

COSETS OF BERSHADSKY-POLYAKOV ALGEBRAS AND RATIONAL \mathcal{W} -ALGEBRAS OF TYPE A

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ABSTRACT. The universal Bershadskey-Polyakov algebra \mathcal{W}^k at level $k \neq -3$ is freely generated by fields J, T, G^+, G^- of weights $1, 2, \frac{3}{2}, \frac{3}{2}$, respectively. The simple quotient of \mathcal{W}^k is denoted by \mathcal{W}_k , and for $p = 5, 7, 9, \dots$, $\mathcal{W}_{p/2-3}$ is known to be C_2 -cofinite and rational. We prove the following results. First, the commutant of J inside \mathcal{W}^k is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for all $k \neq -1, -\frac{3}{2}$. Next, $\mathcal{W}_{p/2-3}$ contains a lattice vertex algebra V_L with $L = \sqrt{3p-9}\mathbb{Z}$, and V_L and its commutant $\mathcal{C}_{p/2-3} = \text{Com}(V_L, \mathcal{W}_{p/2-3})$ form a dual pair inside $\mathcal{W}_{p/2-3}$. Finally, $\mathcal{C}_{p/2-3}$ is isomorphic to the principal, rational $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra with central charge $c = -\frac{3}{p}(p-4)^2$. This was conjectured in the physics literature over 20 years ago. As a byproduct, we construct a new family of rational, C_2 -cofinite vertex superalgebras from $\mathcal{W}_{p/2-3}$ with even generators in weights $1, 2, 3$, and two odd generators in weight 2 .

1. INTRODUCTION

A longstanding problem in vertex algebra theory is to classify rational and C_2 -cofinite vertex algebras. Well-known examples include those associated to even positive definite lattices [D], and simple affine vertex algebras at positive integral level [FZ]. More recently, Kac and Wakimoto conjectured the C_2 -cofiniteness and rationality of many quantum Hamiltonian reductions based on the modularity of their characters [KWII], and these conjectures were proven for the minimal series principal \mathcal{W} -algebras in [ArIV, ArV]. New examples beyond this conjecture and still associated to quantum Hamiltonian reductions have also been found [Ka, ArM]. The classification problem seems to be out of reach at present, but it is still of great interest to find new examples and constructions of such vertex algebras.

There are some standard ways to construct new vertex algebras from old ones. First, the *orbifold construction* begins with a vertex algebra \mathcal{V} and a group of automorphisms G , and considers the invariant subalgebra \mathcal{V}^G and its extensions. The Moonshine vertex algebra is constructed in this way; its full automorphism group is the Monster, which is the largest of the sporadic simple finite groups, and its graded character is $j(\tau) - 744$ where $j(\tau)$ is the modular invariant j -function [FLM]. It is widely believed that if \mathcal{V} is C_2 -cofinite and rational, and G is a finite group, \mathcal{V}^G will inherit these properties. This is known in many examples, but so far there are few general results of this kind. Miyamoto has established the C_2 -cofiniteness of \mathcal{V}^G when G is either cyclic or solvable [MiI], but the rationality is still out of reach.

Similarly, the *coset construction* associates to a vertex algebra \mathcal{V} and a subalgebra \mathcal{A} , the subalgebra $\text{Com}(\mathcal{A}, \mathcal{V}) \subset \mathcal{V}$ which commutes with \mathcal{A} . This was introduced by Frenkel and

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Zhu in [FZ], generalizing earlier constructions in representation theory [KP] and physics [GKO], where it was used to construct the unitary discrete series representations of the Virasoro algebra. It is believed that if both \mathcal{A} and \mathcal{V} are C_2 -cofinite and rational, these properties will be inherited by $\text{Com}(\mathcal{A}, \mathcal{V})$. In the case where \mathcal{V} is a rational affine vertex algebra and \mathcal{A} is a lattice vertex algebra, this has been proven recently in [DR], building on [ALY, DLY, DLWY, DWI, DWII, DWIII].

In this paper, we consider cosets of the Bershadsky-Polyakov algebra. This algebra appeared originally in the physics literature [Ber, Pol], and is the \mathcal{W} -algebra corresponding to the quantum Hamiltonian reduction of \mathfrak{sl}_3 for the non-principal nilpotent embedding of \mathfrak{sl}_2 . The *universal* Bershadsky-Polyakov algebra \mathcal{W}^k at level $k \neq -3$ is freely generated by fields J, T, G^+, G^- of weights $1, 2, \frac{3}{2}, \frac{3}{2}$, and we denote its simple quotient by \mathcal{W}_k . For $k = p/2 - 3$ where $p = 5, 7, 9, \dots$, the C_2 -cofiniteness and rationality $\mathcal{W}_{p/2-3}$ were proven in [ArII]. Let

$$\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k),$$

where \mathcal{H} is the Heisenberg algebra generated by J . It was conjectured over 20 years ago in the physics literature [BEHHHIII] that \mathcal{C}^k should be of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for generic values of k ; in other words, a minimal strong generating set consists of a field in these weights. In this paper it was also conjectured that when $k = p/2 - 3$ for $p = 5, 7, 9, \dots$ as above, the simple coset

$$\mathcal{C}_{p/2-3} = \text{Com}(\mathcal{H}, \mathcal{W}_{p/2-3})$$

should be the principal $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra with central charge $c = -\frac{3}{p}(p-4)^2$, which is rational by [ArV]. So far, this has been proven only very recently by Kawasetsu in the case $p = 5$. These conjectures imply a *uniform truncation* property of this family of simple $\mathcal{W}(\mathfrak{sl}_{p-3})$ algebras for $p > 9$; they are expected to be of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$, even though the universal $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra is of type $\mathcal{W}(2, 3, \dots, p-3)$. This can only happen if there exist decoupling relations expressing the generators of weights $8, 9, \dots, p-3$ as normally ordered polynomials in the ones up to weight 7. The existence of a singular vector of weight 8 in the universal algebra has been known for many years [BEHHHI], and provides further evidence for these conjecture.

In this paper we shall prove all of these conjectures. We begin by studying the $U(1)$ -orbifold $(\mathcal{W}^k)^{U(1)}$ in Section 5. Typically, an orbifold of a vertex algebra \mathcal{A}^k with structure constants depending continuously on a parameter k will have a minimal strong generating set that works for generic values of k . This was established in [CLI] when \mathcal{A}^k is a tensor product of affine and free field algebras, and a similar approach works for \mathcal{W}^k . We show that $(\mathcal{W}^k)^{U(1)}$ is of type $\mathcal{W}(1, 2, 3, 4, 5, 6, 7)$ for generic values of k , and we give a complete description of *nongeneric* set, where additional generators are needed; it consists only of $\{-1, -\frac{3}{2}\}$. In general, it is an important problem to characterize the nongeneric set for such orbifolds, and we expect that our methods will be applicable in a wider setting. In Section 6, using the description of $(\mathcal{W}^k)^{U(1)}$, we obtain

Theorem 1.1. $\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k)$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for all values of k except for -1 and $-\frac{3}{2}$. Here \mathcal{H} is the Heisenberg algebra generated by J .

In Section 7, we show that the Heisenberg algebra $\mathcal{H} \subset \mathcal{W}_{p/2-3}$ is actually part of a rank one lattice vertex algebra V_L for $L = \sqrt{3p-9}\mathbb{Z}$. Moreover, V_L and $\mathcal{C}_{p/2-3}$ form a *Howe pair*, i.e., a pair of mutual commutants, inside $\mathcal{W}_{p/2-3}$. By a theorem of Miyamoto, this immediately implies the C_2 -cofiniteness of $\mathcal{C}_{p/2-3}$. In Section 8, we consider a certain

simple current extension of $\mathcal{W}(\mathfrak{sl}_{p-3}) \otimes V_L$ and show that it is isomorphic to $\mathcal{W}_{p/2-3}$. This implies our main result:

Theorem 1.2. $\mathcal{C}_{p/2-3}$ is isomorphic to the principal, rational $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra with central charge $c = -\frac{3}{p}(p-4)^2$.

We immediately obtain the uniform truncation property of this family of $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebras, and also a new proof of the C_2 -cofiniteness and rationality of $\mathcal{W}_{p/2-3}$, since it is a simple current extension of a rational, C_2 -cofinite vertex algebra. The appearance of principal, rational \mathcal{W} -algebras as cosets of nonprincipal ones is a remarkable coincidence, and it indicates that the principal rational \mathcal{W} -algebras may be important building blocks for more general rational \mathcal{W} -algebras. In Section 9, we obtain some nontrivial character identities as a consequence of the above isomorphism. Finally, in Section 10 we consider cosets of the Heisenberg algebra inside $\mathcal{W}^k \otimes \mathcal{E}$ where \mathcal{E} denotes the rank one bc -system. We obtain a new deformable vertex superalgebra with even generators in weights 1, 2, 3 and two odd generators in weight 2, whose simple quotients for $k = p/2 - 3$ form a new series of rational, C_2 -cofinite vertex superalgebras with integer conformal weights.

Coset vertex algebras have various applications in physics. One of the most interesting recent developments in the AdS/CFT correspondence is the relation between two-dimensional coset conformal field theories and higher spin gravity on three dimensional Anti-de-Sitter space. The main examples are the \mathcal{W}_N -minimal models [GG] and their super conformal analogues [CHR]. Recall that the Bershadsky-Polyakov algebra coincides with the case $n = 3$ in the family of $\mathcal{W}_n^{(2)}$ -algebras constructed by Feigin and Semikhatov [FS], which were studied in this context [ACGHR]. In that article it was observed that the $\mathcal{W}_4^{(2)}$ -algebra at level $-7/3$ is just a rank one lattice vertex algebra and hence rational. We expect that for all $n \geq 4$, there is a similar series of rational $\mathcal{W}_n^{(2)}$ -algebras which contain a rank one lattice vertex algebra V_L , and that the coset of V_L will give rise to another series of principal, rational \mathcal{W} -algebras of type A with a uniform truncation property. Finally, we expect that the appearance of \mathcal{W} -algebras of type A as cosets of the Bershadsky-Polyakov algebra is not limited to the rational levels, and also occurs for certain families of nonrational levels. These ideas will be explored in another paper.

