcal{U}$ *is* $\mathcal{V}$.

Corollary 5.2. *If $\mathcal{U}$ is a Fubini power of some selective ultrafilter $\mathcal{V}$, then there is a finite-to-one map τ from the flat-top front base B for $\mathcal{U}$ into ω such that $\{\tau(T) : T \in \mathfrak{T}\}$ generates $\mathcal{V}$.*

It is useful to point out the connection and contrast between this corollary and the following previously known results. Every Fubini power of some selective ultrafilter is Tukey equivalent to that selective ultrafilter (Corollary 37 in [8]). Thus, the q-point must in the above scenario be the selective ultrafilter. On the other hand, the only ultrafilters Tukey reducible to a selective ultrafilter are those ultrafilters isomorphic to some Fubini power of the selective ultrafilter (Theorem 24 of Todorcevic in [13]). Thus, the collection of ultrafilters Tukey equivalent to a given selective ultrafilter is exactly the collection of Fubini powers of that selective ultrafilter; and Tukey reducibility from a Fubini power of a selective ultrafilter to that same selective ultrafilter already implies Rudin-Keisler reducibility.

In Corollary 56 of [13], Raghavan showed that if $\mathcal{U}$ is some Fubini iterate of p-points and $\mathcal{V}$ is selective, then $\mathcal{V} \leq_T \mathcal{U}$ implies $\mathcal{V} \leq_{RK} \mathcal{U}$. We now generalize this to q-points, though at the cost of assuming $\mathcal{U}$ is only a finite Fubini iterate of p-points.

Theorem 5.3. *Suppose $\mathcal{U}$ is a finite iterate of Fubini products of p-points. If $\mathcal{V}$ is a q-point and $\mathcal{V} \le_T \mathcal{U}$, then $\mathcal{V} \le_{RK} \mathcal{U}$.*

Proof. Let k denote the length of the Fubini iteration, so $[\omega]^k$ is the flat-top front base for $\mathcal{U}$. Let τ be the finite-to-one map from Theorem 5.1, and without loss of generality, assume τ is defined on all of $[\omega]^k$. For each $T \in \mathfrak{T}$, we notice that $\bigcup_{1\le l\le k} \tau(T \cap [\omega]^l) = \tau(T) \in \mathcal{V}$. For each $T \in \mathfrak{T}$, let $L(T) = \{1 \le l \le k : \tau(T \cap [\omega]^l) \in \mathcal{V}\}$. Then there is a $T \in \mathfrak{T}$ such that for all $S \in \mathfrak{T} \restriction T$, $L(S) = L(T)$. Let $l = \max(L(T))$.

Now $\{S \cap [\omega]^l : S \in \mathfrak{T} \restriction T\}$ generates an ultrafilter on base set $[\omega]^l \cap T$; further, for each $S \in \mathfrak{T} \restriction T$, $\tau(S \cap [\omega]^l)$ is a member of $\mathcal{V}$ (since $l \in L(S)$). Thus, $\{\tau(S \cap [\omega]^l) : S \in \mathfrak{T} \restriction T\}$ generates an ultrafilter, and each of these τ-images is in $\mathcal{V}$. It follows that $\{\tau(S \cap [\omega]^l) : S \in \mathfrak{T} \restriction T\}$ generates $\mathcal{V}$. If $l = k$, we are done, and in fact we have a Rudin-Blass map from $\mathcal{U}$ to $\mathcal{V}$. If $l < k$, then define $\sigma : [\omega]^k \to \omega$ by $\sigma(a) = \tau(\pi_l(a))$. Then σ is a Rudin-Keisler map from $\mathcal{U}$ into $\mathcal{V}$. $\square$

We point out that the *basic* maps on the generic ultrafilters $\mathcal{G}_k$ forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, $2 \le k < \omega$, in [5] have exactly the same properties as basic maps on $[\omega]^k$ in this paper. (See Definition 37 and Theorem 38 in [5].) Hence, Theorem 5.1 also applies to these ultrafilters. Since for each $1 \le l \le k$, the projection of $\mathcal{G}_k$ to $[\omega]^l$ yields the generic ultrafilter $\mathcal{G}_l$, the same proof as in Theorem 5.3 yields the following theorem.

