THE MODULAR GROMOV-HAUSDORFF PROPINQUITY

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ABSTRACT. We introduce a metric on Hilbert modules equipped with a generalized form of a differential structure, thus extending Gromov-Hausdorff convergence theory to vector bundles and quantum vector bundles — not convergence as total space but indeed as quantum vector bundle. Our metric is new even in the classical picture, and creates a framework for the study of the moduli spaces of modules over C*-algebras from a metric perspective. We apply our construction, in particular, to the continuity of Heisenberg modules over quantum 2-tori.

CONTENTS

1. INTRODUCTION

Our project in noncommutative metric geometry aims at constructing an analytic framework for the study of entire classes of C*-algebras, seen as objects

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of some larger geometry, thus taking the perspective that new mathematics and mathematical physics will arise from looking at C*-algebras as points of some hyperspace, in the spirit of metric geometry [12], extended into the realm of noncommutative geometry [8]. We laid some of the foundations for this project with the construction of the Gromov-Hausdorff propinquity [31, 27, 24, 29, 28], a noncommutative analogue of the Gromov-Hausdorff distance [11] for quantum metric spaces, i.e. noncommutative generalizations of algebras of Lipschitz functions over metric spaces [7, 40, 41]. We now extend our framework to quantum vector bundles over quantum metric spaces, i.e. to modules over C*-algebras, equipped with appropriate additional structures to encode metric information.

Modules, and in particular finitely generated projective modules over C*-algebras, are crucial to the theory of C*-algebras — from Morita equivalence [36] to K-theory and KK-theory [2] — and vector bundles play a fundamental role in the construction of fields in quantum physics. Of course, vector bundles are a corner stone of topology and geometry. We construct a far-reaching extension of the quantum Gromov-Hausdorff propinquity to Hilbert modules [35] over quantum metric spaces, equipped with a metric generalization of a connection, called a D-norm. We prove that our new metric, called the modular Gromov-Hausdorff propinquity, is indeed a distance on our class of metrized quantum vector bundles up to a strong notion of isomorphism, which preserves the module structure, the inner product — hence the C^*-Hilbert norm — and the D-norm. We check that our new distance extends the topology of the quantum propinquity, since C*-algebras are canonically Hilbert modules over themselves. There are no analogues of the Gromov-Hausdorff distance on vector bundles even classically, hence the modular propinquity introduces new possibilities even in the classical picture. We shall however focus on an application of the modular propinquity to the continuity of Heisenberg modules [6, 37] over quantum tori.

We strongly believe that the modular Gromov-Hausdorff propinquity is a significant progress in our program, and this paper lays the foundation for many research questions. The genesis of our project lies in mathematical physics and the desire to provide a formal approach to various approximation results involving matrix algebras and noncommutative, as well as classical limits (see for instance [46] for some references in physics). In this context, the convergence of matrix algebras to a torus or a quantum torus is not an end, but rather a mean to then discuss convergences of various quantities of physical importance, many of which are defined on modules over the spaces under consideration. These spaces often carry a form of geometry, often given by means of generalized differential calculus, which provide metric data which we can use in our current project. A particularly common object in this context are connections, which do play a key role in motivating the notions of metrics on quantum vector bundles.

The moduli space of finitely generated projective modules over C*-algebra has been a central structure in the study of C*-algebras. Rieffel studied the categorical aspect of this space and introduced Rieffel-Morita equivalence between C*-algebras — a weak form of isomorphism whose role is well-established near the core of our subject. K-theory, then later on its far-reaching extension, Kasparov KK-theory, directly involve the classification of modules under a form of stable isomorphism, and once more, this algebraic-topological aspect of the moduli space of modules is a central topic in C*-algebra research. We now propose to introduce a metric on this moduli space, and thus open many avenues for research on new analytic aspects of the moduli space of modules, so far completely unexplored. We expect not only that our metric will provide new and interesting results within the previously studied categorical and algebraic framework, but also that new ideas will emerge based on analytic constructs and ideas so far out of reach.

Remarkably, our modular propinquity introduces a mean to define a topology on vector bundles over metric spaces which is new even in the classical picture. This is also a very exiting development for us. Indeed, it may well be that the functional analytic perspective taken in the development of the quantum propinquity opened new ways to think of convergence of classical geometry objects such as vector bundles. Once more, this opens obvious new directions for queries within the realm of Riemannian geometry and in particular, the metric aspects of Riemannian geometry. The Gromov-Hausdorff distance has proven a very powerful tool to discuss the geometry of classes of spaces, including singular limits of Riemannian manifolds, by focusing on the underlying metric structure. The modular propinquity, in turn, allows one to discuss convergence of vector bundles, thus of additional structures besides the underlying metric. We are hopeful that once more, new and exiting results lie in the future of the exploration of this particular consequence of the construction we present in this paper.

Our research program began with the observation by A. Connes in [7] that an noncommutative analogue of the Monge-Kantorovich metric could be defined on the state space of a C*-algebra by means of a spectral triple. Rieffel laid the foundation for the study of quantum metric spaces [40], recognizing that a quantum metric is encoded in a type of seminorm whose dual seminorm induces a noncommutative Monge-Kantorovich metric which, crucially, Rieffel requires to induce the weak* topology on the state space, just as its classical counter part. This observation laid the groundwork for the construction of noncommutative analogues of the Gromov-Hausdorff distance — a task fraught with unexpected challenges, and which gave rise to a succession of candidates. The first construction is also due to Rieffel who defined the quantum Gromov-Hausdorff distance [49]. We introduce the dual Gromov-Hausdorff propinquity in [27, 24] as our answer to many of the questions which research in noncommutative metric geometry brought up in the last decade or so. The dual Gromov-Hausdorff propinquity can be specialized — in other words, it really is a family of metrics, a technical flexibility which we will take advantage of in this paper by working with the most important specialization of the dual propinquity: the quantum Gromov-Hausdorff propinquity [31] (which was actually introduced before the more general dual propinquity).

A key motivation behind our introduction of the propinquity was to devise an analogue of the Gromov-Hausdorff distance which was well-behaved with respect to the C*-algebraic structure. In essence, a compact quantum metric space comes with a metric structure, embodied in a particular seminorm which generalizes the Lipschitz seminorm, and a topological structure, encoded in the C*-algebraic structure. We propose to require a connection between these two by imposing a form of the Leibniz inequality to the Lip-norm; quantum compact metric spaces which satisfy this additional requirement are called quasi-Leibniz quantum compact metric spaces [29] (originally, we worked with Leibniz quantum compact metric spaces in [31, 27], but new examples of quantum compact metric spaces led us to the more general notion of quasi-Leibniz Lip-norms). This development actually raises some serious difficulties when attempting to construct an analogue of the Gromov-Hausdorff distance, as seen with the early work of Kerr [19] or, later on, with the quantum proximity of Rieffel [45], where in each case, working exclusively with Leibniz Lip-norms, or even their generalizations, lead to analogues of the Gromov-Hausdorff distance which are not satisfying, as far as we know, the triangle inequality.

As we thus solved the difficulties inherent in extending the Gromov-Hausdorff topology to noncommutative geometry with the introduction of the Gromov-Hausdorff propinquity, we now wish to move our theory forward and study within its context such C*-algebraic related structures as modules, as announced already in [31, 27, 24]. We chose to work with the quantum Gromov-Hausdorff propinquity since it provides us with additional structure compared to the more general dual Gromov-Hausdorff propinquity, which will prove helpful in our first venture into the realm of modules. We thus assume given a sequence of quasi-Leibniz quantum compact metric spaces converging for the quantum propinquity. There are many interesting such sequences already known: quantum tori [25], fuzzy tori converging to quantum tori [25], matrix algebras converging to the sphere [43, 45], certain sequences of AF-algebras [1], finite dimensional quasi-Leibniz quantum compact metric spaces converging to any nuclear quasi-diagonal quasi-Leibniz quantum compact metric space [29], conformal deformations of Leibniz quantum compact metric spaces constructed from spectral triples [28], curved quantum tori [26], noncommutative solenoids [32], to name but a few. Given such a sequence, we wish to make sense of the statement: each quasi-Leibniz quantum compact metric spaces in the sequence carries a module, presumably equipped with additional metric data, such that the resulting sequence of modules converge in some sense to a module over the limit quasi-Leibniz quantum compact metric spaces.

Such a statement immediately raises three entangled questions. First, we wish to make sense of what it means for a sequence of modules to converge at all. Given the context of this question, we expect that there may be a need to introduce some metric data on modules, and this is itself an interesting, second challenge. We may approach this question by expecting to have a canonical mean to extend the metric information of quasi-Leibniz quantum compact metric spaces to their modules (or some class of modules), or we may take the view that this metric data is a new component of our theory. Last, we may wonder, given a module over the limit of a sequence of quasi-Leibniz quantum compact metric spaces, how to select modules over the spaces in the sequence to get the desired convergence.

Rieffel initiated a first approach to the convergence of modules in his pioneering work in [44, 45, 47, 48]; indeed his work spurred our own research project on the quantum propinquity. He proposed to work with finitely generated projective modules over certain Leibniz quantum compact metric spaces. In this picture, modules correspond to projections in matrix algebras over quantum metric spaces. Rieffel suggests in [48] that the metric data needed to work with modules is a canonical extension of the metric data on the base space. He then suggests the fascinating idea that we can measure the "twist" of a vector bundle using that metric data. Using this approach, Rieffel provided a meaning to certain vector bundles over the sphere to be approximated by vector bundles over finite spaces [44], and initiated the same study for approximations by full matrix algebras. Notably, Rieffel's recent work utilizes the quantum propinquity as a starting point [48]. In this picture, the effort is placed on proving that certain projections in matrix algebras over quasi-Leibniz quantum compact metric spaces are close in the sense of a generalized Lip-norm.

We propose an approach in this paper which, we submit, is complementary to Rieffel's approach. Our focus is on extending the quantum propinquity to left Hilbert modules. Thus, we start our investigation with the goal to define a very general notion of convergence, induced by an actual metric. To this end, we do not regard the metric data on modules as canonically associated with the base space, but rather as a new ingredient, which may be given via Rieffel's approach or, as in the main example of this paper, may have a geometric source. Heisenberg modules over quantum tori then provide many examples of convergent sequences.

Our approach is best presented by first discussing it informally in the context of manifolds and their vector bundles. A vector bundle V over a compact, connected Riemannian manifold M, may be equipped with a metric — which in this context, means a smooth section of the bundle of inner products over each fiber of V. In other words, we pick an inner product on the $C(M)$-module Γ of continuous sections of V, where $C(M)$ is the C*-algebra of $\mathbb{C}$-valued continuous functions over M. This is a particular example of a Hilbert module over a C*-algebra, and thus we will work in this paper with left Hilbert modules. Yet, there is one more essential tool of differential geometry when working with modules: namely, the notion of a connection. Indeed, given a metric on a bundle, one may always find a so-called metric connection, and under stronger assumption, this connection is in fact unique. In other words, to a metric corresponds a natural notion of parallel transport.

In noncommutative geometry, there still exists metric connections, appropriately defined, on left Hilbert modules, under rather generous conditions. The matter of uniqueness issue is less clear. In any event, we adopt the perspective that the metric information needed to work with vector bundles include not only an inner product on its module of sections, but also a choice of a connection, or rather an additional norm on the space of smooth functions which encode some aspects of the connection which are of use to define our metric. We do not need the full strength of a connection in this work, and we will address the issue of convergence for differential structures in a forthcoming paper.

For our purpose, therefore, the metric data which we will consider to define a metrized quantum vector bundle includes a Hilbert module equipped with an additional, densely defined norm called a D-norm with a natural topological condition inspired by the commutative picture above. There is a clear relation between our notion of a metrized quantum vector bundle and the notion of a quasi-Leibniz quantum compact metric space which we take as a sign that our approach is sensible. Moreover, a metrized quantum vector bundle will of course be defined over a particular quasi-Leibniz quantum compact metric space, its base space. Just as is the case with proving that certain semi-norms are Lip-norms, establishing that a given norm is a D-norm may be challenging.

Concretely, Heisenberg modules over quantum tori are in fact equipped with a natural connection [6, 9] which plays an important role in the geometry of quantum tori. This connection is indeed what we will use as part of the metric data to turn Heisenberg modules into metrized quantum vector bundles, and then study some convergence properties for our modular propinquity.

The modular Gromov-Hausdorff propinquity is thus defined on the class of all metrized quantum vector bundles. Its construction is inspired by our work on the quantum Gromov-Hausdorff propinquity for two reasons. First of all, we develop the quantum propinquity in the hope to work with C*-algebraic structures, and thus it is for us the natural path to follow here. Second of all, we envisage that when working with the modular Gromov-Hausdorff propinquity, we already have acquired some good understanding of the quantum Gromov-Hausdorff propinquity between the quantum base spaces of our noncommutative bundles. Thus we want the modular propinquity to take advantage, as much as possible, of the work done on the base spaces. This strongly motivates us to design our new metric around concepts which we unearthed when working with the quantum propinquity. Naturally, there are many new challenges raised by working with modules, and this, too, is a reason to take full advantage of our understanding of the quantum propinquity.

In summary, we present in this paper the class of metrized quantum vector bundles, on which we then define a metric akin to a Gromov-Hausdorff distance for (noncommutative) vector bundles. We prove that our metric, the modular Gromov-Hausdorff propinquity, is indeed a metric up to full quantum isometry of metrized quantum vector bundles — i.e. an appropriate notion of morphism between left Hilbert modules over possibly different C*-algebras which also preserves all the metric data. We then prove that we apply our metric to the subclass of the metrized quantum vector bundles canonically constructed from quasi-Leibniz quantum compact metric spaces — extending the observation that any C*-algebra is a left Hilbert module over itself — we recover the topology of the quantum Gromov-Hausdorff topology. We also show that, reassuringly, the modular propinquity between free modules over quasi-Leibniz quantum compact metric spaces behave as expected — close base spaces in the quantum propinquity and same rank of free modules imply close in the modular propinquity. We discuss a sufficient condition for the direct sum of metrized quantum vector bundles to be continuous on certain classes called iso-pivotal.

We then apply our work to exhibit continuity properties for the modular propinquity among the Heisenberg modules over quantum 2-tori. This work involves several step reflective of our new approach: we must construct D-norms on Heisenberg modules, which actually do arise from the natural connection these modules carry. We must establish that the connection does give rise to D-norms, which involve some aspects of the analysis of the Moyal plane. We also must prove that the norms of Heisenberg modules can form continuous fields, and do the same regarding our D-norms. The final proof of our main example exemplifies the idea that we bootstrap our convergence for modules from convergence of the base space.

2. The modular Gromov-Hausdorff propinquity

We propose a structure of *metrized quantum vector bundle* which abstracts the notion of a metric on a vector bundle over a manifold. Our structure builds upon common elements of noncommutative geometry, though it adds a crucial analytic condition.

A metric on a complex vector bundle V over a compact space M is typically defined as a continuous section of the associated bundle of sesquilinear forms over the fibers of V — when M is a manifold, the metric is in fact required to be a smooth section in general. We can thus regard a metric on a vector bundle as defining an inner product, valued in $C(M)$, over the continuous sections of V. In other words, one natural ingredient for our work is a structure of left Hilbert module.

However, a fundamental observation of Riemannian geometry is that the metric is associated with a form of parallel transport. There is in fact a unique metric connection on the cotangent bundle with zero torsion — the Levi-Civita connection and thus a metric provides a canonical notion of transport, curvature, and more. In noncommutative geometry, we do not have in general a unique metric connection, even under conditions of zero torsion. None the less, metric connections can be shown to exist under very mild assumptions. As such, it becomes natural to say that in differential geometry, metric information on bundle comes in the form, not only of a section of fiber-wise inner products, but also in the form of a connection. Involving a connection, however, would seem to suggest that we work on a noncommutative differentiable manifold. While our main example will indeed be in this setting, our modular propinquity is much more general. The reason is that we shall reduce the idea of a connection to a particular norm called a D-norm. Thus, a metrized quantum vector bundle will be a left Hilbert module equipped with a (densely defined) additional norm inspired from the differential notion of a connection. We shall make these statements precise in Example (2.2.10).

We begin this section with a short survey section on quantum compact metric spaces which will provide us with the base space for our metrized quantum vector bundles.

2.1. **Quantum Compact Metric Spaces.**

A quantum compact metric space is a noncommutative generalization of the algebras of Lipschitz functions over a metric space. Our work on the Gromov-Hausdorff propinquity [31, 27, 24, 29, 26] emphasizes the role of a relation between the generalized Lipschitz seminorms and the multiplicative structure of the underlying algebra, though this relation can be quite general. We will thus work in the category of quasi-Leibniz quantum compact metric spaces, defined as follows.

Notation 2.1.1. The norm of a normed vector space E will be denoted by $\|\cdot\|_E$ by default.

Notation 2.1.2. Let $\mathfrak{A}$ be a unital C*-algebra. The space of self-adjoint elements of $\mathfrak{A}$ is denoted by $\mathfrak{sa}(\mathfrak{A})$, while the state space of $\mathfrak{A}$ is denoted by $\mathscr{S}(\mathfrak{A})$. The unit of $\mathfrak{A}$ is denoted by $1_{\mathfrak{A}}$.

Definition 2.1.3 ([29])**.** A function $F : [0,\infty)^4 \to [0,\infty)$ is *admissible* when for all:

$$(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in [0,\infty)^4$$

such that $x_j \leqslant y_j$ for all $j \in \{1,2,3,4\}$, we have:

$$F(x_1, x_2, x_3, x_4) \leqslant F(y_1, y_2, y_3, y_4)$$

and $x_1 x_3 + x_2 x_4 \leqslant F(x_1, x_2, x_3, x_4)$.

Definition 2.1.4 ([40, 41, 31, 29])**.** An *F–quasi-Leibniz quantum compact metric space* $(\mathfrak{A}, \mathsf{L})$, for some admissible function F, is a unital C*-algebra $\mathfrak{A}$ and a seminorm L defined on a dense Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$, such that:

1. $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
2. the *Monge-Kantorovich metric* mk_{L}, defined on the state space $\mathscr{S}(\mathfrak{A})$ by setting for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$:
$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\},$$
metrizes the weak* topology on $\mathscr{S}(\mathfrak{A})$,
3. L is lower semi-continuous, i.e. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1\}$ is norm closed,
4. for all $a, b \in \mathrm{dom}(\mathsf{L})$, we have:
$$\max\{\mathsf{L}(a \circ b), \mathsf{L}(\{a,b\})\} \leqslant F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{L}(b)).$$

The seminorm L of a F–quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ is called an *L-seminorm* of type F. If $F : x, y, l_x, l_y \mapsto x l_y + y l_x$, then $(\mathfrak{A}, \mathsf{L})$ is simply called an Leibniz quantum compact metric space and L is said to be of Leibniz type.

We adopt the following convention in our exposition to keep our notations simple:

Convention 2.1.5. If L is a seminorm defined on a subspace $\mathrm{dom}(\mathsf{L})$ of a vector space E then, for all $x \in E \setminus \mathrm{dom}(\mathsf{L})$, we set $\mathsf{L}(x) = \infty$. Thus $\mathrm{dom}(\mathsf{L}) = \{x \in E : \mathsf{L}(x) < \infty\}$ with this convention. With the additional convention that $0 \cdot \infty = 0$ and $\infty + r = r + \infty = \infty$ for all $r \in [0,\infty]$ when working with seminorms, we can see L as a seminorm on E taking the value ∞.

The classical prototype of a Leibniz quantum compact metric space is given by the ordered pair $(C(X), \mathrm{Lip})$ of the C*-algebra $C(X)$ of $\mathbb{C}$-valued continuous functions over a compact metric space (X, d) and the Lipschitz seminorm induced on $\mathfrak{sa}(C(X))$ — the Banach algebra of $\mathbb{R}$-continuous functions — by the metric d. In this case, the metric mk_{L} is indeed the Monge-Kantorovich metric on the space of regular Borel probability measures over X, a fundamental object introduced by Kantorovich [17] in the study of Monge's transportation problem. The form of the Monge-Kantorovich metric which we generalize in Definition (2.1.4) was discovered by Kantorovich and Rubinstein [18].

Important examples of quasi-Leibniz quantum compact metric spaces include the quantum tori [40, 42], Connes-Landi sphere [33], full C*-algebras of Hyperbolic groups [34], AF-algebras with a faithful tracial state [1], curved quantum tori [26], conformal perturbations of quantum metric spaces obtained from Dirac operators [28], C*-algebras of nilpotent groups [5], noncommutative solenoids [32], among many other. Moreover, finite dimensional C*-algebras can be endowed with many quantum metric structures which play an important role when approximating C*-algebras of continuous functions over coadjoint orbits of semisimple Lie groups [43], quantum tori [21, 25], AF-algebras [1], and arbitrary nuclear quasi-diagonal quasi-Leibniz quantum compact metric spaces [29].

The first occurrence of a noncommutative version of the Monge-Kantorovich metric is due to Connes in [7], where it was observed that a spectral triple give rise to a metric on the state space of a C*-algebra. Rieffel initiated the study of compact quantum metric spaces in [40] by requiring that the Monge-Kantorovich metric in noncommutative geometry should metrize the weak* topology on the state space, and can be built without appealing to the theory of spectral triple, but rather using a generalized Lipschitz seminorm. As a matter of terminology, a seminorm L satisfying properties (1) and (2) is called a *Lip-norm*. We choose our new terminology to avoid writing the rather long expression "quasi-Leibniz lower semi-continuous Lip-norm" too often. A pair $(\mathfrak{A}, \mathsf{L})$ where L is a Lip-norm is called a compact quantum metric space.

Quantum locally compact metric spaces were introduced in [23], building on our work in [22], and provide a far-reaching generalization of Definition (2.1.4).

Definition (2.1.4) evolved with the role of the algebraic structure of a compact quantum metric spaces. In [40], Rieffel's original notion of Lip-norm was defined over normed vector spaces (and the notion of state was replaced with a more general notion). In [41], the focus was on order-unit spaces, and this was the setting for the construction of the quantum Gromov-Hausdorff distance [49], and the first examples of continuity for that metric [43, 21, 33]. As research in noncommutative metric geometry became focused on the relationship between convergence for analogues of the Gromov-Hausdorff distance and C*-algebraic structures — in particular modules [44, 45] — it became apparent that Lip-norms should be connected to the underlying C*-algebraic structure. We proposed Definition (2.1.4) by adapting the idea of F-Leibniz seminorms of Kerr's [19], with two differences.