Another reason to study cosets \mathcal{C}_k at nonrational levels is that they have some interesting connections to *logarithmic conformal field theories*, which are associated to nonrational C_2 -cofinite vertex algebras whose representation category is not semisimple. The best studied examples are the so-called $\mathcal{W}(p)$ -triplet algebras, see e.g. [AM, TW]. Interestingly, $\mathcal{W}(p)$ is related to the coset $\text{Com}(\mathcal{H}, \mathcal{W}_{p-1}^{(2)})$ where \mathcal{H} is a Heisenberg algebra. In particular, $\mathcal{W}(4)$ is a simple current extension of our coset \mathcal{C}_k with $k = -\frac{9}{4}$ [CRW].

2. VERTEX ALGEBRAS

We will assume that the reader is familiar with the basics of vertex algebra theory, which has been discussed from several different points of view in the literature (see for example [Bor, FLM, K, FBZ]). We will follow the formalism developed in [LZ] and partly in [LiI], and we will use the notation of the previous papers of the second and third authors. By a *vertex algebra*, we mean a quantum operator algebra \mathcal{A} in which any two elements a, b are local, i.e., $(z-w)^N[a(z), b(w)] = 0$ for some positive integer N . Here \mathcal{A} is assumed to be $\mathbb{Z}/2\mathbb{Z}$ -graded, and $[\]$ denotes the super bracket. This is well known to be

equivalent to the notion of vertex algebra in [FLM]. The *operators product expansion* (OPE) formula is given by

$$a(z)b(w) \sim \sum_{n \geq 0} a(w) \circ_n b(w) (z-w)^{-n-1}.$$

Here \sim means equal modulo terms which are regular at $z = w$, and \circ_n denotes the n^{th} circle product. A subset $S = \{a_i \mid i \in I\}$ of \mathcal{A} *generates* \mathcal{A} if \mathcal{A} is spanned by words in the letters a_i, \circ_n , for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* \mathcal{A} if \mathcal{A} is spanned by words in the letters a_i, \circ_n for $n < 0$. Equivalently, \mathcal{A} is spanned by

$$\{ : \partial^{k_1} a_{i_1} \cdots \partial^{k_m} a_{i_m} : \mid i_1, \dots, i_m \in I, k_1, \dots, k_m \geq 0 \}.$$

We say that S *freely generates* \mathcal{A} if there are no nontrivial normally ordered polynomial relations among the generators and their derivatives. As a matter of notation, we say that a vertex algebra \mathcal{A} is of type $\mathcal{W}(d_1, \dots, d_r)$ if it has a minimal strong generating set consisting of one field in each weight d_1, \dots, d_r .

3. THE BERSHADSKY-POLYAKOV ALGEBRA

The universal Bershadsky-Polyakov algebra \mathcal{W}^k at level $k \neq -3$ is freely generated by fields T, J, G^+, G^- of weights $1, 2, \frac{3}{2}, \frac{3}{2}$, satisfying the following nontrivial OPE relations.

$$J(x)J(w) \sim \frac{2k+3}{3}(z-w)^{-2}, \quad G^\pm(z)G^\pm(w) \sim 0, \quad J(z)G^\pm(w) \sim \pm G^\pm(w)(z-w)^{-1},$$

$$T(z)T(w) \sim -\frac{(2k+3)(3k+1)}{2(k+3)}(z-w)^{-4} + 2T(w)(z-w)^{-2} + \partial T(w)(z-w)^{-1},$$

$$T(z)G^\pm(w) \sim \frac{3}{2}G^\pm(w)(z-w)^{-2} + \partial G^\pm(w)(z-w)^{-1},$$

$$T(z)J(w) \sim J(w)(z-w)^{-2} + \partial J(w)(z-w)^{-1},$$

$$G^+(z)G^-(w) \sim (k+1)(2k+3)(z-w)^{-3} + 3(k+1)J(w)(z-w)^{-2}$$

$$+ \left(3 : J(w)J(w) : + \frac{3(k+1)}{2} \partial J(w) - (k+3)T(w) \right) (z-w)^{-1}.$$

This algebra is isomorphic to the \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_\theta)$ associated to \mathfrak{sl}_3 with its minimal nilpotent element f_θ . Alternatively, it arises as the algebra $\mathcal{W}_3^{(2)}$ in the family $\mathcal{W}_n^{(2)}$ constructed by Feigin and Semikhatov [FS].

For $k = \frac{p}{2} - 3$ with $p = 3, 5, 7, \dots$, the ideal generated by $(G^\pm)^{p-2}$ is nontrivial, and we denote by \mathcal{W}_k the quotient of \mathcal{W}^k by its maximal ideal. For $p = 3$, $\mathcal{W}_{p/2-3} \cong \mathbb{C}$ and for $p = 5, 7, \dots$, $\mathcal{W}_{p/2-3}$ is C_2 -cofinite and rational [ArII]. For notational convenience, we will replace k with $\ell = k + \frac{3}{2}$. Then $\ell = (p-3)/2$, so $p = 5, 7, \dots$, corresponds to $\ell = 1, 2, \dots$. For the rest of this paper, we will use the notation \mathcal{W}^ℓ to denote $\mathcal{W}^{k+3/2}$, and for $\ell = 1, 2, \dots$ we denote by \mathcal{W}_ℓ the simple rational quotient.

4. WEAK INCREASING FILTRATIONS

A *good increasing filtration* on a vertex algebra \mathcal{A} is a $\mathbb{Z}_{\geq 0}$ -filtration

$$(4.1) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)}$$

such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(r)}, b \in \mathcal{A}_{(s)}$, we have

$$(4.2) \quad a \circ_n b \in \begin{cases} \mathcal{A}_{(r+s)} & n < 0 \\ \mathcal{A}_{(r+s-1)} & n \geq 0 \end{cases}.$$

We set $\mathcal{A}_{(-1)} = \{0\}$, and we say that $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ has degree d . The key property of such filtrations is that the associated graded algebra $\text{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)} / \mathcal{A}_{(d-1)}$ is a $\mathbb{Z}_{\geq 0}$ -graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on \mathcal{A} . For $r \geq 1$ we have the projection

$$(4.3) \quad \varphi_r : \mathcal{A}_{(r)} \rightarrow \mathcal{A}_{(r)} / \mathcal{A}_{(r-1)} \subset \text{gr}(\mathcal{A}).$$

The assignment $\mathcal{A} \mapsto \text{gr}(\mathcal{A})$ is a functor from \mathcal{R} to the category of $\mathbb{Z}_{\geq 0}$ -graded (super)commutative rings with a differential ∂ of degree zero, which we call ∂ -rings. A ∂ -ring is just an *abelian* vertex algebra, that is, a vertex algebra \mathcal{V} in which $[a(z), b(w)] = 0$ for all $a, b \in \mathcal{V}$. A ∂ -ring A is said to be generated by a set $\{a_i \mid i \in I\}$ if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates A as a ring. The key feature of \mathcal{R} is the following reconstruction property [LL].

Lemma 4.1. *Let \mathcal{A} be a vertex algebra in \mathcal{R} and let $\{a_i \mid i \in I\}$ be a set of generators for $\text{gr}(\mathcal{A})$ as a ∂ -ring, where a_i is homogeneous of degree d_i . If $a_i(z) \in \mathcal{A}_{(d_i)}$ are elements satisfying $\varphi_{d_i}(a_i(z)) = a_i$, then \mathcal{A} is strongly generated as a vertex algebra by $\{a_i(z) \mid i \in I\}$.*

In this paper, we will need filtrations with weaker properties. We define a *weak increasing filtration* on a vertex algebra \mathcal{A} to be a $\mathbb{Z}_{\geq 0}$ -filtration

$$(4.4) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)}$$

such that for $a \in \mathcal{A}_{(r)}, b \in \mathcal{A}_{(s)}$, we have

$$(4.5) \quad a \circ_n b \in \mathcal{A}_{r+s}, \quad n \in \mathbb{Z}.$$

This condition guarantees that $\text{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)} / \mathcal{A}_{(d-1)}$ is a vertex algebra, but it is no longer abelian in general. As above, a strong generating set for $\text{gr}(\mathcal{A})$ consisting of homogeneous elements always gives rise to a strong generating set for \mathcal{A} .

We define an increasing filtration

$$\mathcal{W}_{(0)}^\ell \subset \mathcal{W}_{(1)}^\ell \subset \cdots$$

on \mathcal{W}^ℓ as follows: $\mathcal{W}_{(-1)}^\ell = \{0\}$, and $\mathcal{W}_{(r)}^\ell$ is spanned by iterated Wick products of the generators J, T, G^\pm and their derivatives, such that at most r of G^\pm and their derivatives appear. It is clear from the defining OPE relations that this is a weak increasing filtration. Clearly $\mathcal{W}_{(0)}^\ell$ is the Virasoro algebra with generator $T^c = T - \frac{4}{3\ell} : JJ :$, it follows that $\mathcal{W}_{(0)}^\ell$ is the tensor product of \mathcal{H} and the Virasoro algebra with generator T^c .

The associated graded algebra

$$\text{gr}(\mathcal{W}^\ell) = \bigoplus_{d \geq 0} \mathcal{W}_{(d)}^\ell / \mathcal{W}_{(d-1)}^\ell$$

is not abelian since it contains $\mathcal{W}_{(0)}$ as a subalgebra, but $G^+(z)G^-(w) \sim 0$ in this algebra.

5. THE $U(1)$ INVARIANTS IN \mathcal{W}^ℓ

The action of the zero mode J_0 integrates to a $U(1)$ action on \mathcal{W}^ℓ , and the invariant subalgebra $(\mathcal{W}^\ell)^{U(1)}$ coincides with the kernel of J_0 . Since J, T lie in \mathcal{W}^ℓ and $J_0(G^\pm) = \pm G^\pm$, it is immediate that $(\mathcal{W}^\ell)^{U(1)}$ is spanned by all normally ordered monomials of the form

$$(5.1) \quad : (\partial^{a_1} T) \cdots (\partial^{a_i} T) (\partial^{b_1} J) \cdots (\partial^{b_j} J) (\partial^{c_1} G^+) \cdots (\partial^{c_r} G^+) (\partial^{d_1} G^-) \cdots (\partial^{d_r} G^-) :,$$

where $r \geq 0$ and $0 \leq a_1 \leq \cdots \leq a_i$, $0 \leq b_1 \leq \cdots \leq b_j$, $0 \leq c_1 \leq \cdots \leq c_r$, and $0 \leq d_1 \leq \cdots \leq d_s$. We say that $\omega \in (\mathcal{W}^\ell)^{U(1)}$ is in *normal form* if it has been expressed as a linear combination of such monomials. Since \mathcal{W}^ℓ is freely generated by T, J, G^\pm , these monomials form a *basis* of $(\mathcal{W}^\ell)^{U(1)}$, and the normal form is unique.