Theorem 5.4. *Suppose $\mathcal{G}_k$ is a generic ultrafilter forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, for any $2 \le k < \omega$. If $\mathcal{V}$ is a q-point and $\mathcal{V} \le_T \mathcal{G}_k$, then $\mathcal{V} \le_{RK} \mathcal{G}_k$.*

Remark 5.5. We cannot in general weaken the requirement of q-point to rapid in Theorem 5.1. In [10], it is shown that there are Tukey equivalent rapid p-points, and hence a Fubini iterate of such p-points Tukey equivalent to a rapid p-point, which are Rudin-Keisler incomparable.

6. Ultrafilters Tukey reducible to Fubini iterates of p-points have finitely generated Tukey reductions

In this section, we prove the analogue of Theorem 2.6 for the class of all ultrafilters which are Tukey reducible to some Fubini iterate of p-points. Namely, in Theorem 6.3, we prove that every ultrafilter Tukey reducible to some Fubini iterate of p-points has finitely generated Tukey reductions (see Definition 6.2 below). This means that every monotone cofinal map on such an ultrafilter is approximated by some monotone finitary map on a filter base. This sharpens a result of Raghavan (Lemma 16 in [13]) by obtaining finitary maps which generate the original cofinal map on some filter base rather than some possibly different cofinal map. Also, the class on which we obtain finitely generated Tukey reductions is closed under Tukey reduction, whereas the class where his result applies (basically generated ultrafilters) is not known to be closed under Tukey reduction. Theorem 6.3 allows us to extend Theorem 17 of Raghavan in [13] relating Tukey reduction to Rudin-Keisler reduction for basically generated ultrafilters to the class of all ultrafilters Tukey reducible to some Fubini iterate of p-points (see Theorem 6.4 and the discussion preceding it).

The next lemma is the analogue of Lemma 2.4 for the space $2^{\hat{B}}$ in place of 2^ω. As the proof is almost verbatim by making the obvious changes, we omit it.

Lemma 6.1 (Extension lemma for flat-top fronts)**.** *Suppose $\mathcal{U}$ is a non-principal ultrafilter with base set a flat-top front B. Suppose $f:\mathcal{U}\to\mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $\mathcal{C}\subseteq\mathcal{U}$ such that $f\restriction\mathcal{C}$ is basic. Then there is a continuous, monotone $\tilde{f}:2^{\hat{B}}\to 2^\omega$ such that*

(1) $\tilde{f}$ is basic: There is an increasing sequence $(k_m)_{m<\omega}$ and a monotone, level and end-extension preserving function $\hat{f}:\bigcup_{m<\omega}2^{\hat{B}\restriction k_m}\to 2^{<\omega}$ such that for each $\hat{Z}\subseteq\hat{B}$, $\tilde{f}(\hat{Z})=\bigcup_{m<\omega}\hat{f}(\hat{Z}\restriction k_m)$.

Define the function $f':\mathcal{U}\to 2^\omega$ by $f'(U)=\tilde{f}(\hat{U})$, for $U\in\mathcal{U}$. Then

(2) $f'\restriction\mathcal{U}$ is a cofinal map from $\mathcal{U}$ to $\mathcal{V}$; and

(3) $f'\restriction\mathcal{C}=f\restriction\mathcal{C}$.

We now give the following equivalent of Definition 1.2 (2), which will be employed in this section. Though this definition is seemingly weaker, it is not hard to check that any finitary function satisfying the following definition can be extended to one satisfying Definition 1.2 (2).

Definition 6.2 (Finitely generated Tukey reduction)**.** We say that an ultrafilter $\mathcal{V}$ on base set ω has *finitely generated Tukey reductions* if for each monotone cofinal map $f:\mathcal{V}\to\mathcal{W}$, there is a cofinal subset $\mathcal{D}\subseteq\mathcal{V}$, an increasing sequence $(k_m)_{m<\omega}$, and a function $\hat{f}:D\to 2^{<\omega}$, where $D=\{X\restriction k_m : X\in\mathcal{D},\ m<\omega\}$, such that

(1) $\hat{f}$ is level preserving: For each $m<\omega$ and $s\in D$, $|s|=k_m$ implies $|\hat{f}(s)|=m$;

(2) $\hat{f}$ is monotone: For $s,t\in D$, $s\subseteq t$ implies $\hat{f}(s)\subseteq\hat{f}(t)$;

(3) $\hat{f}$ *generates* f on $\mathcal{D}$: For each $X \in \mathcal{D}$,

$$f(X) = \bigcup_{m<\omega} \hat{f}(X \restriction k_m). \tag{6.1}$$

Now we prove the main theorem of this section. This is the extension of Theorem 2.5 (which holds for ultrafilters Tukey reducible to some p-point) to the setting of all ultrafilters Tukey reducible to some Fubini iterate of p-points.