L-seminorms are defined on a dense *-subspace of the self-adjoint part of C*-algebras, and in general $\mathfrak{sa}(\mathfrak{A})$ is not a *-subalgebra of $\mathfrak{A}$ for non-Abelian C*-algebras. It is a Jordan-Lie algebra, and thus we use the Jordan and Lie product in the definition of the quasi-Leibniz property. Our insistence on working with L-seminorms defined only on self-adjoint elements will be justified when discussing quantum isometries later on. Second, we require that the quasi-Leibniz property be no sharper than the Leibniz property, which actually ensures that for any given choice of an admissible function F, the quantum propinquity can be restricted to the class of F–quasi-Leibniz quantum compact metric spaces and never involve any space outside of this class. We refer to [31, 27, 24, 28, 26] for details.

The class of quantum compact metric spaces form a category when morphisms are defined using a natural Lipschitz condition. In fact, there are at least three ideas one may consider when defining a notion of a Lipschitz morphism between compact quantum spaces, and these notions will not agree in general. However, as we impose that L-seminorms are lower-semicontinuous, all three ideas agree for quasi-Leibniz quantum compact metric spaces.

We choose the following definition for morphisms of quasi-Leibniz quantum compact metric spaces.

Definition 2.1.6. A *k-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ between two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-morphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that for all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$:

$$\mathsf{L}_{\mathfrak{B}} \circ \pi(a) \leqslant k\mathsf{L}_{\mathfrak{A}}(a).$$

We then show that other natural ideas for morphisms of quasi-Leibniz quantum compact metric spaces agree with Definition (2.1.6).

Theorem 2.1.7 ([41, 30])**.** *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces and $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ be a *-morphism. The following assertions are equivalent for any $k \geqslant 0$:*

(1) π is a k-Lipschitz morphism,
(2) $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is a k-Lipschitz map from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$,
(3) $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

Assertion (2) in Theorem (2.1.7) was the initial definition of a Lipschitz morphism in [41, 49]. The equivalence between Assertion (1) and Assertion (3) was the subject of [30] while Rieffel proved the equivalence between Assertion (1) and (2) in [41], where the importance of lower semicontinuity for Lip-norms was discovered.

It is straightforward to check that, taking for objects our quasi-Leibniz quantum compact metric spaces and for morphisms our Lipschitz morphisms give rise to a category.

The stronger notion of isometry between quasi-Leibniz quantum compact metric spaces must be well-understood in our context, since the Gromov-Hausdorff propinquity is a metric up to isometry, properly defined. The original notion of isometry for compact quantum metric space [49] did not involve *-morphisms, since two Rieffel's distance could be null between compact quantum metric spaces which were not *-isomorphic.

Rieffel's insight into the proper notion of an isometric embedding rests on Blaschke's theorem [4, Theorem 7.3.8], which states that a *real valued* k-Lipschitz function on some nonempty subset of a metric space can be extended to a k-Lipschitz function on the whole space. For our purpose, the main consequence of Blaschke's theorem is that, if $\pi : (X, \mathsf{d}) \hookrightarrow (Z, \mathsf{m})$ is an injection between two compact metric spaces, then π is an isometry if and only if the Lipschitz seminorm on $C(X)$ induced by d is the quotient seminorm of the Lipschitz seminorm on $C(Z)$ induced by m. This is the origin of the definition of a quantum isometry.

Blaschke's theorem is not valid as stated for $\mathbb{C}$-valued Lipschitz functions: in general, the best statement for $\mathbb{C}$-value Lipschitz functions is that a k-Lipschitz function over a subset can be extended to the whole space as a $\frac{4k}{\pi}$-Lipschitz function [44]. It means that the relationship between Lipschitz seminorms provided, on the $\mathbb{R}$-valued functions, by isometries, does not hold for $\mathbb{C}$-valued function, rendering the generalization of these ideas to the noncommutative realm less obvious, and thus justifying in large part the choice to work with L-seminorms defined only for self-adjoint elements in general.

The construction of the propinquity was in large part motivated by ensuring that *-isomorphism is necessary for distance zero, and thus we arrive at the notion of quantum isometries which we have used since our work in [31]:

Definition 2.1.8 ([49, 31])**.** Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that for all $b \in \mathfrak{B}$ we have:

$$\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* π is a quantum isometry and a *-isomorphism such that π^{-1} is also a quantum isometry.

If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a quantum isometry, then in particular $\mathsf{L}_{\mathfrak{B}} \circ \pi(a) \leqslant \mathsf{L}_{\mathfrak{A}}(a)$ for all $a \in \mathfrak{sa}(\mathfrak{A})$ and thus π is a 1-Lipschitz morphism.

If π is a full quantum isometry and $a \in \mathfrak{sa}(\mathfrak{A})$ then $\mathsf{L}_{\mathfrak{B}} \circ \pi(a) = \mathsf{L}_{\mathfrak{A}}(a)$, since:

$$\mathsf{L}_{\mathfrak{B}}(\pi(a)) \leqslant \mathsf{L}_{\mathfrak{A}}(a) = \mathsf{L}_{\mathfrak{A}}(\pi^{-1}(\pi(a))) \leqslant \mathsf{L}_{\mathfrak{B}}(\pi(a)).$$

One may therefore define a subcategory of quasi-Leibniz quantum compact metric spaces whose morphisms are quantum isometries, as quantum isometries compose to quantum isometries by [49, Proposition 3.7]. In this category, full quantum isometries are the isomorphisms. The Gromov-Hausdorff propinquity is null between two quasi-Leibniz quantum compact metric spaces if and only if they are fully quantum isometric [31, 27, 29].

We now turn to the following question: what metric structure may we equip modules over quasi-Leibniz quantum compact metric spaces, so that we then might develop a generalized notion of convergence for such metrized modules?

2.2. D-norms.

Our work in this article is concerned with the construction of a metric on modules over quasi-Leibniz quantum compact metric spaces, appropriately defined. For the current research, a module over a quasi-Leibniz quantum compact metric space is a left Hilbert $\mathfrak{A}$-module over the underlying C*-algebra, equipped with an additional norm defined on some dense subspace (which is not in general a submodule) satisfying a particular topological requirement, and with various basic inequalities connecting all the ingredients of such a structure. These inequalities generalize the Leibniz inequality for L-seminorms.

As a matter of fixing our notations, we recall the definition of left Hilbert modules:

Definition 2.2.1 ([35])**.** A *left pre-Hilbert module* $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ over a C*-algebra $\mathfrak{A}$ is a left module $\mathscr{M}$ over $\mathfrak{A}$ equipped with a sesquilinear map $\langle\cdot,\cdot\rangle_{\mathscr{M}} : \mathscr{M} \times \mathscr{M} \to \mathfrak{A}$ such that for all $\omega, \eta \in \mathscr{M}$ and $a \in \mathfrak{A}$ we have:

(1) $\langle a\omega, \eta\rangle_{\mathscr{M}} = a\langle\omega, \eta\rangle_{\mathscr{M}}$,
(2) $(\langle\omega, \eta\rangle_{\mathscr{M}})^* = \langle\eta, \omega\rangle_{\mathscr{M}}$,
(3) $\langle\omega, \omega\rangle_{\mathscr{M}} \geqslant 0$.
(4) $\langle\omega, \omega\rangle_{\mathscr{M}} = 0$ if and only if $\omega = 0$.

Let $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ be a left pre-Hilbert module over a C*-algebra $\mathfrak{A}$. We note that Conditions (1) and (2) together prove that $\langle\cdot,\cdot\rangle_{\mathscr{M}}$ possesses a modular form of sesquilinearity, i.e. $\langle\omega, a\eta\rangle_{\mathscr{M}} = \langle\omega,\eta\rangle_{\mathscr{M}} a^*$ for all $a \in \mathfrak{A}$ and $\omega,\eta \in \mathscr{M}$.

A version of the Cauchy-Schwarz inequality is valid for left pre-Hilbert modules, so that for all $\omega,\eta \in \mathscr{M}$ we have:

$$\langle\omega,\eta\rangle_{\mathscr{M}} \langle\eta,\omega\rangle_{\mathscr{M}} \leqslant \|\langle\omega,\omega\rangle_{\mathscr{M}}\|_{\mathfrak{A}} \langle\eta,\eta\rangle_{\mathscr{M}}$$

and thus, together with the rest of the properties of the inner product, we may define a module norm on $\mathscr{M}$ from the inner product $\langle\cdot,\cdot\rangle_{\mathscr{M}}$:

Proposition 2.2.2 ([35])**.** *If $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ is a left pre-Hilbert module over a C*-algebra $\mathfrak{A}$, and if, for all $\omega \in \mathscr{M}$, we set:*

$$\|\omega\|_{\mathscr{M}} = \sqrt{\langle\omega,\omega\rangle_{\mathscr{M}}}$$

then $\|\cdot\|_{\mathscr{M}}$ is a module norm on $\mathscr{M}$. i.e. a norm such that for all $a \in \mathfrak{A}$ and $\omega \in \mathscr{M}$, the following holds:

$$\|a\omega\|_{\mathscr{M}} \leqslant \|a\|_{\mathfrak{A}} \|\omega\|_{\mathscr{M}}.$$

Moreover, for any $\omega,\eta \in \mathscr{M}$, we also have:

$$|\langle\omega,\eta\rangle_{\mathscr{M}}| \leqslant \|\omega\|_{\mathscr{M}} \|\eta\|_{\mathscr{M}}.$$

Notation 2.2.3. For a left pre-Hilbert module $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$, we adopt the convention that $\|\cdot\|_{\mathscr{M}}$ always refer to the norm defined in Proposition (2.2.2) and we call this norm the *Hilbert morm* of $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$.

We thus may require completeness of a left pre-Hilbert module for its C^*-Hilbert norm, leading to the following definition.

Definition 2.2.4 ([35])**.** A *left Hilbert module* $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ over a C*-algebra $\mathfrak{A}$ is a left pre-Hilbert module over $\mathfrak{A}$ which is complete for its C^*-Hilbert norm $\|\cdot\|_{\mathscr{M}}$.

We define our notion of a morphism between left Hilbert modules. Our morphisms can be defined between left Hilbert modules over different base algebras, and this concept will be folded in our notion of a morphism for a metrized quantum vector bundle.

Definition 2.2.5. Let $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ and $(\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}})$ be two left Hilbert modules over, respectively, two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. A *module morphism* (Θ,θ) is given by a a *-morphism $\theta : \mathfrak{A} \to \mathfrak{B}$, and a $\mathbb{C}$-linear map $\Theta : \mathscr{M} \to \mathscr{N}$, such that for all $a \in \mathfrak{A}$ and $\omega,\eta \in \mathscr{M}$, we have:

(1) $\Theta(a\omega) = \theta(a)\Theta(\omega)$,
(2) $\langle\Theta(\omega),\Theta(\eta)\rangle_{\mathscr{N}} = \langle\omega,\eta\rangle_{\mathscr{M}}$.

The module morphism (Θ,θ) is *unital* when θ is a unital *-morphism.

We note that if (Θ,θ) is a module morphism, Θ is continuous of norm 1 by definition.

In continuing with the tradition in noncommutative geometry to name structures after their commutative analogues, we shall thus define a metrized quantum vector bundle as follows. We first extend the notion of an admissible function to a triple of functions, as we shall have three versions of the Leibniz inequality in our definition of metrized quantum vector bundles.

Definition 2.2.6. A triple (F, G, H) is *admissible* when:

(1) $F : [0,\infty)^4 \to [0,\infty)$ is admissible,
(2) $G : [0,\infty)^3 \to [0,\infty)$ satisfies $G(x,y,z) \leqslant G(x',y',z')$ if $x,y,z,x',y',z' \in [0,\infty)$ and $x \leqslant x', y \leqslant y', z \leqslant z'$, while:
$$(x+y)z \leqslant G(x,y,z).$$
(3) $H : [0,\infty)^2 \to [0,\infty)$ satisfies $H(x,y) \leqslant H(x',y')$ if $x,y,x',y' \in [0,\infty)$ and $x \leqslant x'$ and $y \leqslant y'$ while $2xy \leqslant H(x,y)$.

The structure of a metrized quantum vector bundle is thus given by the following definition.

Notation 2.2.7. Let $a \in \mathfrak{A}$ where $\mathfrak{A}$ is a C*-algebra. We denote $\frac{a+a^*}{2}$ by $\Re a$ and $\frac{a-a^*}{2i}$ by $\Im a$. Note that $\Re a, \Im a \in \mathfrak{sa}(\mathfrak{A})$.

Definition 2.2.8. A (F,G,H)*–metrized quantum vector bundle* $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, for some admissible triplet (F,G,H), is given by a F–quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, as well as a left Hilbert module $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ over $\mathfrak{A}$ and a norm $\mathsf{D}_{\mathscr{M}}$ defined on a dense $\mathbb{C}$-subspace $\mathrm{dom}(\mathsf{D}_{\mathscr{M}})$ of $\mathscr{M}$ such that:

(1) we have $\|\cdot\|_{\mathscr{M}} \leqslant \mathsf{D}_{\mathscr{M}}$,
(2) the set:
$$\{\omega \in \mathscr{M} : \mathsf{D}_{\mathscr{M}}(\omega) \leqslant 1\}$$
is compact for $\|\cdot\|_{\mathscr{M}}$.
(3) for all $a \in \mathfrak{sa}(\mathfrak{A})$ and for all $\omega \in \mathscr{M}$, we have:
$$\mathsf{D}_{\mathscr{M}}(a\omega) \leqslant G\left(\|a\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(a), \mathsf{D}_{\mathscr{M}}(\omega)\right),$$
which we call the *inner quasi-Leibniz inequality* for $\mathsf{D}_{\mathscr{M}}$,
(4) for all $\omega, \eta \in \mathscr{M}$, we have:
$$\max\left\{\mathsf{L}_{\mathfrak{A}}\left(\Re\langle\omega,\eta\rangle_{\mathscr{M}}\right), \mathsf{L}_{\mathfrak{A}}\left(\Im\langle\omega,\eta\rangle_{\mathscr{M}}\right)\right\} \leqslant H\left(\mathsf{D}_{\mathscr{M}}(\omega), \mathsf{D}_{\mathscr{M}}(\eta)\right),$$
which we call the *modular quasi-Leibniz inequality* for $\mathsf{D}_{\mathscr{M}}$.

The F–quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is called the *base quantum space* of Ω and is denoted by $\mathrm{bqs}(\Omega)$. The norm $\mathsf{D}_{\mathscr{M}}$ will be called the *D-norm* of Ω.

If Ω is a (F,G,H)–metrized quantum vector bundle then its D-norm is said to be of type (G,H). In general, we say that Ω is a metrized quantum vector bundle when it is a (F,G,H)–metrized quantum vector bundle for some admissible triple (F,G,H). We make a simple observation which also applies, in a simpler form, to our work in [29] as we shall discuss after the proof of Theorem (2.5.11).

Notation 2.2.9. Let (F_1,G_1,H_1) and (F_2,G_2,H_2) be two admissible triple. If $F = \max\{F_1,F_2\}$, $G = \max\{G_1,G_2\}$ and $H = \max\{H_1,H_2\}$, then (F,G,H) is also an admissible triple, denoted by $(F_1,G_1,H_1) \vee (F_2,G_2,H_2)$. In particular, F is admissible. Therefore, given any pair of quasi-Leibniz quantum compact metric spaces or metrized quantum vector bundles, one may always assume that they share the same quasi-Leibniz properties.

The classical picture which inspired our Definition (2.2.8) of a metrized quantum vector bundle is provided by Riemannian geometry and locally trivial complex vector bundles. Our idea is that the choice of a metric connection for a hermitian metric on a complex vector bundle is in fact a part of the metric data of the associated module — and gives rise to a prototypical D-norm.

Example 2.2.10. Let M be a compact connected differentiable manifold of dimension n. As a matter of convention, we assume in this example that vector bundle is meant for locally trivial vector bundle, and that all our vector bundles have complex vector spaces as fibers; in particular, the tangent and cotangent bundles are complexified (by taking their tensor product with the trivial bundle $M \times \mathbb{C}$).

A natural derivation, namely the exterior differential, acts on the dense *-subalgebra $C^1(M)$ of C^1, $\mathbb{C}$-valued functions over M, inside the C*-algebra $C(M)$ of $\mathbb{C}$-valued continuous functions over M. This derivation is valued in the $C(M)$-$C(M)$-bimodule Ω_1 of continuous sections of the cotangent bundle $T^*_{\mathbb{C}}M$ of M. We recall that for all pair f, g of C^1 functions on M, we have $d(fg) = f \wedge dg + df \wedge g = fdg + df \cdot g$.

We are interested in metric structures, and thus we naturally endow M with some Riemannian metric g. Formally, a metric g^V on a vector bundle V is a smooth section of the vector bundle over M of sesquilinear functionals over each fiber of V such that for all $x \in \mathfrak{M}$, the map g^V_x over the fiber V_x of V at x is in fact a hermitian inner product. In particular, a Riemannian metric is given as a metric over the cotangent bundle $T^*_{\mathbb{C}}M$ (or equivalently over the tangent bundle $T_{\mathbb{C}}M$ of M).

Now, if ΓV is the $C(M)$-left module of continuous sections of a vector bundle V over M, then setting, for all $\omega, \eta \in \Gamma V$:

$$x \in M \mapsto g^V_x(\omega_x, \eta_x)$$

defines a $C(X)$-valued inner product on ΓV, which we still denote by slight abuse of notation by g^V. Thus, (Γ, g^V) is a left Hilbert module with norm, for all $\omega \in \Gamma V$:

$$\|\omega\|_{\Gamma V} = \sup_{x \in M} \sqrt{g^V_x(\omega_x, \omega_x)}.$$

Notably, $\|\cdot\|_{\Gamma_V}$ is a module norm, i.e. for all $f \in C(X)$ and $\omega \in \Gamma V$ we can trivially check that $\|f\omega\|_{\Gamma_V} \leqslant \|f\|_{C(X)} \|\omega\|_{\Gamma_V}$.

We note that we will simply write $g^V_x(\omega, \eta)$ for $g^V_x(\omega_x, \eta_x)$ in the rest of this example, whenever $x \in M$, and ω and η are in ΓV.

Let us focus for a moment on the case where V is the cotangent bundle $T^*_{\mathbb{C}}M$ of M, and g some Riemannian metric for M. We note that the right action of $C(M)$ on Ω_1 is by so-called adjoinable operators, and in fact (Ω_1, g) is a Hilbert C*-bimodule over $C(M)$. Consequently, if we define:

$$\mathsf{L} : f \in C^1(M) \mapsto \|df\|_{\Omega_1}$$

then L is a seminorm defined on a dense subalgebra of $C(M)$, taking the value 0 exactly on the constant functions over M since M is connected, and satisfying the Leibniz inequality:

$$\mathsf{L}(fg) \leqslant \|f\|_{C(X)} \mathsf{L}(g) + \mathsf{L}(f) \|g\|_{C(X)}.$$

Thanks to the Arzéla-Ascoli theorem and since M is compact, we note that:

$$\mathfrak{B}\mathfrak{L}_1 = \left\{ f \in C(M) : \mathsf{L}(f) \leqslant 1, \|f\|_{C(X)} \leqslant 1 \right\}$$

is totally bounded in $C(X)$. Moreover, the closure of $\mathfrak{B}\mathfrak{L}_1$ consists of the 1-Lipschitz functions with respect to the Riemannian path distance induced by g on M, and the Lipschitz seminorm for this distance is the Minkowsky gauge functional of the closure of $\mathfrak{B}\mathfrak{L}_1$. As it agrees with L on $C^1(M)$, we denote the Lipschitz seminorm simply by L as well. Its closed unit ball is now compact (as it is closed since L is lower semicontinuous), and it is trivial to check that is satisfies the Leibniz inequality. Thus $(C(X), \mathsf{L})$ is a Leibniz quantum compact metric space.

Our purpose is to introduce data on modules which will allow us to define a Gromov-Hausdorff distance between them, and thus we now return to our discussion of the metric structure of a generic vector bundle V over M. Given a metric g^V on V, a very important fact of Riemannian geometry is the existence of a metric connection, i.e. a connection ∇ on V with the property that for all tangent vector fields X of M, and for all $\omega, \eta \in \Gamma V$, we have:

$$d_X g^V(\omega, \eta) = g^V(\nabla_X \omega, \eta) + g^V(\omega, \nabla_X \eta).$$

If we require the connection to be torsion free when $V = T^*_{\mathbb{C}} M$, then the connection ∇ is unique and called the Lévi-Civita connection; we will however work on general vector bundles and not require any condition on the torsion of our metric connections. Instead, we look at the connection as part of the metric information of our vector bundle V.

Thus, let us fix a complex bundle V over M with a metric g^V and let ∇ be a g^V-metric connection on V. Let ΓV be the $C(M)$-module of continuous sections of V over M and $\Gamma^1 V$ the $C^1(M)$-module of differentiable sections over M.

We already have endowed T^*M with a Riemannian metric g, and thus we also have a metric on the tangent bundle TM by (fiber-wise) duality; we denote this metric by g_*. The connection ∇ defined, for all differentiable section ω of V, the linear map:

$$\nabla \omega : X \in TM \mapsto \nabla_X \omega \in \Gamma V.$$

We define for all differentiable section ω the norm:

$$|||\nabla \omega||| = \sup_{x \in M} \sup \left\{ \sqrt{g^V_x(\nabla_X \omega, \nabla_X \omega)} : X \in TM, g_{*x}(X, X) = 1 \right\},$$

i.e. the operator norm of $\nabla \omega$ for the underlying inner products valued in $C(M)$ on the module of vector fields and the module of sections of V.

For all differentiable $\omega \in \Gamma V$, we set:

$$\mathsf{D}(\omega) = \max \left\{ \|\omega\|_{\Gamma V}, |||\nabla \omega||| \right\}.$$

We now explicit some formulas which we will need. Since M is compact and V is locally trivial, there exists a finite atlas $\mathscr{U}$ such that for any chart $(U, \psi) \in \mathscr{U}$, we also have a local frame for V over U, i.e. k functions $\{e^U_1, \ldots, e^U_k\}$ such that for all $x \in U$, the set $\{e^U_1(X), \ldots, e^U_k(x)\}$ is a basis for the fiber V_x.

We moreover let $\mathscr{V}$ be an open cover of M with the property that for all $O \in \mathscr{V}$, there exists $(U, \psi) \in \mathscr{U}$ such that the closure $\mathrm{cl}(O) \subseteq U$. This can always be achieved since M is compact.

Let us fix $(U, \psi) \in \mathscr{U}$. Let $\{\partial_1, \ldots, \partial_n\}$ be some set of tangent vector fields on U such that for all $x \in U$, the set $\{\partial_1(x), \ldots, \partial_n(x)\}$ is a basis for $T_x M$.

We also note that ∇ restricts to a metric connection for (V, g^V) restricted to U, as a vector bundle over U. We shall tacitly identify ∇ with its restriction.