The filtration on \mathcal{W}^ℓ restricts to a filtration

$$(\mathcal{W}^\ell)_{(0)}^{U(1)} \subset (\mathcal{W}^\ell)_{(1)}^{U(1)} \subset \cdots$$

on $(\mathcal{W}^\ell)^{U(1)}$, where $(\mathcal{W}^\ell)_{(r)}^{U(1)} = (\mathcal{W}^\ell)^{U(1)} \cap \mathcal{W}_{(r)}^\ell$. Define

$$(5.2) \quad U_{i,j} = : \partial^i G^+ \partial^j G^- : \in (\mathcal{W}^\ell)_{(2)}^{U(1)},$$

which clearly has weight $i+j+3$. Since G^+ and G^- commute with each other in $\text{gr}(\mathcal{W}^\ell)$, it follows by induction on filtration degree that each standard monomial (5.1) can be written as a normally ordered polynomial in $\{J, T, U_{i,j} \mid i, j \geq 0\}$ and their derivatives. Therefore $\{J, T, U_{i,j} \mid i, j \geq 0\}$ is a strong generating set for $(\mathcal{W}^\ell)^{U(1)}$. Letting A_m be the span of $\{U_{i,j} \mid i+j = m\}$ and using the relations $\partial U_{i,j} = U_{i+1,j} + U_{i,j+1}$, we see that

$$A_m = \partial(A_{m-1}) \oplus \langle U_{0,m} \rangle,$$

where $\langle U_{0,m} \rangle$ denotes the span of $U_{0,m}$. Then $\text{span}(\{U_{i,j} \mid i, j \geq 0\}) = \text{span}(\{\partial^i U_{0,j} \mid i, j \geq 0\})$. It follows that $(\mathcal{W}^\ell)^{U(1)}$ is strongly generated by the more economical set

$$(5.3) \quad \{J, T, U_{0,m} \mid m \geq 0\}.$$

In terms of the generating set (5.3), $(\mathcal{W}^\ell)_{(2r)}^{U(1)}$ is spanned by elements with at most r of the fields $U_{0,m}$ and $(\mathcal{W}^\ell)_{(2r+1)}^{U(1)} = (\mathcal{W}^\ell)_{(2r)}^{U(1)}$.

Observe next that $(\mathcal{W}^\ell)^{U(1)}$ is not *freely* generated by (5.3). The first weight in which a normally ordered polynomial relation among these generators occurs is weight 8, and this relation has the form

$$(5.4) \quad : U_{0,0} U_{1,1} : - : U_{0,1} U_{1,0} : = \frac{1}{60} \ell(2\ell - 1) U_{0,5} + \cdots .$$

The remaining terms lie in $(\mathcal{W}^\ell)_{(2)}^{U(1)}$ and are normally ordered monomials in $\{J, T, U_{0,i} \mid 0 \leq i \leq 4\}$ and their derivatives. For the reader's convenience, this relation is written down explicitly in the Appendix. As we shall see, the coefficient $\frac{1}{60} \ell(2\ell - 1)$ of $U_{0,5}$ is canonical in the sense that it does not depend on any choices of normal ordering of any of the terms appearing in this relation. Since this coefficient is nonzero for all $\ell \neq 0, \frac{1}{2}$, it follows that $U_{0,5}$ can be expressed as a normally ordered polynomial in $\{J, T, U_{0,i} \mid 0 \leq i \leq 4\}$ for all such ℓ .

In order to construct more “decoupling relations” of this kind, we introduce a certain invariant of elements of $(\mathcal{W}^\ell)_{(2)}^{U(1)}$. Given $\omega \in (\mathcal{W}^\ell)_{(2)}^{U(1)}$ of weight $n + 7$, write ω in normal form. For $i = 0, 1, \dots, n + 4$, let $C_{n,i}(\omega)$ denote the coefficient of $(\partial^i G^+)(\partial^{n+4-i} G^-)$ appearing in the normal form. Define

$$(5.5) \quad C_n(\omega) = \sum_{i=0}^{n+4} (-1)^i C_{n,i}(\omega).$$

Next, since $\{J, T, U_{0,i} \mid i \geq 0\}$ strongly generates $(\mathcal{W}^\ell)^{U(1)}$ and since $U_{0,i}$ has weight $i + 3$, we may express ω as a normally ordered polynomial $P_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4})$ in the subset $\{J, T, U_{0,i} \mid 0 \leq i \leq n + 4\}$ and their derivatives. Since there exist relations among the generators $\{J, T, U_{0,i} \mid 0 \leq i \leq n + 4\}$, as well as different choices of normal ordering, such an expression for ω is not unique. In particular, the coefficients of $\partial^i U_{0,n+4-i}$ for $i > 0$ will depend on the choice of normal ordering in $P_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4})$. We will see that for any ω , the coefficient of $U_{0,n+4}$ in $P_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4})$ is independent of all choices of normal ordering, and coincides with $(-1)^n C_n(\omega)$. In order to prove this, we need several technical lemmas.

Lemma 5.1. *The coefficient of $U_{0,n+4}$ in $P_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4})$ is independent of all choices of normal ordering in $P_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4})$, and coincides with $(-1)^n C_n(\omega)$.*

Proof. Let $\mathcal{J}^\ell \subset (\mathcal{W}^\ell)^{U(1)}$ denote the subspace spanned by elements of the form $a \partial b$ with $a, b \in (\mathcal{W}^\ell)^{U(1)}$. It is well known that Zhu’s commutative algebra $C((\mathcal{W}^\ell)^{U(1)}) = (\mathcal{W}^\ell)^{U(1)} / \mathcal{J}^\ell$ is a commutative, associative algebra with generators corresponding to the strong generators $\{T, J, U_{0,n} \mid n \geq 0\}$. In particular, given $\omega \in (\mathcal{W}^\ell)_{(2)}^{U(1)}$ of weight $n + 7$, and given two expressions

$$\omega = P_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4}) = Q_\omega(J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4})$$

where P, Q are normally ordered polynomials, we have $P_\omega - Q_\omega \in \mathcal{J}^\ell$. In particular, the component of $P_\omega - Q_\omega$ which is a linear combination of $U_{i,n+4-i}$ for $i = 0, \dots, n + 4$ must be a total derivative. It follows that the coefficient of $U_{0,n+4}$ in P_ω and Q_ω is the same. The last statement follows from the fact that

$$U_{i,n+4-i} = \sum_{r=0}^i (-1)^{i+r} \binom{i}{r} \partial^r U_{0,n+4-r}.$$

□

Corollary 5.2. *The coefficient of $U_{0,5}$ in (5.4) coincides with $-C_1(: U_{0,0} U_{1,1} : - : U_{0,1} U_{1,0} :)$ and is independent of all choices of normal ordering.*

Theorem 5.3. *For all $\ell \neq 0$ or $\frac{1}{2}$, $(\mathcal{W}^\ell)^{U(1)}$ has a minimal strong generating set*

$$\{J, T, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}\},$$

and in particular is of type $\mathcal{W}(1, 2, 3, 4, 5, 6, 7)$.

Proof. It suffices to construct decoupling relations of the form

$$U_{0,n+4} = P_n(J, T, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}),$$

for all $n > 1$ and $\ell \neq 0, \frac{1}{2}$, since we already have such a relation for $n = 1$. Since G^\pm commute in $\text{gr}(\mathcal{W}^\ell)$, $: U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} :$ lies in $\mathcal{W}_{(2)}^\ell$, and we can express it as a normally ordered polynomial in the generators $J, T, U_{0,m}$ and their derivatives. We have

$$: U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} : = C_n U_{0,n+4} + \cdots,$$

where $C_n = C_n(: U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} :)$, and the remaining terms are normally ordered polynomials in $\{J, T, U_{0,0}, U_{0,1}, \dots, U_{0,n+4-1}\}$ and their derivatives. It suffices to show that $C_n \neq 0$ for all $\ell \neq 0$ or $\frac{1}{2}$; the claim then follows by induction on n .

First, we have

$$\begin{aligned} : U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} : &= (: G^+G^- :)(: (\partial G^+)\partial^n G^- :): - : G^+G^-(\partial G^+)\partial^n G^- : \\ &+ : G^+G^-(\partial G^+)\partial^n G^- : - : G^+G^-(\partial^n G^-)(\partial G^+) : \\ &+ : G^+G^-(\partial^n G^-)(\partial G^+) : - : G^+(\partial^n G^-)G^-(\partial G^+) : \\ &+ : G^+(\partial^n G^-)G^-(\partial G^+) : - : G^+(\partial^n G^-)(\partial G^+)G^- : \\ &+ : G^+(\partial^n G^-)(\partial G^+)G^- : - : (: G^+(\partial^n G^-) :)(: (\partial G^+)G^- :): . \end{aligned}$$