Theorem 6.3. *Let $\mathcal{U}$ be any Fubini iterate of p-points. If $\mathcal{V} \le_T \mathcal{U}$, then $\mathcal{V}$ has finitely generated Tukey reductions.*

Proof. Suppose that $\mathcal{U}$ is an iteration of Fubini products of p-points and that $\mathcal{V} \le_T \mathcal{U}$. Without loss of generality, assume that ω is the base set for the ultrafilter $\mathcal{V}$. Let B be a flat-top front which is a base for $\mathcal{U}$. By Theorem 4.4, $\mathcal{U}$ has basic Tukey reductions. Applying Lemma 6.1, we obtain a continuous monotone map $\tilde{f} : 2^{\hat{B}} \to 2^\omega$ which is generated by a monotone, level and end-extension preserving map $\hat{f} : \bigcup_{m<\omega} 2^{\hat{B}\restriction k_m} \to 2^{<\omega}$, for some increasing sequence $(k_m)_{m<\omega}$. Hence, for each $\hat{Z} \subseteq \hat{B}$, $\tilde{f}(\hat{Z}) = \bigcup_{m<\omega} \hat{f}(\hat{Z} \restriction k_m)$. Furthermore, defining $f(U) = \tilde{f}(\hat{U})$ for $U \in \mathcal{U}$, we see that $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map.

Suppose $\mathcal{W} \le_T \mathcal{V}$, and let $h : \mathcal{V} \to \mathcal{W}$ be a monotone cofinal map. Extend h to the map $\tilde{h} : 2^\omega \to 2^\omega$ defined as follows: For each $X \in 2^\omega$, let

$$\tilde{h}(X) = \bigcap \{h(V) : V \in \mathcal{V} \text{ and } V \supseteq X\}. \tag{6.2}$$

It follows from h being monotone that $\tilde{h}$ is monotone and that $\tilde{h} \restriction \mathcal{V} = h$.

Letting $\tilde{g}$ denote $\tilde{h} \circ \tilde{f}$, we see that the map $\tilde{g} : 2^{\hat{B}} \to 2^\omega$ is monotone. For $U \in \mathcal{U}$, $\tilde{g}(\hat{U}) = \tilde{h}(\tilde{f}(\hat{U})) = \tilde{h}(f(U)) = h \circ f(U)$. Thus, letting g denote $h \circ f$, we see that $g : \mathcal{U} \to \mathcal{W}$ is a monotone cofinal map with the property that for each $U \in \mathcal{U}$, $g(U) = \tilde{g}(\hat{U})$. By Theorem 4.4, there is a $\vec{\mathcal{U}}$-tree $\tilde{T}$ and an increasing sequence $(k_m)_{m<\omega}$ such that $g \restriction (\mathfrak{T} \restriction \tilde{T})$ is basic, generated by some monotone, level and end-extension preserving map $\hat{g} : C \to 2^{<\omega}$, where

$$C = \{T \restriction k_m : T \in \mathfrak{T} \restriction \tilde{T} \text{ and } m < \omega\}.$$

Without loss of generality, we may assume that the levels k_m are the same for $\hat{f}$ and $\hat{g}$, by taking the minimum of the two m-th levels. For $T \in \mathfrak{T} \restriction \tilde{T}$, each $T \restriction k_m$ represents both the subset of T of members with maximum below m and also the characteristic function (with domain $\tilde{T} \restriction k_m$) of the members of T with maximum below k_m.

Let $\mathcal{C}$ denote the collection of all $[T]$ such that $T \in \mathfrak{T} \restriction \tilde{T}$. Thus, $\mathcal{C}$ is the collection of members of $\mathcal{U}$ which are represented by some $\vec{\mathcal{U}}$-tree $T \subseteq \tilde{T}$. Define

$$D = \{\hat{f}(s) : s \in C\} \text{ and } \mathcal{D} = f''\mathcal{C}. \tag{6.3}$$