For any $\omega \in \Gamma_V$, we now write $\omega = \sum_{j=1}^n \omega_j e_j^U$ for $\omega_1, \ldots, \omega_k \in C^1(M)$. Now, if we write, for all p, r in $\{1, \ldots, k\}$ and $q \in \{1, \ldots, n\}$:

$$\nabla_{\partial_q} e_p = \sum_{r=1}^k \Gamma_{pq}^r e_r$$

noting that $e_1, \ldots, e_k$ are smooth so that the above expression makes sense, then, for all smooth ω:

$$\nabla_{\partial_q} \omega = \sum_{r=1}^k \left(\partial_q \omega_r + \sum_{p=1}^k \Gamma_{pq}^r \omega_p \right) e_r.$$

In particular, for any $x \in U$, $q \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$, we thus have:

$$\left| \left(\partial_q \omega\right)_j (x) \right| \leqslant \left| \left(\nabla_{\partial_q} \omega(x) \right)_j - \sum_{p=1}^k \Gamma_{pq}^r \omega_p \right|.$$

We now need a few estimates. We first note that by construction, if for all $x \in U$ we set:

$$G_x = \begin{pmatrix} g_x^V(e_1^U, e_1^U) & \ldots & g_x^V(e_k^U, e_1^U) \\ \vdots & & \vdots \\ g_x^V(e_1^U, e_k^U) & \ldots & g_x^U(e_k^U, e_k^U) \end{pmatrix}$$

then $G : x \in U \mapsto G_x$ is a continuous function valued in the $k \times k$ (positive symmetric invertible) matrices. We endow the algebra of $k \times k$ matrices with the norm $|||\cdot|||_k$ induced by the usual inner product on $\mathbb{C}^d$.

Moreover by construction, if we set:

$$\left\langle \sum_{j=1}^k \omega_j e_j, \sum_{r=1}^k \eta_k e_k \right\rangle_x = \sum_{j=1}^k \omega_j(x) \overline{\eta_j(x)}$$

then we have $g_x^V(\omega, \eta) = \langle G\omega, \eta \rangle_x$. Now, for all $x \in U$, we have:

$$\begin{aligned} \max\left\{ |\omega_j(x)| : j \in \{1, \ldots, k\} \right\} &\leqslant \sqrt{\langle \omega, \omega \rangle_x} \\ &\leqslant \sqrt{\left|\left|\left| G_x^{-1} \right|\right|\right|_k \langle G\omega, \omega \rangle_x} \\ &= \sqrt{\left|\left|\left| G_x^{-1} \right|\right|\right|_k} \sqrt{g_x^V(\omega, \omega)}. \end{aligned}$$

We now pick any $O \subseteq \mathscr{V}$ such that the closure $\mathrm{cl}(O)$ lies inside our chosen U. In this case, G, and therefore G^{-1}, are continuous on the compact $\mathrm{cl}(O)$ and thus, $x \in \mathrm{O} \mapsto \left|\left|\left| G_x^{-1} \right|\right|\right|_k$ is bounded below and above; since the bounds are reached and G_x is never null, we conclude that there exists $w > 0$ such that for all $x \in \mathrm{cl}(O)$ we have:

$$\max\left\{ |\omega_j(x)| : j \in \{1, \ldots, k\} \right\} \leqslant w \sqrt{\langle G\omega, \eta \rangle_x} = w \sqrt{g_x^V(\omega, \omega)}.$$

In particular, we note that if $\mathsf{D}(\omega) \leqslant 1$ and $p \in \{1, \ldots, k\}$ then:

$$\|\omega_p\|_{C(\mathrm{cl}(O))} \leqslant w. \tag{2.2.1}$$

We also note that the functions $x \in \mathrm{cl}(O) \mapsto \sqrt{g_{*x}(\partial_q, \partial_q)}$ are continuous on a compact as well for all $q \in \{1, \ldots, d\}$, and thus we can choose $K > 0$ such that:

$$\sup\left\{\sqrt{g_{*x}(\partial_q, \partial_q)} : x \in \mathrm{cl}(O), q \in \{1, \ldots, n\}\right\} \leqslant K.$$

Last, we note that since the Christoffel symbols of our connection in our local frame are continuous as well, there exists $K_2 > 0$ such that:

$$\sup\left\{\left|\Gamma^r_{pq}\right| : p, r \in \{1, \ldots, k\}, q \in \{1, \ldots, n\}, x \in \mathrm{cl}(O)\right\} \leqslant K_2.$$

We have thus for all $\omega \in \Gamma V$ with $\mathsf{D}(\omega) \leqslant 1$, $x \in \mathrm{cl}(O)$ and $q \in \{1, \ldots, k\}$:

$$\begin{aligned}
|\nabla_{\partial_q}\omega_j(x)| &\leqslant \max\left\{|\nabla_{\partial_q}\omega_j(x)| : j \in \{1, \ldots, k\}\right\} \\
&\leqslant w\sqrt{g^V_x(\nabla_{\partial_q}\omega, \nabla_{\partial_q}\omega)} \\
&\leqslant w|||\nabla\omega|||\sqrt{g_{*x}(\partial_q, \partial_q)} \\
&\leqslant wK.
\end{aligned}$$

Therefore for all $q \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$, and for all $\omega \in \Gamma V$ with $\mathsf{D}(\omega) \leqslant 1$, we estimate:

(2.2.2)
$$\begin{aligned}
\left\|(\partial_q\omega)_j\right\|_{C(\mathrm{cl}(O))} &= \sup_{x \in \mathrm{cl}(O)} |\partial_q\omega(x)| \\
&\leqslant \sup_{x \in \mathrm{cl}(O)} \left|\left(\nabla_{\partial_q}\omega\right)_j(x)\right| \\
&\quad + k\sup\left\{\left|\Gamma^r_{pq}(x)\right| : p, r \in \{1, \ldots, k\}, q \in \{1, \ldots, n\}, x \in \mathrm{cl}(O)\right\} \\
&\quad \times \max_{p=1}^{k} \|\omega_p\|_{C(\mathrm{cl}(O))} \\
&\leqslant wK + wkK_2.
\end{aligned}$$

Let:

$$\mathscr{D}_1(\mathsf{D}, O) = \left\{\text{the restriction of } \omega_j \text{ to } \mathrm{cl}(O) : \mathsf{D}(\omega) \leqslant 1, j \in \{1, \ldots, k\}\right\}.$$

We deduce from Equations (2.2.2) and (2.2.1) that $\mathscr{D}_1(\mathsf{D}, O)$ is an equicontinuous family (in fact, a collection of $(Kw + wkK_2)$-Lipschitz functions).

We may now apply Arzéla-Ascoli theorem to the set $\mathscr{D}_1(\mathsf{D}, O)$, viewed as an equicontinuous set of functions on the compact O and valued in a fixed compact in $\mathbb{C}$. Thus, $\mathscr{D}_1(\mathsf{D}, O)$ is totally bounded for the uniform norm $\|\cdot\|_{\mathrm{cl}(O)}$.

Now, we also observe that since G is bounded above as well, there exists $w_2 > 0$ such that for all $\omega \in \Gamma_V$ and $x \in \mathrm{cl}(O)$:

$$g^V_x(\omega, \omega) \leqslant |||G_x|||_k\langle\omega, \omega\rangle_x \leqslant k|||G_x|||_k \max\{|\omega_1(x)|, \ldots, |\omega_k(x)|\}.$$

Therefore, we conclude that $\mathscr{D}_1(\mathsf{D})$ is totally bounded for the seminorm:

$$\|\omega\|_{\Gamma V,O} = \omega \in \Gamma V \mapsto \sup_{x\in \mathrm{cl}(O)} \sqrt{g_x^V(\omega,\omega)}.$$

Last, we note that:

$$\|\omega\|_{\Gamma V} = \sup_{x\in M} \sqrt{g_x^V(\omega,\omega)} = \max_{O\in\mathscr{V}} \|\omega\|_{\Gamma V,O}$$

from which it is easy to deduce that $\mathscr{D}_1(\mathsf{D})$ is totally bounded for $\|\cdot\|_{\Gamma V}$.

Our reasoning proves that for all $O \in \mathscr{V}$, the set of all the restrictions of elements $\omega \in \Gamma V$ with $\mathsf{D}(\omega) \leqslant 1$ is equicontinuous and obviously bounded. Since $\mathscr{V}$ is a finite cover of M, we then conclude that $\{\omega \in \Gamma V : \mathsf{D}(\omega) \leqslant 1\}$ is equicontinuous on the compact M and valued in the common compact thus by Arzéla-Ascoli, we conclude that $\{\omega \in \Gamma V : \mathsf{D}(\omega) \leqslant 1\}$ is compact for the supremum norm.

Furthermore, since ∇ is a connection, we check that:

$$\mathsf{D}(f\omega) \leqslant \mathsf{L}(f)\|\omega\|_{\Gamma V} + \|f\|_{C(M)}\mathsf{D}(\omega)$$

for all $f \in C^1(X)$ and smooth $\omega \in \Gamma V$ (and using the fact that $\|\cdot\|_{\Gamma V}$ is a module norm), while since ∇ is a metric connection, we also check that:

$$\mathsf{L}(g^V(\omega,\eta)) \leqslant \mathsf{D}(\omega)\|\eta\|_{\Gamma V} + \|\omega\|_{\Gamma V}\mathsf{D}(\eta)$$

for all smooth $\omega, \eta \in \Gamma V$.

As we did with L, we extend D by defining D_V as the Minkowsky functional of the norm closure of $\mathscr{D}_1(\mathsf{D})$, which is compact. Thus $(\Gamma V, g^V, \mathsf{D}_V, C(M), \mathsf{L})$ is a metrized quantum vector bundle.

We note that it is not clear in general what $\ker\nabla = \{\omega \in \Gamma^1 V : \nabla\omega = 0\}$ might be; it is a key reason why we actually define our D-norms to dominate the underlying norm: by taking the maximum of the module norm and the norm of a connection, we remove the question of what the kernel should be and we can make a clear requirement of compactness for the closed unit ball of our D-norm.

Example (2.2.10) justifies the following terminology:

Definition 2.2.11. A (F,G,H)–metrized quantum vector bundle is *Leibniz* when:

(1) $F = (x,y,l_x,l_y) \in [0,\infty)^4 \mapsto xl_y + yl_x$,
(2) $G = (x,l,d) \in [0,\infty)^3 \mapsto (x+l)d$,
(3) $H = (x,y) \in [0,\infty)^2 \mapsto 2xy$.

We note that a the admissible triple for a Leibniz metrized quantum vector bundle is chosen to be the lower allowed bounds in Definition (2.2.6).

We now turn to examples of metrized quantum vector bundles over general quasi-Leibniz quantum compact metric spaces. We begin with the observation that Definition (2.2.8) contains, in the statement of the inner quasi-Leibniz inequality, a canonical extension of the L-seminorm to a dense *-subalgebra of the entire base quantum space. This extension possess properties which will prove helpful for our next few examples.

Lemma 2.2.12. *Let $(\mathfrak{A}, \mathsf{L})$ be a F–quasi-Leibniz quantum compact metric space for some admissible function F. The seminorm:*

$$\mathrm{M} : a \in \mathfrak{A} \mapsto \max\{\mathsf{L}(\Re a), \mathsf{L}(\Im a)\}$$

satisfies:

$$\mathrm{M}(ab) \leqslant 8F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathrm{M}(a), \mathrm{M}(b)).$$

Moreover the domain of M *is a dense *-subalgebra of* $\mathfrak{A}$*, while* M *satisfies* $\mathrm{M}(a^*) = \mathrm{M}(a)$ *for all* $a \in \mathfrak{A}$ *and* $\{a \in \mathfrak{A} : \mathrm{M}(a) = 0\} = \mathbb{C}1_{\mathfrak{A}}$.

Proof. We first observe that M is a seminorm (which may assume the value ∞, though it is finite on $\mathrm{dom}\,(\mathsf{L}) + i\mathrm{dom}\,(\mathsf{L})$; moreover it is easy to check that $\mathrm{M}(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathfrak{A}}$). Moreover, M restricted to $\mathfrak{sa}\,(\mathfrak{A})$ is of course L. It is similarly straightforward to note that $\mathrm{M}(a^*) = \mathrm{M}(a)$ for all $a \in \mathfrak{A}$.

Let $a, b \in \mathfrak{A}$. We note, since $ab = a \circ b + i\{a, b\}$ and M is a norm:

$$\mathrm{M}(ab) \leqslant \mathrm{M}(a \circ b) + \mathrm{M}(\{a, b\}). \tag{2.2.3}$$

We then have:

$$\begin{aligned}
\mathrm{M}(a \circ b) &\leqslant \mathrm{M}(\Re(a) \circ \Re(b)) + \mathrm{M}(\Re(a) \circ \Im(b)) \\
&\quad + \mathrm{M}(\Im(a) \circ \Re(b)) + \mathrm{M}(\Im(a) \circ \Im(b)) \\
&= \mathsf{L}(\Re(a) \circ \Re(b)) + \mathsf{L}(\Re(a) \circ \Im(b)) \\
&\quad + \mathsf{L}(\Im(a) \circ \Re(b)) + \mathsf{L}(\Im(a) \circ \Im(b)) \\
&\leqslant F(\|\Re(a)\|_{\mathfrak{A}}, \|\Re(b)\|_{\mathfrak{A}}, \mathsf{L}(\Re(a)), \mathsf{L}(\Re(b))) \\
&\quad + F(\|\Re(a)\|_{\mathfrak{A}}, \|\Im(b)\|_{\mathfrak{A}}, \mathsf{L}(\Re(a)), \mathsf{L}(\Im(b))) \\
&\quad + F(\|\Im(a)\|_{\mathfrak{A}}, \|\Re(b)\|_{\mathfrak{A}}, \mathsf{L}(\Im(a)), \mathsf{L}(\Re(b))) \\
&\quad + F(\|\Im(a)\|_{\mathfrak{A}}, \|\Im(b)\|_{\mathfrak{A}}, \mathsf{L}(\Im(a)), \mathsf{L}(\Im(b))) \\
&\leqslant 4F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathrm{M}(a), \mathrm{M}(b)).
\end{aligned}$$

A similar computation allows us to conclude:

$$\mathrm{M}(\{a, b\}) \leqslant 4F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathrm{M}(a), \mathrm{M}(b))$$

and thus, using Inequality (2.2.3), our lemma is proven. □

Remark 2.2.13. Let (X, d) be a compact metric space and L be the seminorm induced on $C(X)$ by d. While L is the Lipschitz seminorm for functions from X valued in $\mathbb{C}$ with its usual hermitian norm, we note that $\mathrm{M} = \max\{\mathsf{L} \circ \Re, \mathsf{L} \circ \Im\}$ is the Lipschitz seminorm of functions form X to $\mathbb{C}$ endowed with the norm $\|x + iy\| = \max\{|x|, |y|\}$ for all $x, y \in \mathbb{R}^2$.

This observation is interesting because Blaschke theorem is valid for the seminorm M: when $\mathbb{C}$ is endowed with $\|\cdot\|$ rather than its standard hermitian norm, a k-Lipschitz function on some subset of X to $\mathbb{C}$ can be extended a k-Lipschitz function over X. Indeed, it is easy to check that for any two quasi-Leibniz quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, a *-morphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is a quantum isometry if and only if $\max\{\mathsf{L}_{\mathfrak{B}} \circ \Re, \mathsf{L}_{\mathfrak{B}} \circ \Im\}$ is the quotient of $\max\{\mathsf{L}_{\mathfrak{A}} \circ \Re, \mathsf{L}_{\mathfrak{A}} \circ \Im\}$ and the notion of full quantum isometry extends similarly. Thus Lemma (2.2.12) provides a rather canonical way to extend L-seminorms while keeping all notions of isometries unchallenged.

We first note that every quasi-Leibniz quantum compact metric space defines a canonical metrized quantum vector bundle over itself. This observation implies that the modular propinquity will provide another metric on quasi-Leibniz quantum compact metric spaces, though we will prove that it is equivalent to the quantum Gromov-Hausdorff propinquity.

Example 2.2.14. Let $(\mathfrak{A}, \mathsf{L})$ be a F–quasi-Leibniz quantum compact metric space. The C*-algebra $\mathfrak{A}$ is of course a left module over itself, using the multiplication of $\mathfrak{A}$ on the left. The C*-algebra $\mathfrak{A}$ is naturally a left $\mathfrak{A}$-Hilbert module by setting for all $a, b \in \mathfrak{A}$:

$$\langle a, b\rangle_{\mathfrak{A}} = ab^*.$$

Note that for every state φ of $\mathfrak{A}$, the completion of the pre-Hilbert space $\mathfrak{A}$ endowed with $\varphi \circ \langle\cdot,\cdot\rangle_{\mathfrak{A}}$ provides the Gel'fand-Naimark-Segal representation associated with φ

The norm of $a \in \mathfrak{A}$ is:

$$\sqrt{\|aa^*\|_{\mathfrak{A}}} = \|a\|_{\mathfrak{A}},$$

and thus the C*-norm $\|\cdot\|_{\mathfrak{A}}$ and the C^*-Hilbert norm $\|\cdot\|_{\mathscr{A}}$ agree. In particular, $(\mathfrak{A}, \langle\cdot,\cdot\rangle_{\mathfrak{A}})$ is complete for its norm.

We can now set $\mathsf{D}_{\mathfrak{A}}(a) = \max\{\mathsf{L}(\Re a), \mathsf{L}(\Im a), \|a\|_{\mathfrak{A}}\}$ for all $a \in \mathfrak{A}$. It is easy to check, using Lemma (2.2.12), that:

$$\Omega(\mathfrak{A}) = (\mathfrak{A}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L})$$

is a $(F, 8F, 8F)$–metrized quantum vector bundle.

We extend Example (2.2.14) to free modules. Free modules are of course basic examples, but they are also important since every finitely generated projective modules lies inside a free module; thus the construction in the next example would provide D-norms to many finitely generated projective modules under appropriate conditions. This being said, our main example of non free, finitely generated projective modules in this paper — Heisenberg modules over quantum 2-tori — will come with a D-norm from a connection, akin to Example (2.2.10) though involving very different techniques. The following example is thus a natural default source of D-norms, while our work may accommodate different D-norms if the context calls for it.

Example 2.2.15. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a F–quasi-Leibniz quantum compact metric space for some admissible function F. Let $d \in \mathbb{N} \setminus \{0\}$. Let $\mathscr{M} = \mathfrak{A}^d$. The map:

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right\rangle_{\mathscr{M}} = \sum_{j=1}^d a_j b_j^*,$$

for all $\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \in \mathfrak{A}^d$, is an $\mathfrak{A}$-inner product, for which $\mathscr{M}$ is complete.

We set $\mathrm{M}_{\mathfrak{A}}(a) = \max\{\mathsf{L}_{\mathfrak{A}}(\Re a), \mathsf{L}_{\mathfrak{A}}(\Im a)\}$ for all $a \in \mathfrak{A}$, and then::

$$\mathsf{L}_{\mathfrak{A}}^d \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = \max\left\{\mathrm{M}(a_j) : j \in \{1,\ldots,d\}\right\}$$

for all $a_1,\ldots,a_d \in \mathfrak{A}$. $\mathsf{L}_{\mathscr{M}}$ is a form of L-seminorm for modules. We note that $\mathsf{L}_{\mathfrak{A}}^d(x) = 0$ if and only if $x \in \mathbb{C}^d$.

We now define $\mathsf{D}_{\mathfrak{A}}^d = \max\left\{\|\cdot\|_{\mathscr{M}}, \mathsf{L}_{\mathfrak{A}}^d\right\}$.

To begin with, we note that for all $a, b_1, \ldots, b_d \in \mathfrak{A}$:

$$\begin{aligned}
\mathsf{L}_{\mathfrak{A}}^d \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right) &= \max\left\{\mathrm{M}(ab_j) : j \in \{1,\ldots,d\}\right\} \\
&= \max\left\{8F(\|a\|_{\mathfrak{A}}, \|b_j\|_{\mathfrak{A}}, \mathrm{M}_{\mathfrak{A}}(a), \mathrm{M}(b_j)) : j \in \{1,\ldots,d\}\right\} \\
&\leqslant 8F\left(\|a\|_{\mathfrak{A}}, \left\| \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right\|_{\mathscr{M}}, \mathrm{M}(a), \mathsf{L}_{\mathfrak{A}}^d \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right) \\
&\leqslant G\left(\|a\|_{\mathfrak{A}}, \mathrm{M}(a), \mathsf{L}_{\mathfrak{A}}^d \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right),
\end{aligned}$$

where $G(x,y,z) = 8F(x,y,z,y)$ for all $x,y,z \geqslant 0$.

Since $\|\cdot\|_{\mathscr{M}}$ is a C^*-Hilbert norm and $\mathrm{M}(a) = \mathsf{L}_{\mathfrak{A}}(a)$ if $a = a^*$, we conclude that if $a \in \mathfrak{sa}(\mathfrak{A})$ then:

$$\mathsf{D}_{\mathfrak{A}}^d \left(a \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right) \leqslant G\left(\|a\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(a), \mathsf{D}_{\mathfrak{A}}^d \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} \right).$$

Moreover, again using Lemma (2.2.12), we also have, for all $a_1,\ldots,a_d,b_1,\ldots,b_d \in \mathscr{M}$, that:

$$\begin{aligned}
\mathsf{L}_{\mathfrak{A}}\left(\Re\left\langle\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix},\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix}\right\rangle_{\mathscr{M}}\right) &= \mathsf{L}_{\mathfrak{A}}\left(\Re\left(\sum_{j=1}^d a_j b_j\right)\right)\\
&\leqslant \mathsf{M}\left(\sum_{j=1}^d a_j b_j\right)\\
&\leqslant 8\sum_{j=1}^d F(\|a_j\|_{\mathfrak{A}},\|b_j\|_{\mathfrak{A}},\mathrm{M}(a_j),\mathrm{M}(b_j))\\
&\leqslant 8dF\left(\left\|\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix}\right\|_{\mathscr{M}},\left\|\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix}\right\|_{\mathscr{M}},\mathsf{L}_{\mathfrak{A}}^d\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix},\mathsf{L}_{\mathfrak{A}}^d\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix}\right)\\
&\leqslant dF\left(\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix},\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix},\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix},\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix}\right)\\
&= H\left(\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix},\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}b_1\\ \vdots\\ b_d\end{pmatrix}\right),
\end{aligned}$$

where $H(x,y) = 8dF(x,y,x,y)$ for all $x,y \geqslant 0$.

If is immediate that (F,G,H) is an admissible triplet. Thus Conditions (1), (3) and (4) of Definition (2.2.8) are met.

Last, let $(a_1^n,\ldots,a_d^n) \in \mathscr{M}^{\mathbb{N}}$ be a sequence such $\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}a_1^n\\ \vdots\\ a_d^n\end{pmatrix} \leqslant 1$ for all $n \in \mathbb{N}$.