It is immediate that $C_{n,i}(: G^+G^-(\partial^n G^-)(\partial G^+) : - : G^+(\partial^n G^-)G^-(\partial G^+) :) = 0$. Let

$$\begin{aligned} C_{n,i}^1 &= C_{n,i} \left((: G^+G^- :)(: (\partial G^+)\partial^n G^- :): - : G^+G^-(\partial G^+)\partial^n G^- : \right), \\ C_{n,i}^2 &= C_{n,i} \left(: G^+G^-(\partial G^+)\partial^n G^- : - : G^+G^-(\partial^n G^-)(\partial G^+) : \right), \\ C_{n,i}^3 &= C_{n,i} \left(G^+(\partial^n G^-)G^-(\partial G^+) : - : G^+(\partial^n G^-)(\partial G^+)G^- : \right), \\ C_{n,i}^4 &= C_{n,i} \left(: G^+(\partial^n G^-)(\partial G^+)G^- : - : (: G^+(\partial^n G^-) :)(: (\partial G^+)G^- :): \right), \end{aligned}$$

so that $C_n = \sum_{i=1}^4 \sum_{j=0}^{n+4} (-1)^j C_{n,j}^i$. Using the OPE formulas, one can check that

$$\begin{aligned} C_{n,0}^1 &= 0, \quad C_{n,1}^1 = \frac{(3+2\ell)(4+4\ell+n+2\ell n)}{4(2+n)(3+n)}, \quad C_{n,2}^1 = -\frac{3(3+2\ell)}{4(n+2)}, \\ C_{n,3}^1 &= \frac{3+2\ell}{2n+2} - \frac{3+10\ell+6n+4\ell n}{12(1+n)}, \\ C_{n,4}^1 &= -\frac{(3+2\ell)(5+6\ell)}{48} - \frac{3+10\ell+6n+4\ell n}{48}, \\ C_{n,j}^1 &= -\frac{(3+10\ell+6n+4\ell n)n!}{2(n+4-j)!j!}, \quad j = 5, \dots, n, \\ C_{n,n+1}^1 &= -\frac{3+10\ell+6n+4\ell n}{12(1+n)}, \quad C_{n,n+2}^1 = -\frac{3+10\ell+6n+4\ell n}{4(1+n)(2+n)}, \\ C_{n,n+3}^1 &= -\frac{3+10\ell+6n+4\ell n}{2(1+n)(2+n)(3+n)}, \quad C_{n,n+4}^1 = -\frac{3(5+2\ell+2n)}{(1+n)(2+n)(3+n)(4+n)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} C_{n,0}^2 &= \frac{18-4\ell+3n+2\ell n}{2(1+n)(2+n)(3+n)(4+n)}, \\ C_{n,j}^2 &= -\frac{6(n!)}{(n+4-j)!(j!)}, \quad j = 1, \dots, n \end{aligned}$$

$$C_{n,n+1}^2 = -\frac{1}{1+n}, \quad C_{n,n+2}^2 = -\frac{3}{(1+n)(2+n)}, \quad C_{n,n+3}^2 = -\frac{6}{(1+n)(2+n)(3+n)},$$

$$C_{n,n+4}^2 = -\frac{-15-2\ell-6n+4\ell n}{2(1+n)(2+n)(3+n)(4+n)}.$$

Next, we have

$$C_{n,0}^3 = -\frac{18-4\ell+3n+2\ell n}{2(1+n)(2+n)(3+n)(4+n)}, \quad C_{n,1}^3 = \frac{6}{(1+n)(2+n)(3+n)},$$

$$C_{n,2}^3 = \frac{3}{(1+n)(2+n)}, \quad C_{n,3}^3 = \frac{1}{1+n}, \quad C_{n,4}^3 = -\frac{5}{16} - \frac{\ell}{24},$$

$$C_{n,i}^3 = 0, \quad 5 \leq i \leq n+4.$$

Finally, we have

$$C_{n,0}^4 = 0, \quad C_{n,1}^4 = -\frac{3(4+n)}{4(2+n)(3+n)} + \frac{\ell(-30-10n+n^2)}{6(2+n)(3+n)} - \frac{\ell^2}{3},$$

$$C_{n,2}^4 = \frac{3(6+4\ell-n+2\ell n)}{8(2+n)}, \quad C_{n,3}^4 = -1,$$

$$C_{n,j}^4 = 0, \quad 4 \leq j \leq n+2,$$

$$C_{n,n+3}^4 = -\frac{(-1)^n(3+2\ell)}{4(3+n)},$$

$$C_{n,n+4}^4 = \frac{(-1)^n(45+40\ell+12\ell^2+25n+47\ell n+22\ell^2 n+14\ell n^2+12\ell^2 n^2-n^3+\ell n^3+2\ell^2 n^3)}{2(1+n)(2+n)(3+n)(4+n)}.$$

Then $C_n(: U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} :) = \sum_{i=1}^4 \sum_{j=0}^{n+4} C_{n,j}^i$. Considerable cancellation occurs, and this turns out to be given by the simple expression

$$C_n(: U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} :) = \frac{n(n+7)(n+2)!}{4!(n+4)!} \ell(2\ell-1),$$

which is clearly nonzero for $\ell \neq 0, \frac{1}{2}$. It follows that for all n , we have a relation

$$: U_{0,0}U_{1,n} : - : U_{0,n}U_{1,0} := (-1)^n \frac{n(n+7)(n+2)!}{4!(n+4)!} \ell(2\ell-1) U_{0,n+4} + \dots,$$

where the remaining terms depend on $\{J, T, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}\}$ and their derivatives. \square

6. THE STRUCTURE OF $\text{COM}(\mathcal{H}, \mathcal{W}^\ell)$

Let $\mathcal{H} \subset \mathcal{W}^\ell$ denote the copy of the Heisenberg vertex algebra generated by J , and let \mathcal{C}^ℓ denote the commutant $\text{Com}(\mathcal{H}, \mathcal{W}^\ell)$. Clearly we have

$$(\mathcal{W}^\ell)^{U(1)} \cong \mathcal{H} \otimes \mathcal{C}^\ell$$

and \mathcal{C}^ℓ has a Virasoro element

$$T^{\mathcal{C}} = T - T^{\mathcal{H}}, \quad T^{\mathcal{H}} = \frac{3}{4\ell} : JJ : .$$

Theorem 6.1. For $0 \leq i \leq 4$, and $\ell \neq 0$, there exist correction terms $\omega_i \in (\mathcal{W}^\ell)_{(2)}^{U(1)}$ such that $U_i^C = U_{0,i} + \omega_i$ lies in \mathcal{C}^ℓ . Therefore \mathcal{C}^ℓ has a minimal strong generating set $\{T^C, U_i^C \mid 0 \leq i \leq 4\}$, and is therefore of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for $\ell \neq 0, \frac{1}{2}$. Moreover, when ℓ is also not a root of any the following quadratic polynomials

$$60x^2 - 104x - 51, \quad 28x^2 - 104x - 107, \quad 4x^2 - 20x - 23, \quad 24x^2 - 22x + 9,$$

$$84x^2 - 220x - 183, \quad 6x^2 - 29x - 33, \quad 60x^2 - 52x + 27, \quad 132x^2 - 832x - 1017,$$

there exist a unique correction term $\omega_i \in (\mathcal{W}^\ell)_{(2)}^{U(1)}$ such that $U_i^C = U_{0,i} + \omega_i$ lies in \mathcal{C}^ℓ and is primary of weight $i + 3$ with respect to T^C .

Proof. The existence and uniqueness of the terms ω_i satisfying the above properties is a straightforward computer calculation. In the explicit formulas for $U_{0,i}^C$ for $i = 1, 2, 3, 4$, all coefficients of normally ordered monomials in $J, T, U_{0,i}$ are rational functions of ℓ , whose denominator only contains the above quadratic factors together with ℓ . \square

An alternative realization of \mathcal{C}^ℓ . Feigin and Semikhatov [FS] constructed a family of vertex algebras which they call $\mathcal{W}_n^{(2)}$ -algebras, and the case $n = 3$ is the Bershadsky-Polyakov algebra. Their first construction is as a centralizer of some quantum super group action and the second is in terms of a coset associated to $V_k(\mathfrak{sl}(n|1))$, that is the affine vertex super algebra of $\mathfrak{sl}(n|1)$ at level k . One consequence is

Theorem 6.2. Let k and k' be complex numbers related by $(k + 2)(k' + 2) = 1$. Then for generic values of k ,

$$\text{Com}(V_{k'}(\mathfrak{gl}_3), V_{k'}(\mathfrak{sl}(3|1))) \cong \mathcal{C}^\ell, \quad \ell = k + \frac{3}{2}.$$

Proof. Feigin and Semikhatov consider $V_{k'}(\mathfrak{sl}(3|1)) \otimes V_L$, where V_L is a certain rank one lattice VOA. They find that the coset by $V_{k'}(\mathfrak{sl}_3) \otimes \mathcal{H}$ for a certain rank one Heisenberg sub VOA \mathcal{H} contains as subalgebra a VOA with same OPE algebra as \mathcal{W}^ℓ for $\ell = k + \frac{3}{2}$. Since the Bershadsky-Polyakov algebra is generically simple [ArII, ArI] the two must be isomorphic for generic level k . Furthermore,

$$\text{Com}(V_{k'}(\mathfrak{gl}_3), V_{k'}(\mathfrak{sl}(3|1))) \cong \text{Com}(V_{k'}(\mathfrak{gl}_3) \otimes \mathcal{H}, V_{k'}(\mathfrak{sl}(3|1)) \otimes V_L)$$

since the Heisenberg coset of a rank one lattice VOA is one-dimensional. In [CLI] we prove that the left-hand side for generic level is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$. This completes the proof since a vertex subalgebra $\mathcal{W} \subset \mathcal{V}$ with same strong generating set as \mathcal{V} implies $\mathcal{W} = \mathcal{V}$. \square

7. THE STRUCTURE OF $\text{COM}(\mathcal{H}, \mathcal{W}_\ell)$

For $\ell \neq -\frac{3}{2}$, let \mathcal{I}_ℓ denote the maximal proper ideal of \mathcal{W}^ℓ , and let \mathcal{W}_ℓ denote the simple quotient $\mathcal{W}^\ell/\mathcal{I}_\ell$. Let \mathcal{C}_ℓ denote the quotient $\mathcal{C}^\ell/(\mathcal{I}_\ell \cap \mathcal{C}^\ell)$.

Lemma 7.1. $\mathcal{C}_\ell = \text{Com}(\mathcal{H}, \mathcal{W}_\ell)$ where $\mathcal{H} \subset \mathcal{W}_\ell$ is the Heisenberg algebra generated by J . In particular, \mathcal{C}_ℓ is simple.

Proof. This is immediate from the fact that \mathcal{W}^ℓ is completely reducible as an \mathcal{H} -module, and \mathcal{I}_ℓ is an \mathcal{H} -submodule of \mathcal{W}^ℓ . \square

As shown by Arakawa [ArII], in the case $\ell = 1, 2, \dots$, \mathcal{I}_ℓ is generated by $(G^\pm)^{2\ell+1}$, and \mathcal{W}_ℓ is C_2 -cofinite and rational.