Then $\mathcal{D}$ is cofinal in $\mathcal{V}$, and every member of $\mathcal{D}$ is a limit of members of D. Letting $\hat{\mathcal{C}}$ denote $\{\hat{X} : X \in \mathcal{C}\}$, we point out that $\hat{\mathcal{C}}$ can be regarded as a subspace of $2^{\tilde{T}}$, which is in turn a subspace of $2^{\hat{B}}$. Then the closure of $\hat{\mathcal{C}}$ in the space $2^{\tilde{T}}$ (or equivalently in $2^{\hat{B}}$) is a compact space. We shall use $\mathcal{C}^*$ to denote the closure of $\hat{\mathcal{C}}$ in $2^{\hat{B}}$. Since $\tilde{f}$ is continuous from the compact space $\mathcal{C}^*$ into 2^ω, it follows that $\overline{\mathcal{D}}$ equals the $\tilde{f}$-image of $\mathcal{C}^*$.

Define a function $\hat{h} : D \to 2^{<\omega}$ as follows: For $t \in D \cap 2^m$, define

$$\hat{h}(t) = \bigcap\{\hat{g}(s) : s \in C \cap 2^{\hat{B}\restriction k_m} \text{ and } \hat{f}(s) = t\}. \tag{6.4}$$

That is, $\hat{h}(t)$ is the function from m into 2 such that for $i \in m$, $\hat{h}(t)(i) = 1$ if and only if $\hat{g}(s)(i) = 1$ for all $s \in C \cap 2^{\hat{B}\restriction k_m}$ satisfying $\hat{f}(s) = t$. By definition, $\hat{h}$ is level preserving. Further, $\hat{h}$ is monotone. This follows easily from the definition of $\hat{h}$ and the fact that $\hat{f}$ and $\hat{g}$ are both end-extension preserving.

Let g^* be the function on $\mathcal{C}^*$ determined by $\hat{g}$ as follows: For $\hat{Z} \in \mathcal{C}^*$, define $g^*(\hat{Z}) = \bigcup_{m<\omega} \hat{g}(\hat{Z} \restriction k_m)$. Since $\hat{g}$ is end-extension preserving, it follows that for each $\hat{Z} \in \mathcal{C}^*$ and each $m < \omega$, $g^*(\hat{Z}) \restriction m = \hat{g}(\hat{Z} \restriction k_m)$.

Claim 1. $g^* = \tilde{g} \restriction \widehat{\mathcal{C}}$. Moreover, for each $\hat{Z} \in \mathcal{C}^*$,

$$g^*(\hat{Z}) = \bigcap\{g(X) : X \in \mathcal{C} \text{ and } \hat{X} \supseteq \hat{Z}\} \supseteq \tilde{g}(\hat{Z}). \tag{6.5}$$

Proof. If $\hat{Z} \in \hat{\mathcal{C}}$, then $g^*(\hat{Z}) = \tilde{g}(\hat{Z})$, since $\hat{g}$ represents $\tilde{g}$ on $\hat{\mathcal{C}}$.

Now let $\hat{Z}$ be any member of $\mathcal{C}^*$. Let m be given and note that $\hat{Z} \restriction k_m \in C$. For any $X \in \mathcal{C}$ such that $\hat{X} \restriction k_m = \hat{Z} \restriction k_m$, we have

$$\hat{g}(\hat{Z} \restriction k_m) = \hat{g}(\hat{X} \restriction k_m) = \tilde{g}(\hat{X}) \restriction m = g(X) \restriction m. \tag{6.6}$$

Since $\mathcal{C} = \{[T] : T \in \mathfrak{T} \restriction \tilde{T}\}$ and $\tilde{g}$ is monotone, there is an $X \in \mathcal{C}$ such that $\hat{X} \supseteq \hat{Z}$ and $\hat{X} \restriction k_m = s$. (This is the key property of $\mathcal{C}$ needed for this proof.) Thus,

$$\hat{g}(\hat{Z} \restriction k_m) = \bigcap\{g(X) \restriction m : X \in \mathcal{C} \text{ and } \hat{X} \supseteq \hat{Z}\} \supseteq \tilde{g}(\hat{Z}) \restriction m. \tag{6.7}$$

Taking the union over all $m < \omega$, the claim follows. $\square$

Claim 2. $\hat{h}$ represents h on $\mathcal{D}$.