Thus $(\Re a_1^n)_{n\in\mathbb{N}}$ lies in the compact $\{a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant 1\}$; we thus may extract a $\|\cdot\|_{\mathfrak{A}}$-convergent subsequence $(\Re a_1^{f_1(n)})_{n\in\mathbb{N}}$ with limit $a_1 \in \mathfrak{sa}(\mathfrak{A})$ such that $\mathsf{L}_{\mathfrak{A}}(a_1) \leqslant 1$ and $\|a_1\|_{\mathfrak{A}} \leqslant 1$ (we used the fact that $\mathsf{L}_{\mathfrak{A}}$ is lower semicontinuous with respect to $\|\cdot\|_{\mathfrak{A}}$). For the same reason, we can then extract convergent subsequences $(\Re a_2^{f_1\circ f_2(n)})_{n\in\mathbb{N}}$ of $(\Re a_2^{f_1(n)})_{n\in\mathbb{N}}$ with limit $\Re a_2 \in \mathfrak{sa}(\mathfrak{A})$, $\ldots$, $(\Re a_d^{f_1\circ f_2\circ\cdots\circ f_d(n)})_{n\in\mathbb{N}}$ from $(\Re a_d^{f_1\circ f_2\circ\cdots\circ f_{d-1}(n)})_{n\in\mathbb{N}}$ with limit $a_d \in \mathfrak{sa}(\mathfrak{A})$; moreover $\max\{\mathsf{L}_{\mathfrak{A}}(a_j),\|a_j\|_{\mathfrak{A}} : j \in \{1,\ldots,n\}\} \leqslant 1$.

If $g : n \in \mathbb{N} \mapsto f_1\circ f_2\circ\cdots\circ f_d(n)$, then $\begin{pmatrix}\Re a_1^{g(n)}\\ \vdots\\ \Re a_d^{g(n)}\end{pmatrix}_{n\in\mathbb{N}}$ converges to $\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix}$, where by construction $\mathsf{D}_{\mathfrak{A}}^d\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix} \leqslant 1$.

Just as easily, we can prove that there exists $b_1, \ldots, b_d \in \mathfrak{sa}(\mathfrak{A})$ and a function $h : \mathbb{N} \to \mathbb{N}$ strictly increasing such that:

$$\lim_{n\to\infty} \begin{pmatrix} \Im a_1^{g(h(n))} \\ \vdots \\ \Im a_d^{g(h(n))} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix}$$

and therefore:

$$\lim_{n\to\infty} \begin{pmatrix} a_1^{g(h(n))} \\ \vdots \\ a_d^{g(h(n))} \end{pmatrix} = \begin{pmatrix} a_1 + ib_1 \\ \vdots \\ a_d + ib_d \end{pmatrix}.$$

By construction, $\mathsf{D} \begin{pmatrix} a_1 + ib_1 \\ \vdots \\ a_d + ib_d \end{pmatrix} \leqslant 1$.

Thus $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}^d_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_A)$ is a (F, G, H)–metrized quantum vector bundle.

We make a few additional comments on our Definition (2.2.8). Condition (3) will be used to prove that distance zero for the modular propinquity will in particular give rise to a module morphism. Condition (4) connects the metric structures of the D-norm and the L-seminorms. Condition (1) is just a normalization condition: indeed, the unit sphere for a D-norm is compact for the C^*-Hilbert norm and thus the norm attains a maximum on it; thus the C^*-Hilbert norm is always less than some constant multiple of the D-norm, thanks to Condition (2). Last, Condition (2) provides us with the compact set we will use to start the construction of a Gromov-Hausdorff distance for modules.

A consequence of Condition (4) is an additional implicit structure in metrized quantum vector bundles:

Remark 2.2.16. Definition (2.2.8) implies that, given a metrized quantum vector bundle $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, the space $\operatorname{dom}(\mathsf{D}_{\mathscr{M}})$ is a left module over the Jordan-Lie algebra $\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$, and that the inner product $\langle\cdot,\cdot\rangle_{\mathscr{M}}$ restricts to an $\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$-valued inner product on $\operatorname{dom}(\mathsf{D}_{\mathscr{M}})$.

Condition (1) (as well as Condition (2)) implies that a D-norm is lower semi-continuous with respect to the C^*-Hilbert norm, which implies:

Remark 2.2.17. Let $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a metrized quantum vector bundle. The $\mathbb{C}$-vector space $(\operatorname{dom}(\mathsf{D}_{\mathscr{M}}), \mathsf{D}_{\mathscr{M}})$ is a Banach space. Indeed, $\mathsf{D}_{\mathscr{M}}$ is lower semi-continuous with respect to $\|\cdot\|_{\mathscr{M}}$ since its unit ball is compact, hence closed, for $\|\cdot\|_{\mathscr{M}}$; moreover $\mathsf{D}_{\mathscr{M}}$ dominates $\|\cdot\|_{\mathscr{M}}$.

The proof that the lower semi-continuity of $\mathsf{D}_{\mathscr{M}}$ implies that $(\mathscr{M}, \mathsf{D}_{\mathscr{M}})$ is a Banach space is identical to the proof that $(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}}), \max\{\|\cdot\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}\})$ is a Banach space, as found in [30].

The category of metrized quantum vector bundles, whose objects are introduced in Definition (2.2.8), is constructed using the following natural notion of morphism.

Definition 2.2.18. Let:

$$\Omega_{\mathfrak{A}} = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}}, \mathsf{D}_{\mathscr{N}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

be two metrized quantum vector bundles. A *morphism* (Θ, θ) from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ is a unital module morphism (Θ, θ) such that:

1. θ is continuous from $(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}}), \mathsf{L}_{\mathfrak{A}})$ to $(\mathrm{dom}(\mathsf{L}_{\mathfrak{B}}), \mathsf{L}_{\mathfrak{B}})$, i.e. there exists $C > 0$ such that $\mathsf{L}_{\mathfrak{B}} \circ \theta \leqslant C\mathsf{L}_{\mathfrak{A}}$ on $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$,
2. Θ is continuous from $(\mathrm{dom}(\mathsf{D}_{\mathscr{M}}), \mathsf{D}_{\mathscr{M}})$ to $(\mathrm{dom}(\mathsf{D}_{\mathscr{N}}), \mathsf{D}_{\mathscr{N}})$, i.e. there exists $M > 0$ such that for all $\omega \in \mathscr{M}$ we have $\mathsf{D}_{\mathscr{N}}(\Theta(\omega)) \leqslant M\mathsf{D}_{\mathscr{M}}(\omega)$.

Such a *morphisms* is an *epimorphism* when both θ and Θ are surjective, and a monomorphism when both θ and Θ are both monomorphisms.

A *isomorphism* is thus given by a morphism (Θ, θ) where θ is a *-isomorphism, Θ is a bijection and $(\Theta^{-1}, \theta^{-1})$ is a morphism from $\Omega_{\mathfrak{B}}$ onto $\Omega_{\mathfrak{A}}$.

As is customary with categories of metric spaces, there are several appropriate of isomorphisms. Inside the general category described via Definitions (2.2.8) and (2.2.18), an isomorphism would be a generalization of a bi-Lipschitz map. For our purpose, a stronger notion of isomorphism will be employed, akin to a notion of isometry. We first define the notion of a full quantum isometry, which is rather self-evident:

Definition 2.2.19. Let:

$$\Omega_{\mathfrak{A}} = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}}, \mathsf{D}_{\mathscr{N}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

be two metrized quantum vector bundles. A *full quantum isometry* (Θ, θ) from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ is a metrized quantum vector bundle isomorphism from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ such that:

1. $\mathsf{L}_{\mathfrak{B}} \circ \theta = \mathsf{L}_{\mathfrak{A}}$ on $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$,
2. $\mathsf{D}_{\mathscr{N}} \circ \Theta = \mathsf{D}_{\mathscr{M}}$ on $\mathrm{dom}(\mathsf{D}_{\mathscr{M}})$.

It is easy to check that the category of metrized quantum vector bundles with quantum isometries as morphisms is a subcategory of the category whose morphisms are given by Definition (2.2.18).

The more delicate question for us regards the notion of a quantum isometry for metrized quantum vector bundles. As we discussed when introducing quantum isometries for quasi-Leibniz quantum compact metric spaces, isometries between L-seminorms rely on working with self-adjoint elements only. We did note in Remark (2.2.13) that we can extend L-seminorms to bypass this issue, though the situation for module requires some idea.

We propose to circumvent this issue by bringing the problem down to the base quantum spaces. Indeed, we take advantage of the observation that the inner quasi-Leibniz inequality implies that for any metrized quantum vector bundle $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ and any $\omega, \eta \in \mathscr{M}$, the elements $\Re\langle\omega,\eta\rangle_{\mathscr{M}}$ and $\Im\langle\omega,\eta\rangle_{\mathscr{M}}$ lies in the domain $\mathrm{dom}(\mathsf{L})$ of the L-seminorm L. Thus, the tools developed for the Gromov-Hausdorff propinquity can be brought to bare to the study of metrized quantum vector bundles.

Now, while the Cauchy-Schwarz inequality for the C^*-Hilbert norm of a left pre-Hilbert module $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ implies that:

$$\|\omega\|_{\mathscr{M}} = \sup\{|\langle\omega,\eta\rangle_{\mathscr{M}}| : \eta \in \mathscr{M}, \|\eta\|_{\mathscr{M}} \leqslant 1\}, \tag{2.2.4}$$

we want to work only with elements bounded for the D-norms. We now study the metric on the domain of D-norm resulting from working with Expression (2.2.4), with the closed unit ball for the C^*-Hilbert norm replaced by the closed unit ball for the D-norm. We begin with a notation we shall use throughout this paper.

Notation 2.2.20. Let $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a (F, G, H)–metrized quantum vector bundle for some admissible (F, G, H). The closed ball of center 0 and radius $r \geqslant 0$ in $(\mathrm{dom}\,(\mathsf{D}_{\mathscr{M}}), \mathsf{D}_{\mathscr{M}})$ is denoted by:

$$\mathscr{D}_r\,(\Omega) = \{\omega \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}}) : \mathsf{D}_{\mathscr{M}}(\omega) \leqslant r\}\,.$$

By Definition (2.2.8), the set $\mathscr{D}_r\,(\Omega)$ is norm compact.

We call the initial topology for a set $\mathcal{F}$ of functions defined on a given set E and valued in a topological space, the smallest topology on E for which all the members of $\mathcal{F}$ are continuous.

Definition 2.2.21. Let $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ be a left Hilbert $\mathfrak{A}$-module. The *A-weak topology* on $\mathscr{M}$ is the initial topology for the set of maps:

$$\{\langle\cdot,\omega\rangle_{\mathscr{M}} : \omega \in \mathscr{M}\}\,.$$

Thus, a net $(\omega_j)_{j\in J}$ converges to some ω in a left Hilbert $\mathfrak{A}$-module $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ when for all $\eta \in \mathscr{M}$, the net $\left(\langle\omega_j,\eta\rangle_{\mathscr{M}}\right)_{j\in J}$ converges to $\langle\omega,\eta\rangle_{\mathscr{M}}$ in $\mathfrak{A}$.

Thus, in particular, for any metrized quantum vector bundle Ω, the set $\mathscr{D}_1\,(\Omega)$ is now endowed with three topologies: the norm topology from the D-norm, the norm topology from the C^*-Hilbert norm inherited from the inner product, and the $\mathfrak{A}$-weak topology where $\mathfrak{A}$ is the base space of Ω. Definition (2.2.8) assures us that the latter two agree on $\mathscr{D}_1\,(\Omega)$.

Lemma 2.2.22. *Let $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L})$ be a metrized quantum vector bundle. The $\mathfrak{A}$-weak topology and the norm topology, induced by $\langle\cdot,\cdot\rangle_{\mathscr{M}}$, agree on $\mathscr{D}_r\,(\Omega)$ for all $r \geqslant 0$.*

Proof. The $\mathfrak{A}$-weak topology is weaker than the topology induced by $\|\cdot\|_{\mathscr{M}}$, yet Hausdorff. On the other hand, $\mathscr{D}_r\,(\Omega)$ is compact for $\|\cdot\|_{\mathscr{M}}$. Therefore, the $\mathfrak{A}$-weak topology and the topology from $\|\cdot\|_{\mathscr{M}}$ agree on $\mathscr{D}_r\,(\Omega)$. □

Our reason to introduce the $\mathfrak{A}$-weak topology is that it is naturally metrized by a metric defined from $\mathscr{D}_1\,(\Omega)$.

Definition 2.2.23. Let $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a metrized quantum vector bundle. The *modular Monge-Kantorovich metric* k_Ω associated with Ω is the metric k_Ω, defined for $\omega, \eta \in \mathscr{M}$ by:

$$\mathsf{k}_\Omega(\omega,\eta) = \sup\left\{\|\langle\omega,\xi\rangle_{\mathscr{M}} - \langle\omega,\xi\rangle_{\mathscr{M}}\|_{\mathfrak{A}} : \xi \in \mathscr{M}, \mathsf{D}(\xi) \leqslant 1\right\}.$$

We note that the Monge-Kantorovich metric on the module of a metrized quantum vector bundle is indeed a metric since the $\mathbb{C}$-linear span of the closed unit ball for the D-norm is dense in the module itself by assumption. We now prove the main property of this metric for us:

Proposition 2.2.24. *Let $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a metrized quantum vector bundle. For all $r > 0$, the* Monge-Kantorovich metric k_Ω *associated with Ω metrizes the $\mathfrak{A}$-weak* topology on $\mathscr{D}_r(\Omega)$, and therefore it metrizes the norm topology on $\mathscr{D}_r(\Omega)$.*

Proof. Let $(\omega_j)_{j\in J}$ be a net in $\mathscr{D}_r(\Omega)$, indexed by $(J, \succ)$ and converging to ω in the $\mathfrak{A}$-weak topology. Let $\varepsilon > 0$. Since $\mathscr{D}_1(\Omega)$ is compact by assumption, there exists a finite subset $F \subseteq \mathscr{D}_1(\Omega)$ which is $\frac{\varepsilon}{3}$-dense in $\mathscr{D}_r(\Omega)$.

There exists $j_0 \in J$ such that for all $j \succ j_0$ we have $\|\langle\omega_j, \xi\rangle_{\mathscr{M}} - \langle\omega, \xi\rangle_{\mathscr{M}}\|_{\mathfrak{A}} \leqslant \frac{\varepsilon}{3}$ for all $\xi \in F$ since F is finite and J is directed. It follows that $\|\langle\omega_j, \xi\rangle_{\mathscr{M}} - \langle\omega, \xi\rangle_{\mathscr{M}}\|_{\mathfrak{A}} \leqslant \varepsilon$. Thus $\mathsf{k}_\Omega(\omega_j, \omega) \leqslant \varepsilon$.

Conversely, if $(\omega_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{D}_r(\Omega)$ converging to some $\omega \in \mathscr{M}$ for k_Ω, then since $\mathscr{D}_1(\Omega)$ is total, we conclude that $(\langle\omega_n, \xi\rangle_{\mathscr{M}})_{n\in\mathbb{N}}$ converges to $\langle\omega, \xi\rangle_{\mathscr{M}}$ for all $\xi \in \mathscr{M}$. Thus ω is the $\mathfrak{A}$-weak limit of $(\omega_n)_{n\in\mathbb{N}}$.

Now, since the $\mathfrak{A}$-weak topology and the norm topology agree on $\mathscr{D}_r(\Omega)$, and $\mathscr{D}_r(\Omega)$ is compact in norm by assumption, we conclude that $\omega \in \mathscr{D}_r(\Omega)$ as well. This concludes our proof. $\square$

We conclude this section with an observation. Let $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L})$ be a metrized quantum vector bundle. Let $\mathfrak{B} = \{a + ib : a, b \in \mathrm{dom}(\mathsf{L})\}$ endowed with the norm $\|\cdot\|_{\mathfrak{B}} = \max\{\|\cdot\|_{\mathfrak{A}}, \mathsf{L}\circ\Re, \mathsf{L}\circ\Im\}$. The norm $\|\cdot\|_{\mathfrak{B}}$ is lower semicontinuous with respect to $\|\cdot\|_{\mathfrak{A}}$ and thus one can prove that $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a Banach algebra (the fact that it is a subalgebra of $\mathfrak{A}$ follows from the fact $\mathrm{dom}(\mathsf{L})$ is a Jordan-Lie subalgebra of $\mathfrak{sa}(\mathfrak{A})$). We note that $\mathrm{dom}(\mathsf{D})$ is a $\mathfrak{B}$-left module thanks to Definition (2.2.8).

We also noted that $(\mathrm{dom}(\mathsf{D}), \mathsf{D})$ is a Banach space as well. Let us call a *current* of Ω a continuous $\mathfrak{B}$-module map from $(\mathrm{dom}(\mathsf{D}), \mathsf{D})$ to $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$. Let $\mathscr{C}(\Omega)$ be the space of all currents of Ω and let $\mathscr{C}_r(\Omega)$ be the closed ball of radius $r > 0$ centered at 0 for the operator norm on $\mathscr{C}(\Omega)$.

Let us call the locally convex topology induced by the seminorms:

$$T \in \mathscr{C}(\Omega) \mapsto \|T(\omega)\|_{\mathfrak{A}}$$

on $\mathscr{C}(\Omega)$ for all $\omega \in \mathrm{dom}(\mathsf{D})$ the $\mathfrak{A}$-*weak* topology*.

Let $\mathfrak{L} = \{a \in \mathfrak{B} : \|a\|_{\mathfrak{B}} \leqslant 1\}$. By assumption on the L-seminorm L, the set $\mathfrak{L}$ is compact in $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$. If we let $\Xi = \prod_{\omega\in\mathrm{dom}(\mathsf{D})} r\mathsf{D}(\omega)\mathfrak{L}$, then Ξ is compact in the product topology by the Tychonoff theorem.

By construction, if $T \in \mathscr{C}_r(\Omega)$ then $\Theta(T) = (T(\omega))_{\omega\in\mathrm{dom}(\mathsf{D})} \in \Xi$. It is straightforward to check that Θ is a continuous injection from the $\mathfrak{A}$-weak* topology to the product topology, whose range is given by:

$$\Xi^r = \bigcap_{b\in\mathfrak{B}, \omega, \eta\in\mathrm{dom}(\mathsf{D})} \left(\pi_{b\omega+\eta} - b\pi_\omega - \pi_\eta\right)^{-1}(\{0\}),$$

where $\pi_\omega : (b_\eta)_{\eta\in\mathrm{dom}(\mathsf{D})} \in \Xi \mapsto b_\omega$ for all $\omega \in \mathrm{dom}(\mathsf{D})$. Of course, by definition of the product topology on Ξ, the maps π_ω are continuous for all $\omega \in \mathrm{dom}(\mathsf{D})$ and thus Ξ^r is closed in Ξ, hence compact. It is easy to see that Θ is an homeomorphism and thus $\mathscr{C}_r(\Omega)$ is actually compact for the $\mathfrak{A}$-weak* topology. These facts did not involve the fact that $\mathscr{D}_1(\Omega)$ is compact for $\|\cdot\|_{\mathscr{M}}$.

Suppose now that S is a seminorm on $\mathscr{M}$ satisfying the inner quasi-Leibniz inequality:

$$\mathsf{L}(\langle\omega,\eta\rangle_{\mathscr{M}}) \leqslant H(\mathsf{S}(\omega),\mathsf{S}(\eta))$$

and $\mathsf{S} \geqslant \|\cdot\|_{\mathscr{M}}$. The latter equation implies that the closed unit ball for S is also closed in the norm $\|\cdot\|_{\mathscr{M}}$.

The inner quasi-Leibniz inequality implies in turn that if $\mathsf{S}(\omega) \leqslant 1$ for $\omega \in \mathscr{M}$ then $\langle\cdot,\omega\rangle_{\mathscr{M}}$ is a $H(1,1)$ current, and thus belongs to some compact set for the $\mathfrak{A}$-weak* topology. It is easy to check that the map $\omega \in \mathscr{M} \mapsto \langle\cdot,\omega\rangle_{\mathscr{M}}$ is an homeomorphism from $\mathscr{M}$ with the $\mathfrak{A}$-weak topology to its range, with the $\mathfrak{A}$-weak* topology. Thus we conclude that $\{\omega \in \mathscr{M} : \mathsf{S}(\omega) \leqslant 1\}$ is totally bounded for the $\mathfrak{A}$-weak topology.

This does not however make S a D-norm, even if it satisfies some form of modular quasi-Leibniz inequality. Indeed, while norm closed, it is unclear whether the closed unit ball of S is also $\mathfrak{A}$-weak closed. Moreover, even it is was, we could not deduce that the closed unit ball for S is norm compact, rather than weakly compact. Thus, it does not appear to be sufficient to assume the inner and modular quasi-Leibniz inequalities and the dominance over the C^*-Hilbert norm to construct a D-norm, and the compactness of the closed ball of a D-norm requires some additional work.

We now turn to the construction of the modular propinquity. The basic ingredient is a notion of a modular bridge which extends the notion of a bridge used as a noncommutative encoding of the idea of an isometric embedding in the construction of the quantum Gromov-Hausdorff propinquity.

2.3. **Modular Bridges.** Bridges between C*-algebras provide a mean to define a particular type of isometric embedding for a any pair of quasi-Leibniz quantum compact metric spaces, from which the quantum Gromov-Hausdorff propinquity is built in [31]. An advantage of bridges is that the quantum propinquity is defined directly from numerical quantities defined using a bridge rather than through the associated isometric embeddings, and thus they are natural to use a foundation for our modular propinquity — bypassing the need for a notion of isometric embeddings of metrized quantum vector bundles. We present our idea on how to extend the notion of bridges to metrized quantum vector bundles in this section. While our presentation will at times refer to [31], we will strive to make it reasonably self-contained.

Bridges involve an element of a unital C*-algebra called a pivot, which allows us to select a particular set of states. We require that this set is not empty. The following definition extends the notion of a state defined on some self-adjoint element [15, Exercise 4.6.16],[16] to general elements.

Definition 2.3.1 ([31, Definition 3.1])**.** The 1-*level set* $\mathscr{S}_1(\mathfrak{D}|x)$ of an element $x \in \mathfrak{sa}(\mathfrak{D})$ of a unital C*-algebra $\mathfrak{D}$ is:

$$\mathscr{S}_1(\mathfrak{D}|x) = \left\{ \varphi \in \mathscr{S}(\mathfrak{D}) \middle| \begin{array}{l} \varphi((1-x)^*(1-x)) = 0 \\ \varphi((1-x)(1-x)^*) = 0 \end{array} \right\}.$$

We make the following remark:

Remark 2.3.2. If $x \in \mathfrak{D}$ for some unital C*-algebra $\mathfrak{D}$, and if $\mathscr{S}_1(\mathfrak{D}|x) \neq \emptyset$, then $\|x\|_{\mathfrak{D}} \geqslant 1$. Indeed, if $\varphi \in \mathscr{S}(\mathfrak{D}|x)$ then $\varphi(x) = \varphi(x^*) = 1$. Thus $\|\Re(x)\|_{\mathfrak{D}} \geqslant 1$ and thus $\|x\|_{\mathfrak{D}} \geqslant \|\Re x\|_{\mathfrak{D}} \geqslant 1$.