Theorem 7.2. *For all positive integers ℓ , \mathcal{W}_ℓ contains a rank one lattice vertex algebra V_L with generators J , $(G^\pm)^{2\ell}$, where $L = \sqrt{6\ell}\mathbb{Z}$. Moreover, $\text{Com}(\text{Com}(\mathcal{H}, \mathcal{W}_\ell)) = V_L$ so V_L and \mathcal{C}_ℓ form a Howe pair inside \mathcal{W}_ℓ .*

Proof. In \mathcal{W}^ℓ we have

$$J(z)(G^\pm)^{2\ell}(w) \sim \pm 2\ell(G^\pm)^{2\ell}(w)(z-w)^{-1},$$

so $(G^\pm)^{2\ell}$ are both primary with respect to J in \mathcal{W}^ℓ , and similarly in \mathcal{W}_ℓ . It suffices to show that $(G^\pm)^{2\ell}$ both lie in the double commutant $\text{Com}(\text{Com}(\mathcal{H}, \mathcal{W}_\ell))$. Equivalently, we need to show that $(G^\pm)^{2\ell}$ both commute with $T^C = T - T^{\mathcal{H}}$ in \mathcal{W}_ℓ . We have the following OPE relations in \mathcal{W}^ℓ .

$$T_0^{\mathcal{H}}(G^\pm)^{2\ell} = 3\ell(G^\pm)^{2\ell}, \quad T_0(G^\pm)^{2\ell} = 3\ell(G^\pm)^{2\ell},$$

so $T_0^C(G^\pm)^{2\ell} = 0$ in \mathcal{W}^ℓ . Similarly,

$$T_{-1}^{\mathcal{H}}(G^\pm)^{2\ell} = \pm 3 : J(G^\pm)^{2\ell} :, \quad T_{-1}(G^\pm)^{2\ell} = \partial(G^\pm)^{2\ell},$$

so $T_{-1}^C(G^\pm)^{2\ell} = \partial(G^\pm)^{2\ell} \mp 3 : J(G^\pm)^{2\ell} :.$ Finally,

$$G^- \circ_1 (G^+)^{2\ell+1} = \frac{(2\ell+1)^2}{2} (\partial((G^+)^{2\ell}) - 3 : J(G^+)^{2\ell} :),$$

so $\partial((G^+)^{2\ell}) - 3 : J(G^+)^{2\ell} : = 0$ in \mathcal{W}_ℓ . Similarly,

$$G^+ \circ_1 (G^-)^{2\ell+1} = -\frac{(2\ell+1)^2}{2} (\partial((G^-)^{2\ell}) + 3 : J(G^-)^{2\ell} :),$$

so $\partial((G^-)^{2\ell}) + 3 : J(G^-)^{2\ell} : = 0$ in \mathcal{W}_ℓ . It is straightforward to check that for all $k \geq 1$, $T_k(G^\pm)^{2\ell} = 0$ and $T_k^{\mathcal{H}}(G^\pm)^{2\ell} = 0$ in \mathcal{W}^ℓ , and hence in \mathcal{W}_ℓ as well. It follows that $(G^\pm)^{2\ell}$ commutes with T^C in \mathcal{W}_ℓ . \square

Corollary 7.3. *For all positive integers ℓ , \mathcal{C}_ℓ is C_2 -cofinite.*

Proof. This follows from Corollary 2 of [Mil]. \square

Since \mathcal{C}^ℓ has strong generators $\{T^C, U_{0,0}^C, U_{0,1}^C, U_{0,2}^C, U_{0,3}^C, U_{0,4}^C\}$ for $\ell \neq 0, \frac{1}{2}$, it follows that for $\ell = 1, 2, \dots$, \mathcal{C}_ℓ is generated by the images of these fields in \mathcal{W}_ℓ , which we also denote by $\{T^C, U_{0,0}^C, U_{0,1}^C, U_{0,2}^C, U_{0,3}^C, U_{0,4}^C\}$ by abuse of notation. For $\ell \geq 4$ this is a minimal strong generating set, but for $\ell = 1, 2, 3$ there are additional decoupling relations and a smaller generating set suffices.

Recently, Kawasetsu proved the following result in the case $\ell = 1$ [Ka].

Theorem 7.4. (Kawasetsu) \mathcal{C}_1 is isomorphic to the rational Virasoro vertex algebra of central charge $c = -\frac{3}{5}$ generated by T^C .

An alternative proof of this result can be given as follows. One can check by computer calculation that for $i = 1, 2, 3, 4$, $U_{0,i}^C \in \mathcal{C}^1$ lies in $\mathcal{I}_1 \cap \mathcal{C}^1$, where $\mathcal{I}_1 \subset \mathcal{W}^1$ denotes the maximal ideal. In particular, $U_{0,i}^C = 0$ as an element of \mathcal{C}_1 , so \mathcal{C}_1 is strongly generated by T^C . Being simple it must be the rational Virasoro algebra with $c = -\frac{3}{5}$.

Similarly, in the case $\ell = 2$, one can check that $U_{0,2}^C$ vanishes in \mathcal{C}_2 , and that $U_{0,3}^C$ and $U_{0,4}^C$ can be expressed as normally ordered polynomials in $\{T^C, U_{0,0}^C, U_{0,1}^C\}$. Since there

are no relations of weight less than 5, it follows that \mathcal{C}_2 is of type $\mathcal{W}(2, 3, 4)$ and has a minimal strong generating set $\{T^C, U_{0,0}^C, U_{0,1}^C\}$. In the case $\ell = 3$, one checks that $U_{0,4}^C$ can be expressed as a normally ordered polynomial in $\{T^C, U_{0,i}^C \mid i = 0, 1, 2, 3\}$. Since there are no relations of weight less than 6, \mathcal{C}_3 is of type $\mathcal{W}(2, 3, 4, 5, 6)$. Finally, for $\ell > 3$, the element of $\mathcal{I}_\ell \cap \mathcal{C}^\ell$ of minimal weight is $(G^+_{\circ_1})^{2\ell+1}(G^-)^{2\ell+1}$, which has weight $2\ell + 1$. Therefore none of the generators of \mathcal{C}_ℓ decouple, so \mathcal{C}_ℓ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$.

In this paper our focus is on \mathcal{C}_ℓ when ℓ is a positive integer, but there are other interesting levels as well. For example,

Lemma 7.5. *Let $\mathcal{A}(3)$ be the rank three symplectic fermion algebra. Then*

$$\mathcal{A}(3)^{GL(3)} \cong \mathcal{C}_\ell, \quad \ell = -\frac{1}{2}.$$

Proof. $\mathcal{A}(3)^{GL(3)}$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ [CLII] and as an orbifold of a simple vertex algebra it is itself simple [DM]. Using the notion of deformable family of vertex algebras introduced in [CLIII] it has been proven [CLI] that

$$\lim_{k' \rightarrow \infty} \text{Com}(V_{k'}(\mathfrak{gl}_3), V_{k'}(\mathfrak{sl}(3|1))) \cong \mathcal{A}(3)^{GL(3)}.$$

By Theorem 6.2, both $\mathcal{A}(3)^{GL(3)}$ and $\mathcal{C}^{-\frac{1}{2}}$ have the same OPE algebra and hence their simple quotients must coincide. \square

8. \mathcal{W} -ALGEBRAS OF TYPE A

In this section, we prove our main result that for all $\ell \geq 1$, \mathcal{C}_ℓ is isomorphic to the principal, rational $\mathcal{W}(\widehat{\mathfrak{sl}}_{2\ell})$ -algebra with central charge $c = -3(2\ell - 1)^2/(2\ell + 3)$, which has level $(2\ell + 3)/(2\ell + 1) - 2\ell$. In the case $\ell = 2$, using the fact that \mathcal{C}_2 is of type $\mathcal{W}(2, 3, 4)$, one can verify by computer calculation that the OPE relations of the generators coincide with the OPE relations of $\mathcal{W}(\widehat{\mathfrak{sl}}_4)$, which appear explicitly in [CZh]. However, for $\ell > 2$ is not practical to check this directly, and a more conceptual approach is needed.

Let \mathcal{V}^ℓ be the universal $\mathcal{W}(\widehat{\mathfrak{sl}}_{2\ell})$ at level $(2\ell + 3)/(2\ell + 1) - 2\ell$ and let \mathcal{V}_ℓ its simple quotient. Its rationality has been proven [ArV]. Simple modules of \mathcal{V}_ℓ are labelled by integrable positive weights of $\widehat{\mathfrak{sl}}_{2\ell}$ at level 3. We denote the fundamental weights of $\widehat{\mathfrak{sl}}_{2\ell}$ by $\Lambda_0, \Lambda_1, \dots, \Lambda_{2\ell-1}$. By Theorem 4.3 of [FKW] (see also [AvE]) \mathcal{V}_ℓ has a simple current $\mathbb{L}_{3\Lambda_1}$ of order 2ℓ and both $\mathbb{L}_{3\Lambda_1}$ and its inverse $\mathbb{L}_{3\Lambda_{2\ell-1}}$ are generated as \mathcal{V}_ℓ modules by a single lowest-weight state of conformal dimension $\frac{3}{2} - \frac{3}{4\ell}$. Further the conformal dimension of the lowest-weight-state of both $\mathbb{L}_{3\Lambda_s}$ and $\mathbb{L}_{3\Lambda_{2\ell-s}}$ for $1 \leq s \leq \ell$ is $\frac{3s}{2} - \frac{3s^2}{4\ell}$. Let $L = \sqrt{6\ell}\mathbb{Z}$ and V_L the lattice vertex algebra associated to L . Then V_L has a simple current $V_{L+\frac{3}{\sqrt{6\ell}}}$ of order 2ℓ with lowest-weight state of conformal dimension $\frac{3}{4\ell}$. It follows that

$$(8.1) \quad \mathcal{B}_\ell := \bigoplus_{s=0}^{2\ell-1} \mathbb{L}_{3\Lambda_s} \otimes V_{L+\frac{3s}{\sqrt{6\ell}}}$$

is a simple current extension of $\mathcal{V}_\ell \otimes V_L$. A priori it is not clear that \mathcal{B}_ℓ is a VOA, but one purpose of [CKL] was to develop a framework (extending earlier works [C, LaLaY]) that decides this type of question.