Proof. Fix $Y \in \mathcal{D}$. Then there is an $X \in \mathcal{C}$ such that $f(X) = Y$. For each $m < \omega$, $\hat{f}(\hat{X} \restriction k_m) = Y \restriction m$, so

(6.8)
$$\hat{h}(Y \restriction m) \subseteq \hat{g}(\hat{X} \restriction k_m) = \tilde{g}(\hat{X}) \restriction m = g(X) \restriction m = h \circ f(X) \restriction m = h(Y) \restriction m.$$

Thus, $\bigcup_{m<\omega} \hat{h}(Y \restriction m) \subseteq h(Y)$.

Next we show that for each $l \in h(Y)$, there is some n such that $l \in \hat{h}(Y \restriction n)$. Let $m = l + 1$ and let $t = Y \restriction m$. Let

$$S_l = \{s \in C \cap 2^{\hat{B} \restriction k_m} : \forall \hat{Z} \in \mathcal{C}^* \text{ with } s \sqsubset \hat{Z},\ \tilde{f}(\hat{Z}) \neq Y\}. \tag{6.9}$$

For each $s \in S_l$, there is an n_s such that each $s' \sqsupset s$ of length n_s has $\hat{f}(s') \not\sqsubset Y$, by compactness of $\mathcal{C}^*$. (For if for all n there were an $s' \in C$ extending s of length n which has $\hat{f}(s') \sqsubset Y$, then by compactness of $\mathcal{C}^*$, there would be a $\hat{Z} \in \mathcal{C}^*$ such that $Z \sqsupset s$ and $\tilde{f}(\hat{Z}) = Y$.) Since S_l is finite, we may take $n = \max\{n_s : s \in S_l\}$. Then for all s' of length k_n, if $\hat{f}(s') \sqsubset Y$ then s' does not end-extend any s in S_l.

We claim that for all $s' \in C$ of length k_n such that $\hat{f}(s') \sqsubset Y$, $l \in \hat{g}(s')$. For each $s' \in C \cap 2^{\hat{B} \restriction k_n}$ satisfying $\hat{f}(s') = Y \restriction n$, we see that $s' \restriction k_m$ is not in S_l. So there is some $\hat{Z} \in \mathcal{C}^*$ such that $\hat{Z} \sqsupset s' \restriction k_m$ and $\tilde{f}(\hat{Z}) = Y$. It follows that

$$\hat{g}(s') \restriction m = \hat{g}(s' \restriction k_m) \supseteq \tilde{g}(\hat{Z}) \restriction m = \tilde{h} \circ \tilde{f}(\hat{Z}) \restriction m = h(Y) \restriction m, \tag{6.10}$$

where the $\supseteq$ follows from Claim 1. Thus, $l \in \hat{g}(s')$, for each $s' \in C \cap 2^{\hat{B} \restriction k_n}$ satisfying $\hat{f}(s') = Y \restriction n$. Therefore, $l \in \hat{h}(Y \restriction n)$.

Thus, for each $l \in h(Y)$, there is an n_l such that $l \in \hat{h}(Y \restriction n_l)$. It follows that, for any $j < \omega$, there is an n such that $\hat{h}(Y \restriction n) \restriction j = h(Y) \restriction j$. This n may be obtained by taking the maximum of the n_s over all $s \in \bigcup\{S_l : l < j\}$, Hence, $\bigcup_{m<\omega} \hat{h}(Y \restriction m) = h(Y)$. □

Thus, $h \restriction \mathcal{D}$ is finitely represented by $\hat{h}$ on D. □

Theorem 6.3 is now applied to extend Theorem 17 of Raghavan in [13] to all ultrafilters Tukey reducible to some Fubini iterate of p-points. Raghavan showed that for any basically generated ultrafilter $\mathcal{U}$, whenever $\mathcal{V} \leq_T \mathcal{U}$ there is a filter $\mathcal{U}(P)$ which is Tukey equivalent to $\mathcal{U}$ such that $\mathcal{V} \leq_{RK} \mathcal{U}(P)$. It is routine to check that the maps in Theorem 6.3 satisfy the conditions of the maps in Theorem 17 of Raghavan in [13]. Thus, we obtain the following.

Theorem 6.4. *If $\mathcal{U}$ is Tukey reducible to a Fubini iterate of p-points, then for each $\mathcal{V} \leq_T \mathcal{U}$, there is a filter $\mathcal{U}(P) \equiv_T \mathcal{U}$ such that $\mathcal{V} \leq_{RK} \mathcal{U}(P)$.*

Here, assuming without loss of generality that the base set for $\mathcal{U}$ is ω, P is the collection of $\sqsubset$-minimal finite subsets s of ω for which $\hat{h}(s) \neq \emptyset$, where

$\hat{h}$ witnesses that a given monotone cofinal $h : \mathcal{U} \to \mathcal{V}$ is finitely generated. $\mathcal{U}(P)$ is the collection of all sets of the form $\{s \in P : s \subseteq U\}$, for $U \in \mathcal{U}$.