In particular, if $\|x\|_{\mathfrak{D}} \leqslant 1$ and $\mathscr{S}(\mathfrak{D}|x) \neq \emptyset$ then $\|x\|_{\mathfrak{D}} = 1$.

We will use Definition (2.3.1) via the following lemma:

Lemma 2.3.3 ([31, Lemma 3.4])**.** *If $\mathfrak{D}$ is a unital C*-algebra and $x \in \mathfrak{D}$, then:*

$$\begin{aligned}\mathscr{S}(\mathfrak{D}|x) &= \{\varphi \in \mathscr{S}(\mathfrak{D}) : \forall d \in \mathfrak{D} \quad \varphi(dx) = \varphi(xd) = \varphi(d)\} \\ &= \{\varphi \in \mathscr{S}(\mathfrak{D}) : \forall d \in \mathfrak{D} \quad \varphi(dx^*) = \varphi(x^*d) = \varphi(d)\}.\end{aligned}$$

Proof. This follows from the Cauchy-Schwarz inequality. □

We first extend the notion of a bridge between quasi-Leibniz quantum compact metric spaces to a modular bridge between metrized quantum vector bundles. While a bridge between to quasi-Leibniz quantum compact metric spaces does not involve any metric information in its definition — the quantum metric information is used to associate numerical quantities to the bridge — a modular bridge between two metrized quantum vector bundles involve the D-norms. Nonetheless, a modular bridge retains the simplicity of a bridge, as it only adds two families of elements from modules.

Definition 2.3.4. Let:

$$\Omega_{\mathfrak{A}} = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}}, \mathsf{D}_{\mathscr{N}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

be two metrized quantum vector bundles.

An *modular bridge*:

$$\gamma = \left(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}, \mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right)$$

from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ is given:

(1) a unital C*-algebra $\mathfrak{D}$,
(2) an element $x \in \mathfrak{D}$ with $\mathscr{S}_1(\mathfrak{D}|x) \neq \emptyset$ and $\|x\|_{\mathfrak{D}} = 1$,
(3) $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$ are two unital *-monomorphisms,
(4) J is some nonempty set,
(5) $(\omega_j)_{j\in J}$ is a family of elements in $\mathscr{D}_1(\Omega_{\mathfrak{A}})$, i.e. $\max\{\mathsf{D}_{\mathscr{M}}(\omega_j) : j \in J\} \leqslant 1$,
(6) $(\eta_j)_{j\in J}$ is a family of elements in $\mathscr{D}_1(\Omega_{\mathfrak{B}})$, i.e. $\max\{\mathsf{D}_{\mathscr{N}}(\eta_j) : j \in J\} \leqslant 1$.

Notation 2.3.5. Let $\gamma = (\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}, \mathfrak{D}, x, \pi, \rho, (\omega_j)_{j\in J}, (\eta_j)_{j\in J})$ be a modular bridge. We will use the following notations and terminology throughout this paper.

(1) The *domain* $\mathrm{dom}(\gamma)$ of γ is $\Omega_{\mathfrak{A}}$.
(2) The *co-domain* $\mathrm{codom}(\gamma)$ of γ is $\Omega_{\mathfrak{B}}$.
(3) The element x is called *the pivot of* γ and is denoted by $\mathrm{pivot}(\gamma)$.
(4) The family $(\omega_j)_{j\in J}$ is the *family of anchors* of γ, denoted by $\mathrm{anchors}(\gamma)$.
(5) The family $(\eta_j)_{j\in J}$ is the *family of co-anchors* of γ, denoted by $\mathrm{coanchors}(\gamma)$.

Notation 2.3.6. Let $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ be two metrized quantum vector bundles. The set of all modular bridges from a $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ is denoted by $\mathscr{Bridges}[\Omega_{\mathfrak{A}} \longrightarrow \Omega_{\mathfrak{B}}]$.

We note that since modular bridges are defined as tuples, the order of their component matter and thus they have a domain and a codomain, though in fact they are quite a symmetric concept. We will remark later that all the quantities defined from modular bridges are in fact symmetric in the domain and the codomain.

We also remark that we include the domain and the codomain of a modular bridge in its very definition. This choice will in fact simplify our notations later on, by removing the need to explicit the quantum metric data as in [31] for various quantities associated to modular bridges.

Last, we note that unlike [31, Definition 3.6], we require the pivot of modular bridges are of norm (at most) 1. This requirement will be essential in the proof of Proposition (2.3.17), which in turn underlies the construction of the modular propinquity.

The basic ingredients to compute the modular propinquity between modules require a lot of notations. We clarify our exposition by grouping some of these notations into a single set of hypothesis which we will use repeatedly in the following definitions and theorems.

Hypothesis 2.3.7. Let:

$$\Omega_{\mathfrak{A}} = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}}, \mathsf{D}_{\mathscr{N}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

be two (F, G, H)–metrized quantum vector bundles.

Let J be some nonempty set and let:

$$\gamma = (\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}, \mathfrak{D}, x, \pi, \rho, (\omega_j)_{j\in J}, (\eta_j)_{j\in J})$$

be a modular bridge from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

The modular propinquity is computed from natural numerical quantities obtained from modular bridges and the quantum metric information encoded in metrized quantum vector bundles. The first quantities we will use are in fact the numerical values introduced in [31] for the canonical bridge from $\mathrm{bqs}(\mathrm{dom}(\gamma))$ to $\mathrm{bqs}(\mathrm{codom}(\gamma))$ associated to any modular bridge γ:

Definition 2.3.8. Let Hypothesis (2.3.7) be given. The *basic bridge* $\gamma_\flat$ from $\mathfrak{A}$ to $\mathfrak{B}$ is given by:

$$\gamma_\flat = (\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}).$$

It is straightforward that Definition (2.3.8) gives a bridge in the sense of [31, Definition 3.6]. Thus, we can compute the reach and height of a basic bridge. We adjust our terminology to fit the setting of this paper in the following definitions of the height and basic reach of a modular bridge.

We start by recalling from [31, Definition 3.10] that a bridge defines an important seminorm:

Definition 2.3.9 ([31, Definition 3.10])**.** Let Hypothesis (2.3.7) be given. The *bridge seminorm* $\mathsf{bn}_\gamma(\cdot,\cdot)$ of the modular bridge γ is the bridge seminorm of the basic bridge $\gamma_\flat$, i.e. the seminorm on $\mathfrak{A}\oplus\mathfrak{B}$ defined for all $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$ by:

$$\mathsf{bn}_\gamma(a,b) = \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}.$$

The bridge seminorm allows us to quantify how far apart two quasi-Leibniz quantum compact metric spaces are from the perspective of a given bridge.

Definition 2.3.10 ([31, Definition 3.14])**.** Let Hypothesis (2.3.7) be given. The *basic reach* $\varrho_\flat(\gamma)$ of the modular bridge γ is the reach of the basic bridge $\gamma_\flat$ with respect to $(\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})$, i.e.

$$\max\left\{\begin{array}{lll} \sup_{a\in\mathfrak{sa}(\mathfrak{A}),\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1} & \inf_{b\in\mathfrak{sa}(\mathfrak{B}),\mathsf{L}_{\mathfrak{B}}(b)\leqslant 1} & \mathsf{bn}_\gamma(a,b) \\ \sup_{b\in\mathfrak{sa}(\mathfrak{B}),\mathsf{L}_{\mathfrak{B}}(b)\leqslant 1} & \inf_{a\in\mathfrak{sa}(\mathfrak{A}),\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1} & \mathsf{bn}_\gamma(a,b) \end{array}\right\}.$$

We provide an alternative expression for the basic reach of a modular bridge. Indeed, the basic reach is where we actually take the Hausdorff distance between quasi-Leibniz quantum compact metric spaces in an appropriate sense. We shall use the following notation for the Hausdorff distance on a pseudo-metric space.

Notation 2.3.11. Let X be a set and d be a pseudo-metric on X. For any nonempty subset $A\subseteq X$ and for any $x\in X$, we set:

$$\mathrm{d}(x,A)=\inf\{\mathrm{d}(x,y): y\in A\}.$$

For any two nonempty sets $A,B\subseteq X$, we then define, following [13]:

$$\mathsf{Haus}_{\mathrm{d}}(A,B)=\sup\left\{\mathrm{d}(x,B),\mathrm{d}(y,A): x\in A, y\in B\right\}.$$

We thus observe, using the notations of Hypothesis (2.3.7) and of Definition (2.3.8), that:

$$\begin{aligned}(2.3.1)\quad \varrho_\flat(\gamma) = \mathsf{Haus}_{\mathsf{bn}_\gamma(\cdot,\cdot)}\big(&\{(a,0)\in\mathfrak{A}\oplus\mathfrak{B} : a\in\mathfrak{sa}(\mathfrak{A}), \mathsf{L}_{\mathfrak{A}}(a)\leqslant 1\}, \\ &\{(0,b): b\in\mathfrak{sa}(\mathfrak{B}),\mathsf{L}_{\mathfrak{B}}(b)\leqslant 1\}\big).\end{aligned}$$

The motivation to use the bridge seminorm, i.e. to involve the pivot, in Equation (2.3.1), in place of the norm $\|\cdot\|_{\mathfrak{D}}$ of $\mathfrak{D}$, is that the pivot allows us to "cut-off" elements and thus may be used as a noncommutative substitute for truncation. This fact is explained and illustrated in [25].

The cost of replacing the norm of $\mathfrak{D}$ by the bridge seminorm in Equation (2.3.1) is measured by the next quantity associated with a modular bridge.

Definition 2.3.12 ([31, Definition 3.16])**.** Let Hypothesis (2.3.7) be given. The *height* $\varsigma(\gamma)$ of the modular bridge γ is the height of the basic bridge $\gamma_\flat$ with respect to $(\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})$, i.e.:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}}\left(\mathscr{S}(\mathfrak{A}),\pi_{\mathfrak{A}}^*\left(\mathscr{S}(\mathfrak{D}|x)\right)\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}}\left(\mathscr{S}(\mathfrak{B}),\pi_{\mathfrak{B}}^*\left(\mathscr{S}(\mathfrak{D}|x)\right)\right)\right\}.$$

The height of a bridge involves computation in each of the domain and codomain of the bridge, but not in between them. Its definition is what justifies that pivot must have nonempty 1-level set.

We now turn to the new quantities which we define for modular bridges, which naturally relate to the module structure. The first of these numerical values, called the reach of a modular bridge, is derived from a new natural seminorm defined by a modular bridge. We continue to choose our terminology from the lexical field of bridges.

Definition 2.3.13. Let Hypothesis (2.3.7) be given. The *deck seminorm* $\mathsf{dn}_\gamma(\cdot,\cdot)$ is the seminorm on $\mathscr{M} \oplus \mathscr{N}$ defined for all $\omega \in \mathscr{M}$ and $\eta \in \mathscr{N}$ by:

$$\mathsf{dn}_\gamma(\omega,\eta) = \max\{\mathsf{bn}_\gamma(\langle\omega,\omega_k\rangle_{\mathscr{M}},\langle\eta,\eta_k\rangle_{\mathscr{N}}), \\ \mathsf{bn}_\gamma(\langle\omega_k,\omega\rangle_{\mathscr{M}},\langle\eta_k,\eta\rangle_{\mathscr{N}}) : k \in J\}.$$

We continue using the notations of Definition (2.3.13). We emphasize that when working with $\mathsf{dn}_\gamma(\cdot,\cdot)$, we only require the structure of vector space on $\mathscr{M} \oplus \mathscr{N}$. We also record the deck seminorm does not involve any explicit need to embed $\mathscr{M}$ and $\mathscr{N}$ is some left Hilbert module. We rely instead on the well understood idea behind noncommutative isometric embeddings of quantum metric spaces and avoid the need to introduce a similar, non-obvious notion for modules.

Furthermore, the deck seminorm is defined with a symmetry in mind, which will prove useful in the notion of the inverse of a bridge defined at the end of this section.

The reach of a modular bridge requires the definition of two additional quantities besides the basic reach. The first quantity, the modular reach, regards the pairing of anchors and co-anchors. We underscore that, in the construction of the deck seminorm, we match anchors and co-anchors with the same index in the modular bridge. Therefore, when constructing of a modular bridge, we must make an astute choice, with the idea that each pair of anchor and co-anchor are expected to be “close” in a sense quantified, ultimately, by the modular reach, via the deck seminorm.

Definition 2.3.14. Let Hypothesis (2.3.7) be given. The *modular reach* $\varrho^\sharp(\gamma)$ is the nonnegative number:

$$\max\left\{\mathsf{dn}_\gamma\left(\omega_j,\eta_j\right) : j \in J\right\}.$$

The last quantity needed to define the reach of a modular bridge is the imprint. The modular bridge only involves anchors and co-anchors, and the cost of this choice, rather than taking some Hausdorff distance between unit balls for D-norms, is measured by the following quantity:

Definition 2.3.15. Let Hypothesis (2.3.7) be given. The *imprint* $\varpi(\gamma)$ of the modular bridge γ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{k}_{\mathsf{D}_{\mathscr{M}}}}\left(\left\{\omega_j : j \in J\right\}, \mathscr{D}_1(\Omega_{\mathfrak{A}})\right), \\ \mathsf{Haus}_{\mathsf{k}_{\mathsf{D}_{\mathscr{N}}}}\left(\left\{\eta_j : j \in J\right\}, \mathscr{D}_1(\Omega_{\mathfrak{B}})\right)\right\}.$$

We now define the reach of a modular bridge as a synthetic valued which adequately combines the basic reach, the modular reach, and the imprint. Our definition is immediately followed with a proposition which, we hope, will clarify the meaning of the reach of a modular bridge — and which will prove crucial for our work in allowing for the definition of target sets for modular bridges, to come shortly.

Definition 2.3.16. Let Hypothesis (2.3.7) be given. The *reach* $\varrho(\gamma)$ of the modular bridge γ is the nonnegative value:

$$\varrho(\gamma) = \max\left\{\varrho_\flat(\gamma), \varrho^\sharp(\gamma) + \varpi(\gamma)\right\}.$$

Proposition 2.3.17. *Let Hypothesis (2.3.7) be given. If $\omega \in \mathscr{M}$ with $\mathsf{D}_{\mathscr{M}}(\omega) \leqslant 1$ and if $j \in J$ is chosen so that $\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega, \omega_j) \leqslant \varpi(\gamma)$, then $\mathsf{dn}_{\gamma}(\omega, \eta_j) \leqslant \varrho(\gamma)$. The result also holds if $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ are switched.*

We then have:

$$\max\left\{\begin{array}{lll} \sup_{a\in\mathfrak{sa}(\mathfrak{A}),\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1} & \inf_{b\in\mathfrak{sa}(\mathfrak{B}),\mathsf{L}_{\mathfrak{B}}(b)\leqslant 1} & \mathsf{bn}_{\gamma}(a,b) \\ \sup_{b\in\mathfrak{sa}(\mathfrak{B}),\mathsf{L}_{\mathfrak{B}}(b)\leqslant 1} & \inf_{a\in\mathfrak{sa}(\mathfrak{A}),\mathsf{L}_{\mathfrak{A}}(a)\leqslant 1} & \mathsf{bn}_{\gamma}(a,b) \\ \sup_{\omega\in\mathscr{M},\mathsf{D}_{\mathscr{M}}(\omega)\leqslant 1} & \inf_{\eta\in\mathscr{N},\mathsf{D}_{\mathscr{N}}(\eta)\leqslant 1} & \mathsf{dn}_{\gamma}(\omega,\eta) \\ \sup_{\eta\in\mathscr{N},\mathsf{D}_{\mathscr{N}}(\eta)\leqslant 1} & \inf_{\omega\in\mathscr{M},\mathsf{D}_{\mathscr{M}}(\omega)\leqslant 1} & \mathsf{dn}_{\gamma}(\omega,\eta) \end{array}\right\} \leqslant \varrho(\gamma). \tag{2.3.2}$$

Proof. By Definition (2.3.10) and Definition (2.3.16), it is sufficient to prove that:

$$\max\left\{\begin{array}{lll} \sup_{\omega\in\mathscr{M},\mathsf{D}_{\mathscr{M}}(\omega)\leqslant 1} & \inf_{\eta\in\mathscr{N},\mathsf{D}_{\mathscr{N}}(\eta)\leqslant 1} & \mathsf{dn}_{\gamma}(\omega,\eta) \\ \sup_{\eta\in\mathscr{N},\mathsf{D}_{\mathscr{N}}(\eta)\leqslant 1} & \inf_{\omega\in\mathscr{M},\mathsf{D}_{\mathscr{M}}(\omega)\leqslant 1} & \mathsf{dn}_{\gamma}(\omega,\eta) \end{array}\right\} \leqslant \varrho(\gamma).$$

Let $\omega \in \mathsf{D}_{\mathscr{M}}$ with $\mathsf{D}_{\mathscr{M}}(\omega) \leqslant 1$. By Definition (2.3.15), there exists $j \in J$ such that:

$$\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega, \omega_j) \leqslant \varpi(\gamma).$$

Now, by Definition (2.3.14), we have:

$$\mathsf{dn}_{\gamma}(\omega_j, \eta_j) \leqslant \varrho^{\sharp}(\gamma).$$

Thus, for any $k \in J$, we compute:

$$\begin{aligned} &\left\|\pi_{\mathfrak{A}}\left(\langle\omega,\omega_k\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\langle\eta_j,\eta_k\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \\ &\qquad\leqslant \left\|\pi_{\mathfrak{A}}\left(\langle\omega,\omega_k\rangle_{\mathscr{M}} - \langle\omega_j,\omega_k\rangle_{\mathscr{M}}\right)x\right\|_{\mathfrak{D}} \\ &\qquad\quad + \left\|\pi_{\mathfrak{A}}\left(\langle\omega_j,\omega_k\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\langle\eta_j,\eta_k\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \\ &\qquad\leqslant \|x\|_{\mathfrak{D}}\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega,\omega_j) + \mathsf{dn}_{\gamma}(\omega_j,\eta_j) \\ &\qquad\leqslant \varpi(\gamma) + \varrho^{\sharp}(\gamma) \leqslant \varrho(\gamma). \end{aligned}$$

We now observe that since the involution of $\mathfrak{D}$ is isometric and $k \in J$:

$$\begin{aligned} &\left\|\pi_{\mathfrak{A}}\left(\langle\omega_k,\omega\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\langle\eta_k,\eta_j\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \\ &\qquad = \left\|x^*\pi_{\mathfrak{A}}\left(\langle\omega,\omega_j\rangle_{\mathscr{M}}\right) - \pi_{\mathfrak{B}}\left(\langle\eta_j,\eta_k\rangle_{\mathscr{N}}\right)x^*\right\|_{\mathfrak{D}}. \end{aligned}$$

Since $\|x^*\|_{\mathfrak{D}} \leqslant 1$, a similar computation then proves that for all $k \in J$:

$$\left\|\pi_{\mathfrak{A}}\left(\langle\omega_k,\omega_j\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\langle\eta_k,\eta_j\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \leqslant \varrho(\gamma).$$

Thus, as desired:

$$\mathsf{dn}_{\gamma}(\omega, \eta_j) \leqslant \varrho(\gamma).$$

In particular, we have shown:

$$\sup_{\omega\in\mathscr{D}_1(\Omega_{\mathfrak{A}})}\inf_{\eta\in\mathscr{D}_1(\Omega_{\mathfrak{B}})}\mathsf{dn}_{\gamma}(\omega,\eta) \leqslant \varrho(\gamma).$$

A similar computation shows that:

$$\sup_{\eta\in\mathscr{D}_1(\Omega_{\mathfrak{B}})}\inf_{\omega\in\mathscr{D}_1(\Omega_{\mathfrak{A}})}\mathsf{dn}_{\gamma}(\omega,\eta) \leqslant \varrho(\gamma).$$

This concludes our proof. ∎

We now pause for a few remarks regarding our Definition (2.3.16) of a reach for the modular bridge. First, unlike in [31], we required in Definition (2.3.4) that pivots have norm at most one. The result in Proposition (2.3.17) is where this additional assumption is needed.

Proposition (2.3.17) suggests a competing candidate for the notion of a bridge, given by the left-hand side of Inequality (2.3.2). This alternate candidate is given as the maximum of Expression (2.3.1) and the following similar expression for modules:

$$\begin{aligned}(2.3.3)\quad \mathsf{Haus}_{\mathsf{dn}_\gamma(\cdot,\cdot)}\left(\{(\omega,0)\in\mathscr{M}\oplus\mathscr{N} : \mathsf{D}_{\mathscr{M}}(\omega)\leqslant 1\},\right.\\ \left.\{(0,\eta)\in\mathscr{M}\oplus\mathscr{N} : \mathsf{D}_{\mathscr{N}}(\eta)\leqslant 1\}\right).\end{aligned}$$

This formulation would more closely resemble the definition of the basic reach. Our preference for Definition (2.3.16) rather than the maximum of Expressions (2.3.1) and (2.3.3) is at the core of our idea for the construction of the modular propinquity. Indeed, Definition (2.3.10) employs the match between anchors and co-anchors. This pairing is essential, because it also appears in the Definition (2.3.13) of the deck seminorm and actually, it is the approach we use to construct a seminorm from a couple of *sesquilinear* maps.

Indeed, we also could have introduced anchors and co-anchors in the construction of the quantum propinquity. Namely, an "anchored" bridge from $(\mathfrak{A},\mathsf{L}_\mathfrak{A})$ to $(\mathfrak{B},\mathsf{L}_\mathfrak{B})$ could be of the form $\gamma=(\mathfrak{D},x,\pi_\mathfrak{A},\pi_\mathfrak{B},(a_j)_{j\in J},(b_j)_{j\in J})$ with $a_j\in\mathfrak{L}_1(\mathsf{L}_\mathfrak{A})$ and $b_j\in\mathfrak{L}_1(\mathsf{L}_\mathfrak{B})$ for all $j\in J$. We then could define the "anchored" reach as we just did for modular bridge, i.e. as the maximum of $\max\{\mathsf{bn}_\gamma(a_j,b_j):j\in J\}$ and of a kind of imprint, i.e. $\max\{\mathsf{Haus}_{\|\cdot\|_\mathfrak{A}}(\{a_j:j\in J\},\mathfrak{L}_1(\mathsf{L}_\mathfrak{A})),\mathsf{Haus}_{\|\cdot\|_\mathfrak{A}}(\{a_j:j\in J\},\mathfrak{L}_1(\mathsf{L}_\mathfrak{A}))\}$. The length of an anchored bridge would then be the maximum of its anchored reach and its height, defined in the usual manner.