Lemma 8.1. *\mathcal{B}_ℓ can be given the structure of a VOA.*

Proof. This follows from [CKL]: A simple current extension is either a VOA or a super VOA by Theorem 3.12 of [CKL] and if the tensor category of the underlying VOA is modular the parity question is decided by the categorical dimension of the generating simple current by Corollary 2.8 of [CKL]. This follows directly from Huang's work on the Verlinde formula [Hu2] and is explained in [CKL]. Since a lattice VOA is a unitary VOA any of its simple currents has categorical dimension one, so we have to determine the categorical dimension of $\mathbb{L}_{3\Lambda_1}$. By Proposition 4.2 together with Proposition 1.1 of [FKW] the categorical dimension is given by

$$\text{qdim}(\mathbb{L}_{3\Lambda_1}) = \frac{S_{3\Lambda_1, 3\Lambda_0}}{S_{3\Lambda_0, 3\Lambda_0}} = \epsilon(3\Lambda_1) = (-1)^{2(\Lambda_1|\rho)}$$

but ($n = 2\ell$ and $\alpha_1, \dots, \alpha_{n-1}$ are the simple roots of \mathfrak{sl}_n)

$$\Lambda_1 = \frac{1}{n}((n-1)\alpha_1 + (n-2)\alpha_2 + \dots + \alpha_{n-1})$$

so that

$$(\Lambda_1|\rho) = \frac{1}{n} \frac{n(n-1)}{2} = \frac{(n-1)}{2}$$

it follows that $\text{qdim}(\mathbb{L}_{3\Lambda_1}) = (-1)^{(n-1)} = -1$ since $n = 2\ell$ is even. Odd quantum dimension and odd twist (half-integer conformal dimension) of a simple current imply that the extension is a VOA by Corollary 2.8 together with Theorems 3.9 and 3.12 of [CKL]. \square

We would like to show that $\mathcal{B}_\ell \cong \mathcal{W}_\ell$. Denote the vertex operator associated to the vector of lowest conformal dimension of $\mathbb{L}_{3\Lambda_1} \otimes V_{L+\frac{3}{\sqrt{6\ell}}}$ by F^+ and the one associated to the vector of lowest conformal dimension of $\mathbb{L}_{3\Lambda_{2\ell-1}} \otimes V_{L+\frac{6\ell-3}{\sqrt{6\ell}}}$ by F^- .

Lemma 8.2. *The subalgebra of \mathcal{B}_ℓ generated under operator product algebra by F^\pm has the same OPE algebra as \mathcal{W}^ℓ .*

Proof. Let

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad G^\pm(z) = \sum_{n \in \mathbb{Z}} G_n^\pm z^{-n-3/2}, \quad T^c(z) = \sum_{n \in \mathbb{Z}} L_n^c z^{-n-2}$$

be the mode expansion of the corresponding fields of \mathcal{W}^ℓ . Here T^c denotes the Virasoro field of \mathcal{C}^ℓ . It is related to the Virasoro field of \mathcal{W}^ℓ by $T = T^c + \frac{1}{2}(J)^2$ where $(J)^2$ denotes the normal ordered product of J with itself. We can thus read off the commutation relations of the mode algebra from the operator product algebra of the fields given in section 3, see also [ArV]. These are

$$\begin{aligned} [J_m, J_n] &= \frac{2\ell}{3} \delta_{n+m, 0} \\ [J_m, G_n^\pm] &= \pm G_{n+m}^\pm \\ [G^+, G^-] &= \frac{9}{4\ell} \left(\ell - \frac{1}{2} \right) (J^2)_{n+m} + \frac{3}{2} \left(\ell - \frac{1}{2} \right) (m-n-1) J_{m+n} - \left(\ell + \frac{3}{2} \right) L_{n+m}^c + \\ &\quad \ell \left(\ell - \frac{1}{2} \right) m(m+1) \delta_{n+m, 0}. \end{aligned}$$

Here we omit the Virasoro algebra relations as they are standard. We want to show that the modes of the fields F^\pm together with those of the Heisenberg field H of V_L and the Virasoro field S of V_ℓ have the same commutation relations. Let

$$H(z) = \sum_{n \in \mathbb{Z}} H_n z^{-n-1}, \quad F^\pm(z) = \sum_{n \in \mathbb{Z}} F_n^\pm z^{-n-3/2}, \quad S(z) = \sum_{n \in \mathbb{Z}} S_n z^{-n-2}$$

be the mode expansion of the corresponding fields. We can normalize H , such that

$$H(z)F^\pm(w) \sim \pm(z-w)^{-1}F^\pm(w),$$

with this normalization the operator product of H with itself is

$$H(z)H(w) \sim \frac{2\ell}{3}(z-w)^{-2}.$$

The following commutation relations follow

$$[H_m, H_n] = \frac{2\ell}{3}\delta_{n+m,0}, \quad [H_m, F_n^\pm] = \pm F_{n+m}^\pm.$$

Since the state of lowest conformal dimension of both $\mathbb{L}_{3\Lambda_2} \otimes V_{L+\frac{6}{\sqrt{6\ell}}}$ and $\mathbb{L}_{3\Lambda_{2\ell-2}} \otimes V_{L+\frac{6\ell-6}{\sqrt{6\ell}}}$ has dimension three the operator product of F^+ with itself as well as the one of F^- with itself is regular. The operator products involving the Virasoro fields are of course the same as the corresponding ones of \mathcal{W}^ℓ . It remains the operator product of F^+ with F^- . The known operator product algebra of lattice VOAs leaves us with two free coefficients a, b for the moment:

$$F^+(z)F^-(w) \sim (z-w)^{-3}a \left(1 + \frac{3}{2\ell}(z-w)H + (z-w)^2 \left(\frac{1}{2} \left(\frac{3}{2\ell} \right)^2 (H^2) + \frac{1}{2} \frac{3}{2\ell} \partial H \right) \right) + b(z-w)^{-1}S.$$

We will see that these coefficients are uniquely determined by the Jacobi identity for the Lie algebra of the modes. First $a \neq 0$ by Proposition 4.1. of [CKL]. We can now rescale the fields F^\pm so that $a = 2\ell(\ell - \frac{1}{2})$. It remains to show that $b = -(\ell + \frac{3}{2})$. Set $b = -(\ell + \frac{3}{2}) + \lambda$. Then using that the Jacobi identity of \mathcal{W}^ℓ holds we get

$$0 = [R_n, [F_m^+, F_r^-]] + [F_m^+, [F_r^-, R_n]] + [F_r^-, [R_n, F_m^+]] = \lambda \frac{c_\ell}{12} (n^3 - n) \delta_{n+m+r,0}.$$

Hence $\lambda = 0$. We thus have shown that the mode algebra of \mathcal{W}^ℓ and the VOA generated by F^\pm coincide, hence their operator product algebras coincide. \square

Theorem 8.3. *The coset \mathcal{C}_ℓ is isomorphic to the simple rational \mathcal{W} -algebra of $\mathfrak{sl}_{2\ell}$, $\mathcal{W}(\mathfrak{sl}_{2\ell})$ at level $(2\ell + 3)/(2\ell + 1) - 2\ell$.*

Proof. By the previous Lemma \mathcal{B}_ℓ is a module for \mathcal{W}^ℓ and it contains a homomorphic image of \mathcal{W}^ℓ and hence the generators of weight 2, 3, 4, 5, 6, 7 that strongly generate the corresponding homomorphic image of \mathcal{C}^ℓ . But under operator product algebra already the dimension two and three fields of \mathcal{V}_ℓ generate \mathcal{V}_ℓ hence the homomorphic image of \mathcal{C}^ℓ must be \mathcal{V}_ℓ . Since it is simple, we have $\mathcal{C}_\ell \cong \mathcal{V}_\ell$. \square

Corollary 8.4. *\mathcal{W}_ℓ is rational and C_2 -cofinite.*

Proof. \mathcal{W}_ℓ is a simple current extension of a rational and C_2 -cofinite VOA of CFT-type, hence itself has these properties [Y]. \square

In other words, we have found a very different proof of the main result of [ArII].

Corollary 8.5. *The simple rational \mathcal{W} -algebra of $\mathfrak{sl}_{2\ell}$, $\mathcal{W}(\mathfrak{sl}_{2\ell})$ at level $k = (2\ell + 3)/(2\ell + 1) - 2\ell$ is a \mathcal{W} -algebra of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for $\ell > 3$. For $\ell = 1$ it is of type $\mathcal{W}(2)$, for $\ell = 2$ it is of type $\mathcal{W}(2, 3, 4)$ and for $\ell = 3$ it is of type $\mathcal{W}(2, 3, 4, 5, 6)$.*

Proof. We have just proven that the simple rational \mathcal{W} -algebra of $\mathfrak{sl}_{2\ell}$, $\mathcal{W}(\mathfrak{sl}_{2\ell})$ at level $k = (2\ell + 3)/(2\ell + 1) - 2\ell$ is isomorphic to \mathcal{C}_ℓ , but the latter is of the corresponding type as proven in the last section. \square

9. A CHARACTER IDENTITY

We first recall the known characters of \mathcal{W}_ℓ and \mathcal{V}_ℓ and their simple currents $\mathbb{L}_{3\Lambda_3}$. In this section $q = e(\tau)$, $z = e(u)$ and τ in the upper half of the complex plane, and $u \in \mathbb{C}$.

Characters of rational Bershadsky-Polyakov algebras. Recall that $\ell = k + \frac{3}{2}$. Let $\mathcal{W}_\ell = \mathcal{W}_k(\mathfrak{sl}_3, f_\theta)$, the simple Bershadsky-Polyakov algebra at level k . We have

$$\mathcal{W}_\ell = H_{DS, f_\theta}^0(L_k(\mathfrak{sl}_3))$$

provided that $k \notin \mathbb{Z}_{\geq 0}$ ([ArI]). Here $H_{DS, f_\theta}^0(?)$ is the Drinfeld-Sokolov reduction with respect to the nilpotent element f_θ ([KRW]) and $L_k(\mathfrak{sl}_3)$ is the simple affine vertex algebra associate with \mathfrak{sl}_3 at level k . This gives

$$\begin{aligned} \text{ch } \mathcal{W}_\ell &:= \text{tr}_{\mathcal{W}_\ell} (z^{J_0} q^{L_0 - \frac{c}{24}}) \\ &= q^{-\frac{c}{24}} \left(\text{ch } L_k(\mathfrak{sl}_3) \prod_{\alpha \in \hat{\Delta}_+^e} (1 - e^{-\alpha}) \frac{1}{\prod_{n \geq 0} (1 - e^{-\alpha_1 - n\delta})(1 - e^{-\alpha_2 - n\delta})} \right) \Bigg|_{\substack{e^{-\delta} \rightarrow q, \\ e^{\alpha_1} \rightarrow z^{-1} q^{-1/2}, \\ e^{\alpha_2} \rightarrow z q^{-1/2}}} \end{aligned}$$