Remark 6.5. The same proofs of Theorems 6.3 works for the basic cofinal maps for the generic ultrafilters $\mathcal{G}_k$ forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$, $2 \leq k < \omega$, in [5]. Thus, Theorem 6.4 also holds when $\mathcal{U}$ is an ultrafilter forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$.

7. Open problems

We conclude this paper by highlighting some of the more important open problems in this area. Theorem 2.6 showed that every ultrafilter Tukey below a p-point has continuous Tukey reductions.

Problem 7.1. Determine the class of all ultrafilters that have continuous Tukey reductions.

In particular, are there ultrafilters not Tukey reducible to a p-point which satisfy the conditions of Theorem 2.5?

By Theorem 56 of Dobrinen and Trujillo in [7], under very mild conditions, any ultrafilter selective for some topological Ramsey space has continuous (with respect to the topological Ramsey space) Tukey reductions. This is especially of interest when the ultrafilters associated with the topological Ramsey space is not a p-point. It should be the case that by arguments similar to those in this paper one can prove the following.

Problem 7.2. Prove the analogues of Theorems 2.5 and 4.4 for stable ordered union ultrafilters and their iterated Fubini products, and more generally for ultrafilters selective for some topological Ramsey space, with respect to the correct topologies.

More generally, we would like to know the following.

Problem 7.3. Determine the class of all ultrafilters which have finitely generated Tukey reductions. Is this the same as the class of all ultrafilters with Tukey type strictly below the maximum Tukey type?

In Section 5, we applied Theorem 5.1 find more examples when Tukey reducibility implies Rudin-Keisler reducibility. Theorem 5.3 improves on one aspect of Corollary 56 in[13] of Raghavan provided that there are q-points which are not selective and which are Tukey below some Fubini iterate of p-points. Do such ultrafilters ever exist?

Problem 7.4. Is there a q-point which is not selective which is Tukey reducible to some finite Fubini iterate of p-points? Or does $\mathcal{V} \le_T \mathcal{U}$ with $\mathcal{U}$ a Fubini iterate of p-points and $\mathcal{V}$ a q-point imply that $\mathcal{V}$ is actually selective?

Problem 7.5. Can Theorem 5.3 be extended to all countable iterates of Fubini products of p-points? Are similar results true for all ultrafilters Tukey reducible to some Fubini iterate of p-points?

Question 25 in [8] asks whether asks whether every ultrafilter Tukey reducible to a p-point is basically generated. Question 26 in [8] asks whether the classes of basically generated and Fubini iterates of p-points the same, or whether the former is strictly larger than the latter? Though these questions in general are still open, we ask the even more general questions.

Problem 7.6. Is the property of being basically generated inherited under Tukey reduction? That is, if $\mathcal{V}$ is Tukey reducible to a basically generated ultrafilter, is it necessarily basically generated?

Or the possibly weaker problem: If $\mathcal{V}$ is Tukey reducible to some Fubini iterate of p-points, it is necessarily basically generated?

There are certain collections of p-points, in particular those associated with the topological Ramsey spaces in [9], [10], and [7], for which every ultrafilter Tukey below some Fubini iterate of these p-points is again a Fubini iterate of these p-points and hence basically generated. However, the above questions are in general still open.

Work in this paper and work in [13] found conditions when Tukey reducibility implies Rudin-Keisler or even Rudin-Blass reducibility.

Problem 7.7. When in general does $\mathcal{U} \ge_T \mathcal{V}$ imply $\mathcal{U} \ge_{RK} \mathcal{V}$ or $\mathcal{U} \ge_{RB} \mathcal{V}$?

Finally, how closely related are the properties of having finitely generated Tukey reductions and having Tukey type below the maximum?

Problem 7.8. Does $\mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$ imply that $\mathcal{U}$ has finitely generated Tukey reductions?

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Department of Mathematics, University of Denver, 2280 S Vine St., Denver, CO 80208 U.S.A.

E-mail address: natasha.dobrinen@du.edu

URL: http://web.cs.du.edu/~ndobrine