Yet such a definition of a bridge reach would not change our construction of the quantum propinquity. Indeed, Proposition (2.3.17) could be adapted to prove that the reach of the bridge $(\mathfrak{D},x,\pi_\mathfrak{A},\pi_\mathfrak{B})$ is lesser or equal than the anchored reach of γ. It is also easy to check that given a bridge $(\mathfrak{D},x,\pi_\mathfrak{A},\pi_\mathfrak{B})$ in the sense of [31, Definition 3.6], there always is a mean to construct an anchor bridge with the same length. We refer briefly to [31] for various notions which we will extend in a moment to modular bridges, and the reader may skip the following few details as they are just a side observation. Using the notion of target sets introduced in [31, Definition 5.1], we can, for all $a\in\mathfrak{L}_1(\mathsf{L}_\mathfrak{A})$, choose some $b_a\in\mathfrak{t}_\gamma(a|1)$, and similarly by symmetry, for all $b\in\mathfrak{L}_1(\mathsf{L}_\mathfrak{B})$, choose $a_b\in\mathfrak{t}_{\gamma^{-1}}(b|1)$. With these notations, if $J=\mathfrak{L}_1(\mathsf{L}_\mathfrak{A})\coprod\mathfrak{L}_1(\mathsf{L}_\mathfrak{B})$, and if we set $a_a=a$ and $b_b=b$ for all $a\in\mathfrak{L}_1(\mathsf{L}_\mathfrak{A})$ and $b\in\mathfrak{L}_1(\mathsf{L}_\mathfrak{B})$, then:

$$\left(\mathfrak{D},x,\pi_\mathfrak{A},\pi_\mathfrak{B},(a_j)_{j\in J},(b_j)_{j\in J}\right)$$

is an anchored bridge with the same length than the bridge $(\mathfrak{D},x,\pi_\mathfrak{A},\pi_\mathfrak{B})$. Thus there is no need for anchors and co-anchors in the construction of the quantum propinquity.

If such is the case, then why did we introduce anchors in our current work? The reason lies with the fact that the bridge seminorm is indeed a seminorm because the maps $\pi_\mathfrak{A}$ and $\pi_\mathfrak{B}$ are linear. However the inner products $\langle\cdot,\cdot\rangle_\mathscr{M}$ and $\langle\cdot,\cdot\rangle_\mathscr{N}$

are sesquilinear, and thus, to construct our deck seminorm as indeed a seminorm, we discovered the idea of employing pairs of anchors-co-anchors. While this idea would not change anything for the quantum propinquity, it becomes essential for the modular propinquity.

The length of a modular bridge is the synthetic numerical value which summarizes all the information contained in the basic reach, modular reach, height and imprint of the modular bridge, and from which the modular propinquity will be computed.

Definition 2.3.18. Let Hypothesis (2.3.7) be given. The length $\lambda(\gamma)$ of the modular bridge γ is the maximum of its reach, its height and its imprint:

$$\lambda(\gamma) = \max\{\varsigma(\gamma), \varrho(\gamma)\}.$$

We note that a modular bridge always has a finite length.

Lemma 2.3.19. *If γ is a modular bridge then $\lambda(\gamma) < \infty$.*

Proof. The imprint and the height of a modular bridge are both defined as the Hausdorff distance between two compact sets and thus are finite.

Now, if $\omega \in \mathscr{D}_1(\Omega_{\mathfrak{A}})$ and $\eta \in \mathscr{D}_1(\Omega_{\mathfrak{B}})$ then since $\|x\|_{\mathfrak{D}} \leqslant 1$, we have:

$$\mathsf{dn}_\gamma(\omega, \eta) = \max_{j \in J} \|\pi_{\mathfrak{A}}(\langle \omega, \omega_k \rangle_{\mathscr{M}}) x - x \pi_{\mathfrak{A}}(\langle \omega, \omega_k \rangle_{\mathscr{M}})\|_{\mathfrak{D}} \leqslant 2.$$

Thus $\varrho^\sharp(\gamma) \leqslant 2$. The reach of γ is thus the maximum of the (finite) basic bridge reach and the sum of the (finite) imprint and the (finite) modular reach. Thus $\varrho(\gamma) < \infty$ and thus our proposition is proven. □

Modular bridges are a type of morphism between metrized quantum vector bundles — though we shall address the question of composition for modular bridges in our next section with the introduction of modular treks. In the rest of this section, we formalize the idea that bridges posses some properties akin to some form of multi-valued morphism. These properties are the essential reason behind the fact that, if the modular propinquity is null between two metrized quantum vector bundles, then they are fully quantum isometric.

A modular bridge from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$, with $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ two metrized quantum vector bundles, defines maps from the domain of the L-seminorm of $\Omega_{\mathfrak{A}}$ to the power set of domain of the L-seminorm of $\Omega_{\mathfrak{B}}$.

Definition 2.3.20. Let Hypothesis (2.3.7) be given. For any $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, we define the *$l$-target set* $\mathfrak{t}_\gamma(a|l)$ of a for γ as:

$$\mathfrak{t}_\gamma(a|l) = \left\{ b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \middle| \begin{array}{l} \mathsf{L}_{\mathfrak{B}}(b) \leqslant l \\ \mathsf{bn}_\gamma(a, b) \leqslant l\varrho(\gamma_\flat). \end{array} \right\}.$$

Definition (2.3.20) ensures that $\mathfrak{t}_\gamma(a|l) = \mathfrak{t}_{\gamma_\flat}(a|l)$, where the right hand side is defined in [31, Definition 5.1]. It actually would not matter in our subsequent work if instead, we had used $\varrho(\gamma)$ in place of $\varrho(\gamma_\flat)$ in Definition (2.3.20). On the other hand, thanks to our choice, we can invoke our work in [31] to immediately conclude:

Proposition 2.3.21. *Let Hypothesis (2.3.7) be given. For all $a, a' \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$, if $b \in \mathfrak{t}_{\gamma}(a|l)$ and $b' \in \mathfrak{t}_{\gamma}(a'|l)$ then:*

(1) *$\mathfrak{t}_{\gamma}(a|l)$ is a nonempty compact subset of $\mathfrak{sa}(\mathfrak{B})$,*
(2) *$\|b\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{A}} + 2l\lambda(\gamma)$,*
(3) *for all $t \in \mathbb{R}$ we have $b + tb' \in \mathfrak{t}_{\gamma}(a + ta'|(1+|t|)l)$,*
(4) *$\|b - b'\|_{\mathfrak{B}} \leqslant \|a - a'\|_{\mathfrak{A}} + 4l\lambda(\gamma)$,*
(5) *We have:*

$$b \circ b' \in \mathfrak{t}_{\gamma}\left(a \circ a' \middle| F(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), \|a'\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l)\right)$$

and

$$\{b, b'\} \in \mathfrak{t}_{\gamma}\left(\{b, b'\} \middle| F(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), \|a'\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l)\right).$$

In particular, for all $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, we have:

$$\operatorname{diam}(\mathfrak{t}_{\gamma}(a|l),) \leqslant 4l\lambda(\gamma).$$

Proof. Assertion (1) is [31, Lemma 5.2] and since it is a closed subset of the norm compact $\mathfrak{L}_l(\mathsf{L}_{\mathfrak{B}})$. Assertion (2) follows from [31, Proposition 5.3]. Assertion (3) follows from [31, Proposition 5.4]. Assertion (4) follows from Assertion (2) and Assertion (3). Assertion (5) is established by noting:

$$\mathsf{L}_{\mathfrak{B}}(b \circ b') \leqslant F(\|b\|_{\mathfrak{B}}, \|b'\|_{\mathfrak{B}}, l, l) \leqslant F(\|a\|_{\mathfrak{A}} + 2\lambda(\gamma), \|a'\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l),$$

and similarly for the Lie product. Setting $a = a'$ gives us the given estimate on the diameter of $\mathfrak{t}_{\gamma}(a|l)$. □

We now define the target set for elements in the domain of a D-norm.

Definition 2.3.22. Let Hypothesis (2.3.7) be given. For any $\omega \in \mathscr{M}$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$, we define the *$l$-modular target set* of ω for γ as:

$$\mathfrak{t}_{\gamma}(\omega|l) = \left\{ \eta \in \mathscr{N} \,\middle|\, \begin{array}{l} \mathsf{D}_{\mathscr{N}}(\eta) \leqslant l \\ \mathsf{dn}_{\gamma}(\omega, \eta) \leqslant l\varrho(\gamma) \end{array} \right\}.$$

We begin by observing that modular target sets are compact and non-empty.

Proposition 2.3.23. *Let Hypothesis (2.3.7) be given. For any $\omega \in \operatorname{dom}(\mathsf{D}_{\mathscr{M}})$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$, the set $\mathfrak{t}_{\gamma}(\omega|l)$ is a nonempty compact for $\|\cdot\|_{\mathscr{N}}$ (or equivalently for $\mathsf{k}_{\Omega_{\mathfrak{B}}}$).*

Proof. By Proposition (2.3.17), for all $\omega \in \mathscr{D}_1(\Omega_{\mathfrak{A}})$ there exists $\eta \in \mathscr{D}_1(\Omega_{\mathfrak{B}})$ such that $\mathsf{dn}_{\gamma}(\omega, \eta) \leqslant \varrho(\gamma)$. Thus, if $\mathsf{D}_{\mathscr{M}}(\omega) \leqslant l$ for some $\omega \in \mathscr{M}$, it follows from homogeneity that there exists $\eta \in \mathscr{D}_l(\Omega_{\mathfrak{B}})$ such that $\mathsf{dn}_{\gamma}(\omega, \eta) \leqslant l\varrho(\gamma)$ since $\mathsf{dn}_{\gamma}(\cdot)$ is a seminorm. Therefore, $\mathfrak{t}_{\gamma}(\omega|l) \neq \emptyset$.

By construction $\mathfrak{t}_{\gamma}(\omega|l)$ is a subset of the compact set $\mathscr{D}_l(\Omega_{\mathfrak{B}})$ (for $\|\cdot\|_{\mathscr{N}}$ or for $\mathsf{k}_{\Omega_{\mathfrak{B}}}$, as both give the same topology on $\mathscr{D}_l(\Omega_{\mathfrak{B}})$). Thus it is sufficient to prove that $\mathfrak{t}_{\gamma}(\omega|l)$ is closed.

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{t}_{\gamma}(\omega|l)$, converging to some η for $\|\cdot\|_{\mathscr{N}}$. Since $\mathsf{D}_{\mathscr{N}}$ is lower semi-continuous with respect to $\|\cdot\|_{\mathscr{N}}$, we have $\mathsf{D}_{\mathscr{N}}(\eta) \leqslant l$.

Moreover, by continuity, $\mathsf{dn}_{\gamma}(\omega, \eta) \leqslant l\varrho(\gamma)$ since $\mathsf{dn}_{\gamma}(\omega, \eta) \leqslant l\varrho(\gamma)$. This proves that $\eta \in \mathfrak{t}_{\gamma}(\omega|l)$ as desired. □

The fundamental property of modular target sets for a modular bridge γ is that their diameter in the modular Monge-Kantorovich metric is controlled by the length of γ — and, in contrast with target sets for basic bridges, not their diameter in the C^*-Hilbert norm. We begin with a well-known lemma included for convenience.

Lemma 2.3.24. *If $\mathfrak{A}$ is a C^*-algebra, $a \in \mathfrak{A}$, and there exists $M \geqslant 0$ such that for all $\varphi \in \mathscr{S}(\mathfrak{A})$ we have:*

$$\max\{|\varphi(\Re(a))|, |\varphi(\Im(a))|\} \leqslant M,$$

then:

$$\|a\|_{\mathfrak{A}} \leqslant \sqrt{2}M.$$

Proof. Let $b \in \mathfrak{sa}(\mathfrak{A})$ then the functional calculus implies that $\|b\|_{\mathfrak{A}} = \sup\{|\varphi(b)| : \varphi \in \mathscr{S}(\mathfrak{A})\}$. Thus for all $a \in \mathfrak{A}$, we compute:

$$\begin{aligned}
\|a\|_{\mathfrak{A}}^2 &= \|aa^*\|_{\mathfrak{A}} \\
&= \|(\Re(a) + i\Im(a))(\Re(a) + i\Im(a))^*\|_{\mathfrak{A}} \\
&= \|\Re(a)^2 + \Im(a)^2\|_{\mathfrak{A}} \\
&\leqslant \|\Re(a)\|^2 + \|\Im(a)\|^2 \\
&= (\sup\{|\varphi(\Re(a)) : \varphi \in \mathscr{S}(\mathfrak{A})\})^2 + (\sup\{|\varphi(\Im(a)) : \varphi \in \mathscr{S}(\mathfrak{A})\})^2 \\
&\leqslant 2M^2.
\end{aligned}$$

This concludes our lemma. □

Proposition 2.3.25. *Let Hypothesis (2.3.7) be given.*

If $\omega, \omega' \in \mathscr{M}$, $l \geqslant \max\{\mathsf{D}_{\mathscr{M}}(\omega), \mathsf{D}_{\mathscr{M}}(\omega')\}$, $\eta \in \mathfrak{t}_\gamma(\omega|l)$ and $\eta' \in \mathfrak{t}_\gamma(\omega'|l)$, then:

$$\mathsf{k}_{\mathsf{D}_{\mathscr{N}}}(\eta, \eta') \leqslant \sqrt{2}\left(\mathsf{k}_{\mathsf{D}_{\mathscr{M}}}(\omega, \omega') + (4l + H(2l, 1))\lambda(\gamma)\right).$$

In particular:

$$\operatorname{diam}\left(\mathfrak{t}_\gamma(\omega|l), \mathsf{k}_{\mathsf{L}_{\mathscr{N}}}\right) \leqslant \sqrt{2}(4l + H(2l, 1))\lambda(\gamma).$$

Proof. Let $\theta = \eta - \eta'$ and $\zeta = \omega - \omega'$. Note that:

$$\max\{\mathsf{D}_{\mathscr{N}}(\theta), \mathsf{D}_{\mathscr{M}}(\zeta)\} \leqslant 2l.$$

Let $\varphi \in \mathscr{S}(\mathfrak{B})$ and let $\nu \in \mathscr{D}_1(\Omega_{\mathfrak{A}})$.

There exists $j \in J$ such that $\mathsf{k}_{\mathsf{D}_{\mathscr{N}}}(\nu, \eta_j) \leqslant \varpi(\gamma)$ by Definition (2.3.15).

By Definition (2.3.12), there exists $\psi \in \mathscr{S}(\mathfrak{D}|x)$ with $\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}(\varphi, \psi \circ \pi_{\mathfrak{B}}) \leqslant \varsigma(\gamma)$.

Note that $\mathsf{D}_{\mathscr{N}}(\eta_j) \leqslant 1$, and therefore, using the inner quasi-Leibniz inequality, we have:

$$\max\left\{\mathsf{L}_{\mathfrak{B}}\left(\Re\langle\theta, \eta_j\rangle_{\mathscr{N}}\right), \mathsf{L}\left(\Im\langle\theta, \eta_j\rangle_{\mathscr{N}}\right)\right\} \leqslant H(2l, 1).$$

We also note that since ψ is a state, we have:

$$|\psi(\Re(d))| = |\Re(\psi(d))| \leqslant |\psi(d)| \text{ and, similarly: } |\psi(\Im(d))| \leqslant |\psi(d)|$$

for all $d \in \mathfrak{D}$.

Now, letting $m = H(2l, 1)$:

$$
\begin{aligned}
|\varphi(\Re\langle\theta,\nu\rangle_{\mathscr{N}})| &\leqslant 2l\lambda(\gamma) + \left|\varphi\left(\Re\langle\theta,\eta_j\rangle_{\mathscr{N}}\right)\right| \text{ by Def. (2.3.15)},\\
&\leqslant (2l+m)\lambda(\gamma) + \left|\psi\circ\pi_{\mathfrak{B}}\left(\Re\langle\theta,\eta_j\rangle_{\mathscr{N}}\right)\right| \text{ by choice of } \psi,\\
&\leqslant (2l+m)\lambda(\gamma) + \left|\psi\circ\pi_{\mathfrak{B}}\left(\langle\theta,\eta_j\rangle_{\mathscr{N}}\right)\right|\\
&\leqslant (2l+m)\lambda(\gamma) + \left|\psi\left(\pi_{\mathfrak{B}}(\langle\theta,\eta_j\rangle_{\mathscr{N}})x\right)\right| \text{ by Def. (2.3.1)},\\
&\leqslant (4l+m)\lambda(\gamma) + \left|\psi\left(x\pi_{\mathfrak{A}}(\langle\zeta,\omega_j\rangle_{\mathscr{M}})\right)\right| \text{ by Prop. (2.3.17)},\\
&\leqslant (4l+m)\lambda(\gamma) + \left|\psi\circ\pi_{\mathfrak{A}}\left(\langle\zeta,\omega_j\rangle_{\mathscr{M}}\right)\right| \text{ by Def. (2.3.1)},\\
&\leqslant (4l+m)\lambda(\gamma) + \left\|\langle\zeta,\omega_j\rangle_{\mathscr{M}}\right\|_{\mathfrak{A}}\\
&\leqslant (4l+m)\lambda(\gamma) + \left\|\langle\omega,\omega_j\rangle_{\mathscr{M}} - \langle\omega',\omega_j\rangle_{\mathscr{M}}\right\|_{\mathfrak{A}}\\
&\leqslant (4l+m)\lambda(\gamma) + \mathsf{k}_{\mathsf{L}_\omega}(\omega,\omega').
\end{aligned}
$$

The same computation holds for $\Re$ replaced with $\Im$, and thus we record:

$$|\varphi(\Im\langle\theta,\nu\rangle_{\mathscr{N}})| \leqslant (4l+m)\lambda(\gamma) + \mathsf{k}_{\mathsf{L}_\omega}(\omega,\omega'),$$

and therefore by Lemma (2.3.24), we conclude:

$$\|\langle\theta,\nu\rangle_{\mathscr{N}}\|_{\mathfrak{B}} \leqslant \sqrt{2}\left((4l+m)\lambda(\gamma) + \mathsf{k}_{\mathsf{D}}(\omega,\omega')\right).$$

Thus:

$$
\begin{aligned}
\mathsf{k}_{\mathsf{D}_{\mathscr{N}}}(\eta,\eta') &= \sup\left\{\|\langle\theta,\nu\rangle_{\mathscr{N}}\|_{\mathfrak{B}} : \nu \in \mathscr{D}_1(\Omega_{\mathfrak{B}})\right\}\\
&\leqslant \sqrt{2}\left((4l+m)\lambda(\gamma) + \mathsf{k}_{\mathsf{D}}(\omega,\omega')\right).
\end{aligned}
$$

Our proof is thus complete. The assertion on the diameter is obtained simply by letting $\omega = \omega'$. □

Our first relation between target sets on modules and the module algebraic structure concerns linearity, as expressed in the following proposition.

Proposition 2.3.26. *Let Hypothesis (2.3.7) be given. If:*

(1) $\omega, \omega' \in \mathscr{M}$,
(2) $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$ *and* $l' \geqslant \mathsf{D}_{\mathscr{M}}(\omega')$,
(3) $\eta \in \mathsf{t}_\gamma(l|\omega)$ *and* $\eta' \in \mathsf{t}_\gamma(\omega'|l')$,
(4) $t \in \mathbb{R}$,

then:

$$\eta + t\eta' \in \mathsf{t}_\gamma\left(\omega + t\omega'\middle|l + |t|l'\right).$$

Proof. Since $\mathsf{D}_{\mathscr{N}}(\eta) \leqslant l$ and $\mathsf{D}_{\mathscr{N}}(\eta') \leqslant l'$, we have $\mathsf{D}_{\mathscr{N}}(\eta + t\eta') \leqslant l + |t|l'$.

On the other hand, we note that since $\mathsf{dn}_\gamma(\cdot)$ is a seminorm $\mathscr{M} \oplus \mathscr{N}$, we conclude that:

$$
\begin{aligned}
\mathsf{dn}_\gamma(\omega + t\omega', \eta + t\eta') &\leqslant \mathsf{dn}_\gamma(\omega,\eta) + |t|\mathsf{dn}_\gamma(\omega',\eta')\\
&\leqslant (l + |t|l')\varrho(\gamma).
\end{aligned}
$$

This completes our proof. □

We now prove that target sets also behave predictably with respect to the left action on the module. This proposition is where the modular quasi-Leibniz inequality plays its role.

Proposition 2.3.27. *Let Hypothesis (2.3.7) be given. Let $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$, $\omega \in \operatorname{dom}(\mathsf{D}_{\mathscr{M}})$, and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$ and $l' \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. Let $b \in \mathfrak{t}_{\gamma}(a|l')$ and $\eta \in \mathfrak{t}_{\gamma}(\omega|l)$. Then:*

$$b\eta \in \mathfrak{t}_{\gamma}\left(a\omega \middle| G(\|a\|_{\mathfrak{A}} + 2\lambda(\gamma), l, l')\right).$$

Proof. We begin with the observation that:

$$\begin{aligned}\mathsf{D}_{\mathscr{N}}(b\eta) &\leqslant G\left(\|b\|_{\mathfrak{B}}, \mathsf{L}_{\mathfrak{B}}(b), \mathsf{D}_{\mathscr{M}}(\eta)\right)\\ &\leqslant G(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), l', l),\end{aligned}$$

using Proposition (2.3.21).