On the other hand if k is admissible we have [KWI]

$$\text{ch } L(\lambda) = \frac{\sum_{w \in \widehat{W}(k\Lambda_0)} e^{w \circ k\Lambda_0}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \widehat{\mathfrak{g}}_\alpha}},$$

where $\widehat{W}(k\Lambda_0)$ is the integrable Weyl group of $k\Lambda_0$ generated by reflections s_α such that $\langle k\Lambda_0, \alpha^\vee \rangle \in \mathbb{Z}$. The nontrivial rational levels are $k = p/2 - 3 = \ell - 3/2$ with $p = 5, 7, 9, \dots$ and thus $\ell = 1, 2, 3, \dots$. Then $k\Lambda_0$ is admissible and

$$\widehat{W}(k\Lambda_0) = W \ltimes 2Q^\vee (\cong \widehat{W} = W \ltimes Q^\vee),$$

where $W = \mathfrak{S}_3$ and Q^\vee is the coroot lattice of $\mathfrak{g} = \mathfrak{sl}_3$. Thus

$$\text{ch } \mathcal{W}_\ell = \frac{1}{\eta(\tau) \vartheta(\tau, u)} \left(\sum_{w \in W \ltimes 2Q^\vee} e^{w \circ k\Lambda_0} \right) \Bigg|_{\substack{e^{-\delta} \rightarrow q, \\ e^{\alpha_1} \rightarrow z^{-1} q^{-1/2}, \\ e^{\alpha_2} \rightarrow z q^{-1/2}}}$$

where

$$\vartheta(\tau, u) = \prod_{i \geq 1} (1 - q^i) \prod_{j \geq 1} (1 - z q^{1/2+j}) (1 - z^{-1} q^{1/2+j}), \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{i \geq 1} (1 - q^i).$$

Characters of simple modules over $\mathcal{W}_k(\mathfrak{sl}_n)$. Let $\mathcal{W}_k(\mathfrak{sl}_n)$ be the simple principal \mathcal{W} -algebra associated with \mathfrak{sl}_n at level k . We consider the special case $k + n = \frac{n+p}{n+1}$ (for us $p = 3$). Then [FKW, ArIII, ArV],

$$\widehat{P}_+^p \xrightarrow{\sim} \{\text{simple } \mathcal{W}^k(\mathfrak{sl}_n)\text{-modules}\}, \quad \lambda \mapsto \mathbb{L}_\lambda := H_{DS, f_{prin}}^0(L(\lambda + (k-p)\Lambda_0)),$$

where \widehat{P}_+^p is the set of dominant integrable weight of $\mathfrak{g} = \mathfrak{sl}_n$ level p , and

$$\text{ch } \mathbb{L}_\lambda = \frac{1}{\eta(\tau)^{n-1}} \sum_{w \in \widehat{W}} \epsilon(w) q^{\frac{(n+p)(n+1)}{2} \left| \frac{w(\lambda+\rho)}{n+p} - \frac{\rho}{n+1} \right|^2}.$$

Here \widehat{W} is the affine Weyl group of $\widehat{\mathfrak{sl}}_n$.

Lattice VOA characters. We also need the characters of the lattice VOA V_L , they are just quotients of theta and eta functions:

$$\text{ch}[V_{L+\frac{s}{\sqrt{6\ell}}}] (\tau, u) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(\sqrt{6\ell}n + \frac{s}{\sqrt{6\ell}})^2} z^{3\ell(n + \frac{s}{3\ell})} =: \frac{\theta_s(\tau, u)}{\eta(\tau)}.$$

Decomposing the character of \mathcal{W}_ℓ . We can now put everything together to decompose the character of the VOA \mathcal{W}_ℓ .

Theorem 9.1. *The character of the simple current U^s of the type A \mathcal{W} -algebra \mathcal{V}_ℓ is in terms of the character of the simple Bershadsky-Polyakov algebra \mathcal{W}_ℓ*

$$\text{ch}[\mathbb{L}_{3\Lambda_s}] (\tau) = \frac{1}{2\ell} \frac{\eta(\tau)}{\theta_s(\tau, 0)} \sum_{t=0}^{2\ell-1} e^{-2\pi i t s} \text{ch}[\mathcal{W}_\ell] \left(\tau, \frac{t}{3\ell} \right).$$

Proof. By Theorem 8.3 together with equation (8.1) we have that

$$\text{ch}[\mathcal{W}_\ell] (\tau, u) = \sum_{s=0}^{2\ell-1} \text{ch}[\mathbb{L}_{3\Lambda_s}] (\tau) \text{ch}[V_{L+\frac{s}{\sqrt{6\ell}}}] (\tau, u).$$

The theorem now follows directly by using $e^{-2\pi i t s} \theta_{s'}(\tau, u + \frac{t}{3\ell}) = e^{-2\pi i t (s-s')} \theta_{s'}(\tau, u)$ which implies

$$\frac{1}{2\ell} \sum_{t=0}^{2\ell-1} e^{-2\pi i t s} \theta_{s'}(\tau, u + \frac{t}{3\ell}) = \frac{1}{2\ell} \sum_{t=0}^{2\ell-1} e^{-2\pi i t (s-s')} \theta_{s'}(\tau, u) = \delta_{s,s'} \theta_s(\tau, u).$$

□

Plugging in the explicit form of the characters, we get the identities of different lattice theta functions:

Corollary 9.2. *For $s = 0, \dots, 2\ell - 1$ the following identity holds*

$$\sum_{w \in \widehat{W}} \epsilon(w) q^{\frac{(n+p)(n+1)}{2} \left| \frac{w(3\Lambda_s + \rho)}{n+p} - \frac{\rho}{n+1} \right|^2} = \frac{1}{2\ell} \frac{\eta(\tau)^{n-1}}{\theta_s(\tau, 0)} \sum_{t=0}^{2\ell-1} \frac{e^{-2\pi i t s}}{\vartheta\left(\tau, \frac{t}{3\ell}\right)} \left(\sum_{w \in W \times 2Q^\vee} e^{w \circ k \Lambda_0} \right) \Bigg|_{\substack{e^{-\delta} \rightarrow q, \\ e^{\alpha_1} \rightarrow e^{-\frac{2\pi i t}{3\ell}} q^{-1/2}, \\ e^{\alpha_2} \rightarrow e^{\frac{2\pi i t}{3\ell}} q^{-1/2}}}$$

where on the left-hand side we have the affine Weyl group of \mathfrak{sl}_n while on the right-hand side it is the Weyl group and coroot lattice of \mathfrak{sl}_3 .

10. A FAMILY OF SUPER \mathcal{W} -ALGEBRAS

We consider the tensor product $\mathcal{W}^\ell \otimes \mathcal{E}$, where \mathcal{E} is the rank one bc -system, or equivalently the lattice vertex algebra $V_{\mathbb{Z}}$. It has odd generators b, c satisfying

$$b(z)c(w) \sim (z-w)^{-1},$$

a Heisenberg element $J^\mathcal{E} = - : bc :$ and a Virasoro element $T^\mathcal{E} = - : b\partial c : + : (\partial b)c :$ of central charge 1, under which b, c are primary of weight $\frac{1}{2}$.

Let $J^{\text{diag}} = J + J^\mathcal{E} \in \mathcal{W}^\ell \otimes \mathcal{E}$ denote the diagonal Heisenberg element, which satisfies

$$J^{\text{diag}}(z)J^{\text{diag}}(w) \sim \frac{3+2\ell}{3}(z-w)^{-2}.$$

We will be interested in the commutant $\text{Com}(J^{\text{diag}}, \mathcal{W}^\ell \otimes \mathcal{E})$. As with our study of \mathcal{C}^ℓ , we begin by describing $(\mathcal{W}^\ell \otimes \mathcal{E})^{U(1)}$ where the action of $U(1)$ is generated by the zero mode of J^{diag} . It is easy to see that $(\mathcal{W}^\ell \otimes \mathcal{E})^{U(1)}$ has the following strong generating set

$$\{T, J, J^\mathcal{E}, U_{0,n}, \varphi_{0,n}^+, \varphi_{0,n}^- \mid n \geq 0\},$$

where $\varphi_{0,n}^+ = : b\partial^n G^+ :$ and $\varphi_{0,n}^- = : c\partial^n G^- :$.

Observe that $\varphi_{0,1}^\pm = \pm : \varphi_{0,0}^\pm J^\mathcal{E} :$, and $\varphi_{0,n}^\pm = \pm : \varphi_{0,n-1}^\pm J^\mathcal{E} :$, for $n \geq 0$, so the generators $\{\varphi_{0,n}^\pm \mid n > 0\}$ are not necessary. From the previous relations, we know that $\{U_{0,n} \mid n \geq 5\}$ are not necessary as long as $\ell \neq 0$ or $\frac{1}{2}$. In fact, we have the relation

$$\begin{aligned} U_{0,1} = & - : \varphi_{0,0}^+ \varphi_{0,0}^- : - : J^\mathcal{E} U_{0,0} : - \frac{3}{2} : JJ\partial J^\mathcal{E} : + \frac{3+2\ell}{4} : T\partial J^\mathcal{E} : + \frac{3-6\ell}{8} : (\partial J)\partial J^\mathcal{E} : \\ & + \frac{1-2\ell}{4} : J\partial^2 J^\mathcal{E} : + \frac{\ell-2\ell^2}{24} \partial^3 J^\mathcal{E} + \partial U_{0,0} + \frac{3}{2} : JJJ^\mathcal{E} J^\mathcal{E} : - \frac{3-6\ell}{8} : (\partial J)J^\mathcal{E} J^\mathcal{E} : \\ & - \frac{3-6\ell}{4} : J(\partial J^\mathcal{E})J^\mathcal{E} : - \frac{\ell-2\ell^2}{24} : JJJJ : + \frac{\ell-2\ell^2}{4} : (\partial J^\mathcal{E})J^\mathcal{E} J^\mathcal{E} : - \frac{\ell-2\ell^2}{6} : (\partial^2 J^\mathcal{E})J^\mathcal{E} : \\ & - \frac{\ell-2\ell^2}{8} : (\partial J^\mathcal{E})\partial J^\mathcal{E} : + \frac{1-2\ell}{4} : JJ^\mathcal{E} J^\mathcal{E} J^\mathcal{E} : - \frac{3+2\ell}{4} : TJ^\mathcal{E} J^\mathcal{E} :, \end{aligned}$$

so for all $\ell \neq -\frac{3}{2}$, $U_{0,1}$ is not necessary. We have similar relations

$$U_{0,n+1} = (-1)^n : (\varphi^+(J^\mathcal{E}(\dots(J^\mathcal{E}(\varphi^-))\dots))) : - : J^\mathcal{E} U_{0,n} + \dots$$

for all $n \geq 1$, where the remaining terms depend only on $J, T, J^\mathcal{E}$ and their derivatives, and have coefficients given by polynomials in ℓ . This proves

Lemma 10.1. *For all $\ell \neq -\frac{3}{2}$, $(\mathcal{W}^\ell \otimes \mathcal{E})^{U(1)}$ has a minimal strong generating set*

$$\{\varphi_{0,0}^\pm, U_{0,0}, T, J, J^\mathcal{E}\}.$$

Let \mathcal{D}^ℓ denote the commutant $\text{Com}(J^{\text{diag}}, \mathcal{W}^\ell \otimes \mathcal{E})$. Note that

$$(\mathcal{W}^\ell \otimes \mathcal{E})^{U(1)} \cong \mathcal{H} \otimes \mathcal{D}^\ell,$$

where \mathcal{H} is generated by J^{diag} . Also, \mathcal{D}^ℓ has a Virasoro element

$$T^{\mathcal{D}} = T + T^\mathcal{E} - \frac{3}{2(3+2\ell)} : J^{\text{diag}} J^{\text{diag}} :$$

and a Heisenberg field

$$J^{\mathcal{D}} = J - \frac{2\ell}{3}J^{\mathcal{E}}.$$

The odd weight 2 elements $\varphi^{\pm} = \varphi_{0,0}^{\pm}$, as well as the even weight 3 element $U^{\mathcal{D}} = U_{0,0}^{\mathcal{C}}$, are easily seen to lie in \mathcal{D}^{ℓ} as well. As in the previous section, Lemma 10.1 implies the following result.

Theorem 10.2. \mathcal{D}^{ℓ} has a minimal strong generating set

$$\{J^{\mathcal{D}}, T^{\mathcal{D}}, \varphi^{\pm}, U^{\mathcal{D}}\}.$$

The full OPE algebra of \mathcal{D}^{ℓ} is easy to compute from the explicit formulas for the generators, and it would be interesting to relate it to a super \mathcal{W} -algebra associated to $\mathfrak{sl}(3|1)$.

Next, we consider the simple vertex algebra $\mathcal{W}_{\ell} \otimes \mathcal{E}$, and we let \mathcal{D}_{ℓ} denote the commutant $\text{Com}(J^{\text{diag}}, \mathcal{W}_{\ell} \otimes \mathcal{E})$. By Corollary 2 of [Mil], \mathcal{D}_{ℓ} is C_2 -cofinite for all $\ell \in \mathbb{Z}_{>0}$.

Theorem 10.3. \mathcal{D}_{ℓ} is rational for all positive integers ℓ .

Proof. \mathcal{D}_{ℓ} is an extension of $\mathcal{C}_{\ell} \otimes V_{\sqrt{2\ell(2\ell+3)\mathbb{Z}}}$. The lattice vertex algebra is rational and for $\ell = 1, 2$ we also know that \mathcal{C}_{ℓ} is rational. The discriminant $\mathbb{Z}/2\ell(2\ell+3)\mathbb{Z}$ of the lattice $\sqrt{2\ell(2\ell+3)\mathbb{Z}}$ acts on \mathcal{D}_{ℓ} as automorphism subgroup. The orbifold is $\mathcal{C}_{\ell} \otimes V_{\sqrt{2\ell(2\ell+3)\mathbb{Z}}}$ and as a module for the orbifold

$$\mathcal{D}_{\ell} = \bigoplus_{n=0}^{2\ell(2\ell+3)-1} M_{\ell},$$

where each M_{ℓ} is a simple $\mathcal{C}_{\ell} \otimes V_{\sqrt{2\ell(2\ell+3)\mathbb{Z}}}$ -module [DM]. It is in fact also C_1 -cofinite as the orbifold is C_2 -cofinite, hence Proposition 20 of [MilII] implies that each M_{ℓ} is a simple current. We thus have a simple current extension of a rational, C_2 -cofinite vertex algebra of CFT-type and hence by [Y] the extension is also rational. \square

Remark 10.4. \mathcal{D}_{ℓ} is an example of a rational, C_2 -cofinite vertex superalgebra whose conformal weight grading is by \mathbb{N} , not $\frac{1}{2}\mathbb{N}$.

11. APPENDIX

Here we give the explicit relation of minimal weight 8 among the generators of $(\mathcal{W}^{\ell})^{U(1)}$.

$$\begin{aligned} & : U_{0,0}U_{1,1} : - : U_{0,1}U_{1,0} : - \frac{3}{4} : (\partial^4 J)JJJ : + 3 : (\partial^3 J)(\partial J)JJ : - \frac{9}{4} : (\partial^2 J)(\partial^2 J)JJ : \\ & + 18 : (\partial^2 J)(\partial J)(\partial J)J : - \frac{3}{32}(-1+2\ell) : (\partial^5 J)JJ : - \frac{3}{16}(-1+2\ell) : (\partial^4 J)(\partial J)J : \\ & - \frac{3}{4}(1-2\ell) : (\partial^3 J)(\partial J)\partial J : + \frac{45}{16}(-1+2\ell) : (\partial^2 J)(\partial^2 J)\partial J : + \frac{1}{16}(3+2\ell) : (\partial^4 T)JJ : \\ & - \left(\frac{3}{2} + \ell\right) : (\partial^3 T)(\partial J)J : - (3/2 + \ell) : (\partial T)(\partial^3 J)J : - \frac{3}{2}(3+2\ell) : (\partial T)(\partial^2 J)\partial J : \\ & + \frac{1}{8}(3+2\ell) : T\partial^4 J)J : + \frac{1}{2}(3+2\ell) : T\partial^3 J)\partial J : + \frac{3}{8}(3+2\ell) : T(\partial^2 J)\partial^2 J : \\ & - \frac{27-36\ell+28\ell^2}{960} : (\partial^6 J)J : - \frac{93-84\ell+52\ell^2}{640} : (\partial^5 J)\partial J : - \frac{3+\ell+2\ell^2}{8} : (\partial^4 J)\partial^2 J : \end{aligned}$$

$$\begin{aligned}
& -\frac{3+\ell+2\ell^2}{12} : (\partial^3 J)\partial^3 J : -\frac{3-4\ell-4\ell^2}{160} : (\partial^5 T)J : +\frac{3-4\ell-4\ell^2}{64} : (\partial^4 T)\partial J : \\
& +\frac{3-4\ell-4\ell^2}{16} : (\partial^3 T)\partial^2 J : -\frac{3-4\ell-4\ell^2}{320} : T\partial^5 J : -\frac{9+12\ell+4\ell^2}{96} : (\partial^4 T)T : \\
& +\frac{9+12\ell+4\ell^2}{24} : (\partial^3 T)\partial T : +\frac{3-13\ell+16\ell^2-4\ell^3}{2240}\partial^7 J -\frac{3\ell-4\ell^2-4\ell^3}{1440}\partial^6 T \\
& - : (\partial^3 J)JU_{0,0} : -3 : (\partial^2 J)(\partial J)U_{0,0} : +3 : (\partial^2 J)JU_{0,1} : +3 : (\partial J)(\partial J)U_{0,1} : \\
& +3 : (\partial J)J\partial^2 U_{0,0} : -6 : (\partial J)J\partial U_{0,1} : +3 : (\partial J)JU_{0,2} : + : JJ\partial^3 U_{0,0} : -\frac{3}{2} : JJ\partial^2 U_{0,1} : \\
& +\frac{3}{2} : JJ\partial U_{0,2} : - : JJU_{0,3} : -\left(\frac{7}{8} + \frac{\ell}{4}\right) : (\partial^3 J)U_{0,1} : -\frac{3}{8}(3+2\ell) : (\partial^2 J)\partial^2 U_{0,0} : \\
& -\left(\frac{9}{4} + \frac{3\ell}{2}\right) : (\partial^2 J)\partial U_{0,1} : +\left(\frac{27}{8} + \frac{9\ell}{4}\right) : (\partial^2 J)U_{0,2} : -\frac{3}{4}(3+2\ell) : (\partial J)\partial^3 U_{0,0} : \\
& +\left(\frac{15}{8} + \frac{9\ell}{4}\right) : (\partial J)\partial^2 U_{0,1} : +\left(\frac{3}{8} - \frac{3\ell}{4}\right) : (\partial J)\partial U_{0,2} : -\frac{3}{8}(3+2\ell) : J\partial^4 U_{0,0} : \\
& \quad + (2+2\ell) : J\partial^3 U_{0,1} : -\left(\frac{3}{4} + \frac{3\ell}{2}\right) : J\partial^2 U_{0,2} : -\left(\frac{1}{4} - \frac{\ell}{2}\right) : J\partial U_{0,3} : \\
& \quad +\left(\frac{1}{8} - \frac{\ell}{4}\right) : JU_{0,4} : +\frac{1}{12}(3+2\ell) : (\partial^3 T)U_{0,0} : -\left(\frac{3}{4} + \frac{\ell}{2}\right) : (\partial^2 T)U_{0,1} : \\
& \quad -\frac{1}{4}(3+2\ell) : (\partial T)\partial^2 U_{0,0} : +\left(\frac{3}{2} + \ell\right) : (\partial T)\partial U_{0,1} : -\left(\frac{3}{4} + \frac{\ell}{2}\right) : (\partial T)U_{0,2} : \\
& \quad -\frac{1}{6}(3+2\ell) : T\partial^3 U_{0,0} : +\left(\frac{3}{4} + \frac{\ell}{2}\right) : T\partial^2 U_{0,1} : -\left(\frac{3}{4} + \frac{\ell}{2}\right) : T\partial U_{0,2} : \\
& \quad +\left(\frac{1}{2} + \frac{\ell}{3}\right) : TU_{0,3} : +\left(\frac{45+58\ell+24\ell^2}{120}\right)\partial^5 U_{0,0} -\left(\frac{8+13\ell+6\ell^2}{8}\right)\partial^4 U_{0,1} \\
& +\left(\frac{7+14\ell+8\ell^2}{8}\right)\partial^3 U_{0,2} -\left(\frac{3+7\ell+6\ell^2}{12}\right)\partial^2 U_{0,3} -\left(\frac{\ell(1-2\ell)}{24}\right)\partial U_{0,4} +\left(\frac{\ell(1-2\ell)}{60}\right)U_{0,5}.
\end{aligned}$$

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