We also note that for any $j \in J$:

$$\begin{aligned}&\left\|\pi_{\mathfrak{A}}\left(\left\langle a\omega, \omega_j\right\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\left\langle b\eta, \eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}}\\ &\quad= \left\|\pi_{\mathfrak{A}}(a)\pi_{\mathfrak{A}}\left(\left\langle \omega, \omega_j\right\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}(b)\pi_{\mathfrak{B}}\left(\left\langle \eta, \eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}}\\ &\quad\leqslant \left\|\pi_{\mathfrak{A}}(a)\pi_{\mathfrak{A}}\left(\left\langle \omega, \omega_j\right\rangle_{\mathscr{M}}\right)x - \pi_{\mathfrak{A}}(a)x\pi_{\mathfrak{B}}\left(\left\langle \eta, \eta_j\right\rangle_{\mathscr{M}}\right)\right\|_{\mathfrak{D}}\\ &\qquad+ \left\|\pi_{\mathfrak{A}}(a)x\pi_{\mathfrak{B}}\left(\left\langle \eta, \eta_j\right\rangle_{\mathscr{M}}\right) - x\pi_{\mathfrak{B}}(b)\pi_{\mathfrak{B}}\left(\left\langle \eta, \eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}}\\ &\quad\leqslant \|a\|_{\mathfrak{A}}\left\|\pi_{\mathfrak{A}}\left(\left\langle \omega, \omega_j\right\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\left\langle \eta, \eta_j\right\rangle_{\mathscr{M}}\right)\right\|_{\mathfrak{D}}\\ &\qquad+ \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}\left\|\pi_{\mathfrak{B}}\left(\left\langle \eta, \eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}}\\ &\quad\leqslant \|a\|_{\mathfrak{A}}\mathsf{dn}_{\gamma}(\omega, \eta) + \mathsf{bn}_{\gamma}(a, b)\|\eta\|_{\mathscr{N}}\\ &\quad\leqslant \|a\|_{\mathfrak{A}}l\varrho(\gamma) + l'\varrho(\gamma)\mathsf{D}_{\mathscr{N}}(\eta)\\ &\quad\leqslant \lambda(\gamma)\left(\|a\|_{\mathfrak{A}}l + l'l\right)\\ &\quad\leqslant \lambda(\gamma)G(\|a\|_{\mathfrak{A}}, l, l') \text{ by Def. (2.2.6)},\\ &\quad\leqslant \lambda(\gamma)G(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l').\end{aligned}$$

A similar computation proves that:

$$\begin{aligned}&\left\|\pi_{\mathfrak{B}}\left(\left\langle \omega_j, a\omega\right\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{A}}\left(\left\langle \eta_j, b\eta\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}}\\ &\quad= \left\|\pi_{\mathfrak{B}}\left(\left\langle b\eta, \eta_j\right\rangle_{\mathscr{N}}\right)x^* - x^*\pi_{\mathfrak{A}}\left(\left\langle a\omega, \omega_j\right\rangle_{\mathscr{M}}\right)\right\|_{\mathfrak{D}}\\ &\quad\leqslant \lambda(\gamma)G(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l').\end{aligned}$$

Therefore, $\mathsf{dn}_{\gamma}(a\omega, b\eta) \leqslant \lambda(\gamma)G(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l')$ since $j \in J$ is arbitrary.

Thus $b\eta \in \mathfrak{t}_{\gamma}(\omega|G(\|a\|_{\mathfrak{A}} + 2l\lambda(\gamma), l, l'))$. ∎

We now relate modular bridges and the inner products on modules, which illustrate the role of the inner quasi-Leibniz inner inequality.

Proposition 2.3.28. *Let Hypothesis (2.3.7) be given. Let $\omega \in \mathscr{M}$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$. If $\eta \in \mathfrak{t}_{\gamma}(\omega|l)$ and $b \in \mathfrak{t}_{\gamma}(\langle\omega, \omega\rangle_{\mathscr{M}}|H(l, l))$ then:*

$$\|b - \langle\eta, \eta\rangle_{\mathscr{N}}\|_{\mathfrak{B}} \leqslant (8l\sqrt{2} + H(2l, 2l) + 2H(l, l) + 2\sqrt{2}H(2l, 1))\lambda(\gamma).$$

Proof. If $l = 0$ then $\omega = 0$, $\eta = 0$ and $b = 0$ thus the proposition is trivial. Let us assume $l > 0$.

Let $\omega \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$. Let $\eta \in \mathsf{t}_{\Gamma}\,(\omega|l)$. We note that:

$$\max\left\{\mathsf{L}_{\mathfrak{A}}\left(\langle\omega,\omega\rangle_{\Omega_{\mathfrak{A}}}\right), \mathsf{L}_{\mathfrak{B}}\left(\langle\eta,\eta\rangle_{\Omega_{\mathfrak{B}}}\right)\right\} \leqslant H(l,l),$$

noting $\langle\omega,\omega\rangle_{\Omega_{\mathfrak{A}}}$ and $\langle\eta,\eta\rangle_{\Omega_{\mathfrak{B}}}$ are self-adjoint.

Let $b \in \mathsf{t}_{\gamma}\,(\langle\omega,\omega\rangle_{\mathscr{M}}|H(l,l))$.

By Definition (2.3.15) of $\varpi\,(\gamma)$, there exists $j \in J$ such that:

$$\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega, l\omega_j) \leqslant l\varpi\,(\gamma).$$

It follows that $\left\langle\omega,\omega - l\omega_j\right\rangle_{\mathscr{M}} = l\left\langle l^{-1}\omega,\omega - l\omega_j\right\rangle_{\mathscr{M}} \leqslant l\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega, l\omega_j) \leqslant l^2\varpi\,(\gamma)$.

Moreover, by Proposition (2.3.17), we have:

$$\mathsf{dn}_{\gamma}\left(\omega, l\eta_j\right) \leqslant l\varrho\,(\gamma) \leqslant l\lambda\,(\gamma).$$

We then have, since $\|x\|_{\mathfrak{D}} \leqslant 1$:

$$\begin{aligned}
&\left\|\pi_{\mathfrak{A}}\left(\langle\omega,\omega\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\left\langle l\eta_j, l\eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \\
&\qquad\leqslant l^2\varpi\,(\gamma) + \left\|\pi_{\mathfrak{A}}\left(\left\langle\omega, l\omega_j\right\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\left\langle l\eta_j, l\eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \\
&\qquad\leqslant l^2\varpi\,(\gamma) + l\left\|\pi_{\mathfrak{A}}\left(\left\langle\omega, \omega_j\right\rangle_{\mathscr{M}}\right)x - x\pi_{\mathfrak{B}}\left(\left\langle l\eta_j, \eta_j\right\rangle_{\mathscr{N}}\right)\right\|_{\mathfrak{D}} \\
&\qquad\leqslant l^2\lambda\,(\gamma) + l\mathsf{dn}_{\gamma}\left(\omega, l\eta_j\right) \\
&\qquad\leqslant 2l^2\lambda\,(\gamma).
\end{aligned}$$

Now, since $l\eta_j \in \mathsf{t}_{\gamma}\,(\omega|l)$ (again Proposition (2.3.17)), we have by Proposition (2.3.25):

$$\mathsf{k}_{\Omega_{\mathfrak{B}}}(\eta, l\eta_j) \leqslant \sqrt{2}\,(4l + H(2l,1))\,\lambda\,(\gamma).$$

Let $\varphi \in \mathscr{S}(\mathfrak{B})$. By Definition (2.3.12), there exists $\psi \in \mathscr{S}(\mathfrak{D})$ such that $\mathsf{k}_{\Omega_{\mathfrak{B}}}(\varphi, \psi \circ \pi_{\mathfrak{B}}) \leqslant \varsigma\,(\gamma)$. We then have:

$$|\varphi(b - \langle\eta,\eta\rangle_{\mathscr{M}})|$$

$$
\begin{aligned}
&\leqslant H(2l,2l)\varsigma(\gamma) + |\psi\circ\pi_{\mathfrak{B}}(b - \langle\eta,\eta\rangle_{\mathscr{N}})| \\
&\leqslant H(2l,2l)\varsigma(\gamma) + \left|\psi\circ\pi_{\mathfrak{B}}(b - \langle l\eta_j, l\eta_j\rangle_{\mathscr{N}})\right| + \left|\psi\circ\pi_{\mathfrak{B}}(\langle l\eta_j, l\eta_j\rangle_{\mathscr{N}} - \langle\eta,\eta\rangle_{\mathscr{N}})\right| \\
&\leqslant H(2l,2l)\varsigma(\gamma) + \left|\psi\left(x\pi_{\mathfrak{B}}\left(b - \langle l\eta_j, l\eta_j\rangle_{\mathscr{M}}\right)\right)\right| + \left\|\langle l\eta_j, l\eta_j\rangle_{\mathscr{N}} - \langle\eta,\eta\rangle_{\mathscr{N}}\right\|_{\mathfrak{B}} \\
&\leqslant H(2l,2l)\varsigma(\gamma) + \left|\psi\left(x\pi_{\mathfrak{B}}\left(b - \langle l\eta_j, l\eta_j\rangle_{\mathscr{M}}\right)\right)\right| + \\
&\quad + \left\|\langle\eta,\eta - l\eta_j\rangle_{\mathscr{N}} + \langle\eta - l\eta_j, l\eta_j\rangle_{\mathscr{N}}\right\|_{\mathfrak{B}} \\
&\leqslant H(2l,2l)\varsigma(\gamma) + \left|\psi\left(x\pi_{\mathfrak{B}}\left(b - \langle l\eta_j, l\eta_j\rangle_{\mathscr{M}}\right)\right)\right| + \\
&\quad + l\mathsf{k}_{\Omega_{\mathfrak{B}}}(\eta, l\eta_j) + l\mathsf{k}_{\Omega_{\mathfrak{B}}}(\eta, l\eta_j) \\
&\leqslant H(2l,2l)\varsigma(\gamma) + \left|\psi\left(x\pi_{\mathfrak{B}}\left(b - \langle l\eta_j, l\eta_j\rangle_{\mathscr{M}}\right)\right)\right. \\
&\quad \left. -\psi(\pi_{\mathfrak{A}}(\langle\omega,\omega\rangle_{\mathscr{M}})x) + \psi(\pi_{\mathfrak{A}}(\langle\omega,\omega\rangle_{\mathscr{M}})x)\right| \\
&\quad + 2l\sqrt{2}\,(4l + H(2l,1))\,\lambda(\gamma) \\
&\leqslant H(2l,2l)\varsigma(\gamma) + |\psi(x\pi_{\mathfrak{B}}(b) - \pi_{\mathfrak{A}}(\langle\omega,\omega\rangle_{\mathscr{M}})x)| \\
&\quad + \left|\psi\left(x\pi_{\mathfrak{B}}\left(\langle l\eta_j, l\eta_j\rangle_{\mathscr{M}}\right)\right) - \psi(\pi_{\mathfrak{A}}(\langle\omega,\omega\rangle_{\mathscr{M}})x)\right| \\
&\quad + 2l\sqrt{2}\,(4l + H(2l,1))\,\lambda(\gamma) \\
&\leqslant H(2l,2l)\lambda(\gamma) + H(l,l)\lambda(\gamma) + H(l,l)\lambda(\gamma) + 2\sqrt{2}l\,(4l + H(2l,1))\,\lambda(\gamma) \\
&\leqslant (8l\sqrt{2} + H(2l,2l) + 2H(l,l) + 2\sqrt{2}H(2l,1))\lambda(\gamma).
\end{aligned}
$$

This concludes our proposition since $b - \langle\eta,\eta\rangle_{\mathscr{N}}$ is self-adjoint in $\mathfrak{B}$. ☐

We now check that modular bridges are essentially symmetric objects. We shall avoid the term inverse as we shall see that for modular bridges, in contrast to bridges, the following notion is not quite an inverse in the sense of morphisms.

Definition 2.3.29. The *reverse bridge* of a bridge deck $\gamma = (\mathfrak{D}, x, \pi, \rho, (\omega)_{j\in J}, (\eta_j)_{j\in J})$ is $\gamma^* = (\mathfrak{D}, x^*, \rho, \pi, (\eta_j)_{j\in J}, (\omega_j)_{j\in J})$.

Lemma 2.3.30. *If $\gamma \in \mathscr{Bridges}[\Omega_{\mathfrak{A}} \longrightarrow \Omega_{\mathfrak{B}}]$ for any two metrized quantum vector bundles $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$, then $\gamma^* \in \mathscr{Bridges}[\Omega_{\mathfrak{B}} \longrightarrow \Omega_{\mathfrak{A}}]$ and $\lambda(\gamma^*) = \lambda(\gamma)$.*

Proof. We use the notations of Hypothesis (2.3.7). We note that for all $\omega \in \mathscr{M}$ and $\eta \in \mathscr{N}$ we have:

$$\mathsf{dn}_{\gamma^*}(\eta,\omega) = \mathsf{dn}_{\gamma}(\omega,\eta)$$

by construction. This observation justifies the particular symmetry in Definition (2.3.13).

Moreover for all $a \in \mathfrak{sa}(\mathfrak{A})$ and $b \in \mathfrak{sa}(\mathfrak{B})$, we have:

$$
\begin{aligned}
\mathsf{bn}_{\gamma^*}(b,a) &= \|\pi_{\mathfrak{B}}(b)x^* - x^*\pi_{\mathfrak{A}}(a)\|_{\mathfrak{D}} \\
&= \|(\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b))^*\|_{\mathfrak{D}} = \mathsf{bn}_{\gamma}(a,b).
\end{aligned}
$$

Thus $\varrho(\gamma) = \varrho(\gamma^*)$. The other claims of our lemma are self-evident. ☐

We remark that for any modular bridge γ we have $\gamma_\flat^{-1} = (\gamma^*)_\flat$, by [31, Proposition 4.7].

We will observe in the next section that we do not need the full generality afforded to us by Definition (2.3.4), as we could limit ourselves to working only with finite families of anchors (and hence of co-anchors). The reason for this observation is the following lemma.

Lemma 2.3.31. *Let Hypothesis (2.3.7) be given. For any $\varepsilon > 0$, there exists a modular bridge γ_ε from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ such that:*

(1) $\lambda(\gamma_\varepsilon) \leqslant \lambda(\gamma) + \varepsilon$,

(2) $\mathrm{anchors}(\gamma)$ *and* $\mathrm{coanchors}(\gamma)$ *are finite families.*

Proof. Let $\varepsilon > 0$. Since:

$$\mathscr{D}_1(\Omega_{\mathfrak{A}}) = \bigcup_{\omega \in \mathscr{D}_1(\Omega_{\mathfrak{A}})} \mathscr{M}(\omega, \varepsilon)$$

by Definition (2.3.15), and since $\mathscr{D}_1(\Omega_{\mathfrak{A}})$ is compact, there exists a finite set $J_1 \subseteq \mathscr{D}_1(\Omega_{\mathfrak{A}})$ such that:

$$\mathscr{D}_1(\Omega_{\mathfrak{A}}) = \bigcup_{\omega \in J_1} \mathscr{M}(\omega, \varepsilon).$$

Similarly, there exists a finite subset J_2 of $\mathscr{D}_1(\Omega_{\mathfrak{B}})$ such that:

$$\mathscr{D}_1(\Omega_{\mathfrak{B}}) = \bigcup_{\eta \in J_1} \mathscr{N}(\eta, \varepsilon).$$

Let $J_3 = J_1 \coprod J_2$ be the disjoint union of J_1 and J_2, itself a finite set.

If $j \in J_1$ then we write $\omega_j = j$ and we choose $\eta_j \in \mathfrak{t}_\gamma(j|1)$. If $j \in J_2$ then we write $\eta_j = j$ and we choose $\omega_j \in \mathfrak{t}_{\gamma^*}(j|1)$. These choices are possible since by Proposition (2.3.23), the target sets involved are all nonempty (and as customary in functional analysis, we work within ZFC).

Let $\gamma_\varepsilon = (\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}, \mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j \in J_3}, (\eta_j)_{j \in J_3})$.

We now make a few simple observations. We have $\mathsf{dn}_\gamma(\omega_j, \eta_j) \leqslant \varrho(\gamma)$ for all $j \in J_3$, and thus $\varrho^\sharp(\gamma_\varepsilon) \leqslant \varrho(\gamma)$. On the other hand, by construction, $\varpi(\gamma_\varepsilon) \leqslant \varepsilon$. Last, we obviously have $\gamma_\circ = (\gamma_\varepsilon)_\circ$ by construction.

Therefore, $\varrho(\gamma_\varepsilon) \leqslant \varrho(\gamma) + \varepsilon$. This concludes our proof since $\varsigma(\gamma) = \varsigma(\gamma_\varepsilon)$. □

The generality of Definition (2.3.4) is however useful to describe modular bridges as morphisms in a category, as we shall do now. Indeed, the we are now ready to introduce the category of metrized quantum vector bundles with modular treks, which generalize modular bridges and which carry a notion of length, from which the modular propinquity is computed.

2.4. **The modular propinquity.** The modular propinquity is constructed using certain morphisms for metrized quantum vector bundles, called modular treks, which extend the notion of modular bridges to allow for the definition of composition. A modular trek, informally, is a finite path made of modular bridges whose codomains match the domain of the next modular bridge in the trek. It is immediate to define the length of a modular trek as the sum of the lengths of its constituent modular bridges. The length of a modular bridge, and by extension of a modular trek, replaces the notion of distortion for a correspondence sometimes used to define the Gromov-Hausdorff distance [4]. The modular propinquity between any two metrized quantum vector bundles $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ is the infimum of the lengths of any modular trek between $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$. Concatenation of treks provide a notion of composition which translates to the fact that the modular propinquity satisfies the triangle inequality. Symmetry of the modular propinquity follow from the fact that treks are always reversible, in a sense to be made precise below. We will handle the more complicated coincidence axiom in the next section.

Since modular treks involve choices of modular bridges, just as with treks in the construction of the dual Gromov-Hausdorff propinquity [27], we have much freedom in defining the modular propinquity to best suits a given context. Indeed, we may reduce the class of allowed modular bridges which may appear in a given modular trek by imposing additional constraints, such as asking the L-seminorms involved to be defined on a dense domain in the entire C*-algebra, additional Leibniz conditions such as the strong Leibniz property, or other additional requirements on D-norms, pivots, anchors or co-anchors (requirements on anchors and co-anchors should be symmetric to ensure that we obtain a metric). This flexibility proved helpful with the dual propinquity and will likely be as well for the modular propinquity.

Let us thus define modular treks formally:

Definition 2.4.1. Let $\mathcal{B}$ be a nonempty class of modular bridges. A *modular $\mathcal{B}$-trek* $\Gamma = \left(\gamma^j\right)_{j\in\{1,\ldots,n\}}$ is given by $n \in \mathbb{N}\setminus\{0\}$ modular bridges $\gamma^0, \ldots, \gamma^n$ such that:

$$\mathrm{dom}\left(\gamma^{j+1}\right) = \mathrm{codom}\left(\gamma^j\right) \text{ for all } j \in \{1, \ldots, n-1\}.$$

The *domain* $\mathrm{dom}(\Gamma)$ of Γ is $\mathrm{dom}\left(\gamma^1\right)$ and the *codomain* $\mathrm{codom}(\Gamma)$ of Γ is $\mathrm{codom}\left(\gamma^n\right)$.

A modular trek is a modular $\mathcal{B}$-trek for some nonempty class $\mathcal{B}$ of modular bridges.

We associate the following natural notion of length to modular treks:

Definition 2.4.2. The *length* of a modular trek $\Gamma = (\gamma^j)_{j\in\{1,\ldots,n\}}$ is:

$$\lambda(\Gamma) = \sum_{j=1}^{n} \lambda(\gamma).$$

Before introducing the modular propinquity, we first assemble the conditions needed on a class of modular bridges to allow for the construction of an actual metric in the following definition, which extends on [27, Definition 3.10].

Definition 2.4.3. Let (F, G, H) be an admissible triple. Let $\mathcal{C}$ be a nonempty class of (F, G, H)–metrized quantum vector bundles. A class $\mathcal{B}$ of modular bridges is *compatible* with $\mathcal{C}$ when:

(1) for all $\gamma \in \mathcal{T}$, we have $\mathrm{dom}(\gamma), \mathrm{codom}(\gamma) \in \mathcal{C}$,
(2) for all $\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}} \in \mathcal{C}$, there exists a modular $\mathcal{B}$-trek from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$,
(3) for all $\gamma \in \mathcal{T}$, we have $\gamma^* \in \mathcal{T}$,
(4) for all $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ in $\mathcal{C}$, if there exists a full quantum isometry $\Theta : \Omega_{\mathfrak{A}} \to \Omega_{\mathfrak{A}}$ then for all $\varepsilon > 0$, there exists a modular $\mathcal{B}$-trek Γ_ε from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ with $\lambda(\Gamma_\varepsilon) < \varepsilon$.

Example 2.4.4. Let (F, G, H) be an admissible triple. Let $\mathcal{C}$ be the class of all (F, G, H)–metrized quantum vector bundles and let $\mathcal{B}$ be the class of all bridges between elements of $\mathcal{C}$. Note that a modular $\mathcal{B}$-trek consists of modular bridges which only involve (F, G, H)–metrized quantum vector bundles. We check that $\mathcal{B}$ is compatible with $\mathcal{C}$.

Assertions (1) and (3) of Definition (2.4.3) are trivial in this case.

Assertion (2) gives us a chance to observe that a bridge gives rise to a modular bridge. Let us use the notations of Hypothesis (2.3.7), with the additional assumption that $\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}} \in \mathcal{C}$.

Let $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a bridge from $\mathfrak{A}$ to $\mathfrak{B}$ with $\|x\|_{\mathfrak{D}} \leqslant 1$. If we pick any $\omega \in \mathscr{D}_1(\Omega_{\mathfrak{A}})$ and $\eta \in \mathscr{D}_1(\Omega_{\mathfrak{B}})$, then $(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}, \mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega, \eta)$ is a modular bridge in $\mathcal{B}$ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ (identifying family of a single element with the element itself).

Now by [31, Proposition 4.6], there does exist a bridge from $\mathfrak{A}$ to $\mathfrak{B}$ with a (self-adjoint) pivot of norm 1. Thus, Assertion (2) holds as well.

Last, keeping the same notations, assume that (Θ, θ) is a full quantum isometry from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$. We simply define the following one-bridge trek:

$$\left(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}, \mathfrak{B}, 1_{\mathfrak{B}}, \theta, \mathrm{id}_{\mathfrak{B}}, (a)_{a \in \mathscr{D}_1(\Omega_{\mathfrak{A}})}, (\Theta(a))_{a \in \mathscr{D}_1(\Omega_{\mathfrak{A}})}\right)$$

where $\mathrm{id}_{\mathfrak{B}}$ is the identity of $\mathfrak{B}$. A straightforward computation shows that $\lambda(\gamma) = 0$.

We will find the following notation helpful.

Notation 2.4.5. Let $\mathcal{C}$ be nonempty class of metrized quantum vector bundles and let $\mathcal{B}$ be a compatible class of modular bridges. Let $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ be chosen in $\mathcal{C}$. The class of all modular $\mathcal{B}$-treks from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ is denoted by:

$$\mathscr{Treks}\left[\Omega_{\mathfrak{A}} \xrightarrow{\mathcal{B}} \Omega_{\mathfrak{B}}\right].$$

We are now ready to introduce the main definition of this work.

Definition 2.4.6. Let $\mathcal{C}$ be a nonempty class of (F, G, H)–metrized quantum vector bundles for some admissible triple (F, G, H) and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. The *modular Gromov-Hausdorff $\mathcal{B}$-propinquity* between two metrized quantum vector bundles $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ in $\mathcal{C}$ is:

$$\boxed{\Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) = \inf\left\{\lambda(\Gamma) : \Gamma \in \mathscr{Treks}\left[\Omega_{\mathfrak{A}} \xrightarrow{\mathcal{B}} \Omega_{\mathfrak{B}}\right]\right\}.}$$

Notation 2.4.7. If $\mathcal{C}$ is the class of all Leibniz metrized quantum vector bundles and $\mathcal{B}$ is the class of all modular bridges, then $\Lambda^{\mathsf{mod}}_{\mathcal{B}}$ is simply denoted Λ^{mod}.

We now proceed to prove that the modular propinquity is a metric up to full quantum isometry, for any compatible class of modular bridges. In the process, we will show that modular treks are morphisms in some category of metrized quantum vector bundles. The coincidence axiom is by far the most involved property to establish, and will be the subject of the next section.

We begin by observing that the modular propinquity is always finite, and it dominates the quantum Gromov-Hausdorff propinquity. We begin with the natural definition of a basic trek.

Definition 2.4.8. If $\Gamma = (\gamma^j)_{j\in\{1,\ldots,n\}}$ is a modular trek, then:

$$\Gamma_\flat = \left(\mathrm{bqs}\left(\mathrm{dom}\left(\gamma^j\right)\right), \gamma^j_\flat, \mathrm{bqs}\left(\mathrm{codom}\left(\gamma^j\right)\right) : j \in \{1,\ldots,n\}\right)$$

is a trek from $\mathrm{bqs}(\mathrm{dom}(\Gamma))$ to $\mathrm{bqs}(\mathrm{codom}(\Gamma))$.

Remark 2.4.9. In [31, Definition 3.20], treks explicitly included domains and codomains of bridges while bridges did not in [31, Definition 3.6]. We have made a different choice of notation, and thus our treks need not include the domain and codomain information already contained in modular bridges.

Proposition 2.4.10. *Let $\mathcal{C}$ be a nonempty class of (F,G,H)–metrized quantum vector bundles for some admissible triple (F,G,H) and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. If:*

$$\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

are two metrized quantum vector bundles in $\mathcal{C}$, and if $\Gamma \in \mathscr{T}\!\mathit{reks}\left[\Omega_{\mathfrak{A}} \xrightarrow{\mathcal{B}} \Omega_{\mathfrak{B}}\right]$, then:

$$\lambda(\Gamma_\flat) \leqslant \lambda(\Gamma),$$

and thus:

$$\Lambda_{\mathcal{C}}((\mathfrak{A},\mathsf{L}_{\mathfrak{A}}),(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})) \leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}\mathcal{T}(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}) < \infty.$$

Proof. By Definition (2.3.8), if $\gamma \in \mathcal{B}$ then $\gamma_\flat \in \mathscr{B}\!\mathit{ridges}[\mathfrak{A} \longrightarrow \mathfrak{B}]$. Moreover $\lambda(\gamma_\flat) \leqslant \lambda(\gamma)$ since $\varsigma(\gamma_\flat) = \varsigma(\gamma)$ by Definition (2.3.12) while $\varrho(\gamma_\flat) = \varrho_\flat(\gamma) \leqslant \varrho(\gamma)$ by Definition (2.3.10).

Now, if $\Gamma = (\gamma^j)_{j\in\{1,\ldots,n\}} \in \mathscr{T}\!\mathit{reks}\left[\Omega_{\mathfrak{A}} \xrightarrow{\mathcal{B}} \Omega_{\mathfrak{B}}\right]$ then $\gamma_\flat = (\gamma^j_\flat)_{j\in\{1,\ldots,n\}}$ is a trek from $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ and:

$$\lambda(\gamma_\flat) = \sum_{j=1}^n \lambda\left(\gamma^j_\circ\right) \leqslant \sum_{j=1}^n \lambda\left(\gamma^j\right) = \lambda(\Gamma).$$

This proves that by definition:

$$\Lambda_((\mathfrak{A},\mathsf{L}_{\mathfrak{A}}),(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})) \leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}).$$

The modular propinquity is finite since there exists at least one modular trek from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ in $\mathcal{B}$ by Definition (2.4.3). Now, a modular trek always has finite length, since modular bridges always have finite length by Lemma (2.3.19). ☐

We now prove that the modular propinquity is symmetric in its arguments and satisfies the triangle inequality. These facts rely on the fact that treks can be reversed and composed.

Definition 2.4.11. The *reverse* of a modular trek $\Gamma = (\gamma_j)_{j\in\{1,\ldots,n\}}$ is the modular trek $\Gamma^* = \left(\gamma^*_{n+1-j}\right)_{j\in\{1,\ldots,n\}}$.

Lemma 2.4.12. *Let $\mathcal{C}$ be a nonempty class of (F,G,H)–metrized quantum vector bundles, where (F,G,H) is an admissible triple, and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. If Γ is a modular $\mathcal{B}$-trek then Γ^* is a modular $\mathcal{B}$-trek from* $\mathrm{codom}(\Gamma)$ *to* $\mathrm{dom}(\Gamma)$*; moreover $\lambda(\Gamma) = \lambda(\Gamma^*)$.*

Proof. This statement is immediate since a compatible class of modular bridges is closed by inversion of modular bridges by Definition (2.4.3), and by Lemma (2.3.30). □

We do not have a direct mean to compose modular bridges — similarly as the situation with bridges in [31]. However, we can easily compose modular treks.

Definition 2.4.13. Let $\Gamma_1 = \left(\gamma_j^1\right)_{j\in\{1,\ldots,n\}}$ and $\Gamma_2 = \left(\gamma_j^2\right)_{j\in\{1,\ldots,m\}}$ be two modular treks. The *composed modular trek* $\Gamma_1 \star \Gamma_2$ is the trek from $\mathrm{dom}(\Gamma_1)$ to $\mathrm{codom}(\Gamma_2)$ given by $\left(\gamma_1^1,\ldots,\gamma_n^1,\gamma_1^2,\ldots,\gamma_m^2\right)$.

Lemma 2.4.14. *Let $\mathcal{C}$ be a nonempty class of (F,G,H)–metrized quantum vector bundles, with (F,G,H) is an admissible triple, and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. If Γ_1 and Γ_2 are two modular $\mathcal{B}$-treks, then $\Gamma_1\star\Gamma_2$ is a modular $\mathcal{B}$-trek and:*

$$\lambda\left(\Gamma_1\star\Gamma_2\right) = \lambda\left(\Gamma_1\right) + \lambda\left(\Gamma_2\right).$$

Proof. The result follows immediately from the Definition (2.4.2) of the length of a modular trek and Definition (2.4.13). □

Proposition 2.4.15. *Let $\mathcal{C}$ be a nonempty class of (F,G,H)–metrized quantum vector bundles, with (F,G,H) an admissible triple, and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. If $\Omega_{\mathfrak{A}}$, $\Omega_{\mathfrak{B}}$, and $\Omega_{\mathfrak{D}}$ are three metrized quantum vector bundles in $\mathcal{C}$, then:*

$$\Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}\right) \leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{D}}\right) + \Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{D}},\Omega_{\mathfrak{B}}\right),$$

and

$$\Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}\right) = \Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{B}},\Omega_{\mathfrak{A}}\right).$$

Proof. Let $\varepsilon > 0$. There exists modular treks Γ_1 and Γ_2, respectively from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ and $\Omega_{\mathfrak{B}}$ to $\Omega_{\mathfrak{D}}$, such that:

$$\lambda\left(\Gamma_1\right) \leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}) + \frac{\varepsilon}{2} \text{ and } \lambda\left(\Gamma_2\right) \leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{B}},\Omega_{\mathfrak{D}}) + \frac{\varepsilon}{2}.$$

Let $\Gamma = \Gamma_1\star\Gamma_2$. Then:

$$\begin{aligned}\Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{D}}) &\leqslant \lambda\left(\Gamma\right)\\ &= \lambda\left(\Gamma_1\right) + \lambda\left(\Gamma_2\right)\\ &\leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}) + \Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{B}},\Omega_{\mathfrak{D}}) + \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we conclude that:

$$\Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}\right) \leqslant \Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{D}}\right) + \Lambda^{\mathsf{mod}}_{\mathcal{B}}\left(\Omega_{\mathfrak{D}},\Omega_{\mathfrak{B}}\right),$$

as desired.

Symmetry follows from Lemma (2.4.12). □

We conclude by observing that the modular propinquity is a pseudo-metric, i.e. in addition to being finite, symmetric and satisfy the triangle inequality, it is null whenever two metrized quantum vector bundles are full quantum isometric.

Proposition 2.4.16. *Let $\mathcal{C}$ be a nonempty class of (F, G, H)–metrized quantum vector bundles, with (F, G, H) an admissible triple, and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. Let:*

$$\Omega_{\mathfrak{A}} = (\mathscr{M}_{\mathfrak{A}}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{M}_{\mathfrak{B}}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

be two metrized quantum vector bundles in $\mathcal{C}$.

If there exists a full quantum isometry (θ, Θ) from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$, then $\Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) = 0$.

Proof. By Definition (2.4.3), for all $\varepsilon > 0$, there exists a modular $\mathcal{B}$-trek Γ_ε from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ such that $\lambda(\Gamma_\varepsilon) < \varepsilon$. Thus $\Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}}, \Omega_{\mathfrak{B}}) < \varepsilon$. This proves our result. □

We pause for an observation which formalizes the intuition we have followed when working with treks. If $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a metrized quantum vector bundle then we may define a canonical modular bridge $\mathrm{idbridge}_{\Omega}$ from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$ by setting:

$$\mathrm{idbridge}_{\Omega} = \left(\mathfrak{A}, 1_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}}, (\omega)_{\omega\in\mathscr{D}_1(\Omega)}, (\omega)_{\omega\in\mathscr{D}_1(\Omega)}\right) \in \mathscr{Bridges}\,[\Omega \longrightarrow \Omega],$$

where $\mathrm{id}_{\mathfrak{A}}$ is the identity *-automorphism of $\mathfrak{A}$. We immediately that $\lambda(\mathrm{idbridge}) = 0$, and it is natural to think of idbridge as the identity bridge of Ω.

Identifying modular bridges with modular treks reduced to a single bridge, we thus seem to have gathered many key ingredients for a category: modular treks compose, and we have an identity modular trek for any metrized quantum vector bundles. Moreover, we will extend in the next section the various properties of target sets for modular bridges to modular treks; while not a part of the requirement to define a category, these morphism-like properties certainly push forth the idea that modular treks ought to be considered a type of morphisms of metrized quantum vector bundles.

There are two small issues to deal with to complete this picture. First of all, we must work with modular treks up to a notion of reduction. Indeed, even composition a trek with the identity trek of its domain or co-domain does not lead to the same modular trek with our definitions. It is however easy to define a notion of a *reduced modular trek*, which is a trek with no loop. Formally, if $\Gamma = (\gamma_j)_{j\in\{1,\ldots,n\}}$ is a modular trek, then we shall say that Γ is reduced there exists no $j < k \in \{1, \ldots, n\}$ such that $\mathrm{dom}(\gamma_j) = \mathrm{codom}(\gamma_k)$ and $\mathrm{anchors}(\gamma_j) = \mathrm{coanchors}(\gamma_k)$. It is trivial to prove that any modular trek can be reduced, i.e. it admits a subfamily which is a reduced trek with the same domain and codomain. We note that a modular trek with a single bridge is by definition reduced.

Now, we can compose two reduced modular treks to a reduced modular trek simply by reducing their composition as defined in Definition (2.4.13). It is a simple exercise to check that composition of reduced treks thus defined is associative and that the identity treks act as units for the composition.

The second small issue is that our morphism sets for our prospective category are not sets. There are simply too many possible modular treks between any two metrized quantum vector bundles. However, this is a very minor issue. The simplest and often sufficient mean to fix this is to restrict which class of metrized quantum vector bundles we work with in a given context, making sure this class is a set, and then use modular treks formed only with metrized quantum vector bundles in this set.

When working with treks, rather than modular treks, a similar construction in [31] led to a category with reduced treks as morphisms over the class of quasi-Leibniz quantum compact metric spaces, and all reduced treks were isomorphisms — i.e. invertible. We note that in our current modular version, modular treks may not be invertible, as being invertible requires that the sets of anchors and co-anchors be the entire closed unit balls of the D-norms of their domain and codomain. In particular, there are many single-bridge modular treks which are not an identity bridge.

Now, the length of a modular trek is larger than the length of its reduction, and thus we could define the modular propinquity with reduced treks only if desired without changing its value. This would introduce unneeded complications, but it is worth noting that we can bring our construction within this framework. Indeed, it really shows that the modular propinquity is constructed via a sort of generalized correspondences in the metric sense.

We now turn to proving that the modular propinquity is indeed a metric up to full quantum isometry.

2.5. **Distance Zero.** We continue our study of the morphism-like properties of modular treks. We extend the notion of a target set from modular bridges to modular treks, using the notion of an itinerary. There are two kind of target sets for treks: one defined for elements in modules and one defined for elements in quasi-Leibniz quantum compact metric spaces. The latter follows the same ideas as in [31].

Once more, we will group certain common notations and hypothesis for multiple use in this section.

Hypothesis 2.5.1. Let $\mathcal{C}$ be a nonempty class of (F,G,H)–metrized quantum vector bundles, with (F,G,H) be an admissible triple, and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. Let:

$$\Omega_{\mathfrak{A}} = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ and } \Omega_{\mathfrak{B}} = (\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}}, \mathsf{D}_{\mathscr{N}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$$

be two metrized quantum vector bundles in $\mathcal{C}$. Let $l \geqslant 0$.

Let $\Gamma = \left(\gamma_j\right)_{j\in\{1,\ldots,n\}}$ be a modular trek from $\Omega_{\mathfrak{A}}$ to $\Omega_{\mathfrak{B}}$.

We begin by recalling [31, Definition 5.7], adjusted to our context.

Definition 2.5.2. Let Hypothesis (2.5.1) be given. Let $l \geqslant 0$. An *l-itinerary* from $a \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right)$ to $b \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{B}}\right)$ along the modular trek Γ is an l-itinerary from a to b along the basic trek $\Gamma_\flat$, i.e. a family $(d_j)_{j\in\{0,\ldots,n\}}$ such that:

1. $d_0 = a$,
2. $d_n = b$,
3. $d_{j+1} \in \mathfrak{t}_{\gamma_{j+1}}\left(d_j \middle| l\right)$ for all $j \in \{0,\ldots,n-1\}$.

The set of all l-itineraries along Γ starting at a and ending at b is denoted by:

$$\mathscr{Itineraries}\left(a \xrightarrow{\Gamma} b \middle| l\right).$$

We now generalize the notion of itinerary to modules.

Definition 2.5.3. Let Hypothesis (2.5.1) be given and $l \geqslant 0$. An *l-itinerary* from $\omega \in \mathrm{dom}(\mathsf{D}_{\mathscr{M}})$ to $\eta \in \mathrm{dom}(\mathsf{D}_{\mathscr{N}})$ along the modular trek Γ is a family $(\xi_j)_{j\in\{0,\ldots,n\}}$ such that:

(1) $\xi_0 = \omega$,
(2) $\xi_n = \eta$,
(3) $\xi_{j+1} \in \mathfrak{t}_{\gamma_{j+1}}\left(\xi_j \middle| l\right)$ for all $j \in \{0,\ldots,n-1\}$.

The set of all itineraries along Γ starting at ω and ending at η is denoted by:

$$\mathscr{Itineraries}\left(\omega \xrightarrow{\Gamma} \eta \middle| l\right).$$

Itineraries allow us to extend the notion of a target set from modular bridges to modular treks.

Definition 2.5.4. Let Hypothesis (2.5.1) be given. The *target set* for some $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$ along the modular trek Γ is:

$$\mathfrak{T}_{\Gamma}(\omega|l) = \left\{ b : \mathscr{Itineraries}\left(a \xrightarrow{\Gamma} b \middle| l\right) \neq \emptyset \right\}.$$

Definition 2.5.5. Let Hypothesis (2.5.1) be given. The *target set* for some $\omega \in \mathrm{dom}(\mathsf{D}_{\mathscr{M}})$ and $l \geqslant \mathsf{D}_{\mathscr{M}}$ along the modular trek Γ is:

$$\mathfrak{T}_{\Gamma}(\omega|l) = \left\{ \eta : \mathscr{Itineraries}\left(\omega \xrightarrow{\Gamma} \eta \middle| l\right) \neq \emptyset \right\}.$$

Definition (2.3.20) was chosen to ensure that, given a modular trek Γ, for all $a \in \mathrm{bqs}(\mathrm{dom}(\Gamma))$, we have $\mathfrak{T}_{\Gamma}(a|l) = \mathfrak{T}_{\Gamma_\flat}(a|l)$, thus allowing us to directly invoke [31] to conclude:

Proposition 2.5.6 ([31, Propositions 5.11 and 5.12])**.** *Let us assume Hypothesis (2.5.1). Let $a, a' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and let $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$. If $b \in \mathfrak{T}_{\Gamma}(a|l)$ and $b' \in \mathfrak{T}_{\Gamma}(a'|l)$ then the following assertions hold:*

1. $\|b - b'\|_{\mathfrak{B}} \leqslant \|a - a'\|_{\mathfrak{A}} + 4l\lambda(\Gamma)$.
2. $\mathrm{diam}\left(\mathfrak{T}_{\gamma_\flat}(a|l), \|\cdot\|_{\mathfrak{B}}\right) \leqslant 4l\lambda(\Gamma_\flat)$.
3. *for all $t \in \mathbb{R}$, we have:*
$$\eta t \eta' \in \mathfrak{T}_{\Gamma}\left(a + ta' \middle| l(1+|t|)\right)$$
4. *we have:*
$$b \circ b' \in \mathfrak{T}_{\Gamma}\left(a \circ a' \middle| F(\|a\|_{\mathfrak{A}} + 2l\lambda(\Gamma), \|a'\|_{\mathfrak{A}} + 2l\lambda(\Gamma), l, l)\right)$$
and:
$$\{b, b'\} \in \mathfrak{T}_{\Gamma}\left(\{a, a'\} \middle| F(\|a\|_{\mathfrak{A}} + 2l\lambda(\Gamma), \|a'\|_{\mathfrak{A}} + 2l\lambda(\gamma_\flat), l, l)\right).$$
5. $\mathfrak{T}_{\Gamma}(a|l)$ *is a nonempty subset of* $\mathfrak{L}_l(\mathsf{L}_{\mathfrak{B}})$.

Proof. Note that $\gamma_\flat = \left(\gamma_\circ^j\right)_{j\in\{1,\ldots,n\}}$ is a trek from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, and $\mathfrak{T}_{\Gamma}(a|l) = \mathfrak{T}_{\gamma_\flat}(a|l)$, while $\lambda(\Gamma_\flat) \leqslant \lambda(\Gamma)$. Thus we may apply our work in [31].

Alternatively, all the statements in this proposition follow from similar techniques to Proposition (2.5.7) applied to Proposition (2.3.21) and Proposition (2.5.8). $\square$

With our notion of target sets in hand, we now can generalize Propositions (2.3.23), (2.3.25), (2.3.26) and (2.3.27) from modular bridges to modular treks.

Proposition 2.5.7. *Let us assume Hypothesis (2.5.1), and let us assume that $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ are (F,G,H)–metrized quantum vector bundle for some admissible triple (F,G,H).*

Let $\omega,\omega' \in \mathrm{dom}(\mathsf{D}_{\mathscr{M}})$ and let $l \geqslant \max\{\mathsf{D}_{\mathscr{M}}(\omega),\mathsf{D}_{\mathscr{M}}(\omega')\}$. If $\eta \in \mathfrak{T}_{\Gamma}(\omega|l)$ and $\eta' \in \mathfrak{T}_{\Gamma}(\omega'|l)$ then the following assertions hold:

1. $\mathsf{k}_{\Omega_{\mathfrak{B}}}(\eta,\eta') \leqslant \sqrt{2}\left(\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega,\omega') + (4l+H(2l,1))\lambda(\Gamma)\right).$
2. $\mathrm{diam}\left(\mathfrak{T}_{\Gamma}(\omega|l),\mathsf{mk}_{\Omega_{\mathfrak{B}}}\right) \leqslant \sqrt{2}(4l+H(2l,1))\lambda(\Gamma).$
3. *for all $t \in \mathbb{C}$, we have:*
$$\eta + t\eta' \in \mathfrak{T}_{\Gamma}\left(\omega + t\omega' \middle| l(1+|t|)\right)$$
4. *for all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and for all $l' \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, if $b \in \mathfrak{T}_{\Gamma}(a|l')$, then we have:*
$$b\eta \in \mathfrak{T}_{\Gamma}\left(a\omega \middle| G(\|a\|_{\mathfrak{A}} + 2l'\lambda(\Gamma), l', l)\right).$$
5. *If $b \in \mathfrak{T}_{\Gamma}\left(\langle\omega,\omega\rangle_{\mathscr{M}} | H(l,l)\right)$ then:*
$$\|b - \langle\eta,\eta\rangle_{\mathscr{N}}\|_{\mathfrak{B}} \leqslant \left(8l\sqrt{2} + H(2l,2l) + 6H(l,l) + 2\sqrt{2}H(2l,1)\right)\lambda(\Gamma).$$

Proof. We write $\Omega_j = \mathrm{codom}\left(\gamma^j\right)$ for all $j \in \{1,\ldots,n\}$ and $\Omega_0 = \Omega_{\mathfrak{A}}$.

Let $(\xi_0,\ldots,\xi_n) \in \mathscr{Itineraries}\left(\omega \xrightarrow{\Gamma} \eta \middle| l\right)$ and $(\xi'_0,\ldots,\xi'_n) \in \mathscr{Itineraries}\left(\omega' \xrightarrow{\Gamma} \eta' \middle| l\right)$. Since $\xi_{j+1} \in \mathsf{t}_{\gamma_{j+1}}\left(\xi_j | l\right)$, Proposition (2.3.25) gives us:
$$\mathsf{k}_{\Omega_{j+1}}(\xi_{j+1},\xi'_{j+1}) \leqslant \sqrt{2}\left(\mathsf{k}_{\Omega_j}(\xi_j,\xi'_j) + (4l+H(2l,1))\lambda\left(\gamma_{j+1}\right)\right).$$
Thus by induction, we get:
$$\begin{aligned}\mathsf{k}_{\Omega_{\mathfrak{B}}}(\eta,\eta') &\leqslant \sqrt{2}\left(\mathsf{mk}_{\Omega_{\mathfrak{A}}}(\omega,\omega') + (4l+H(2l,1))\sum_{j=1}^{n}\lambda\left(\gamma_j\right)\right)\\ &= \sqrt{2}\left(\mathsf{mk}_{\Omega_{\mathfrak{A}}}(\omega,\omega') + (4l+H(2l,1))\lambda(\Gamma)\right).\end{aligned}$$
If $\omega=\omega'$, then we obtain that $\mathrm{diam}\left(\mathfrak{T}_{\Gamma}(\omega|l),\mathsf{mk}_{\Omega_{\mathfrak{B}}}\right) \leqslant \sqrt{2}(4l+H(2l,1))\lambda(\Gamma)$.

We also have $\eta_{j+1} + t\eta'_{j+1} \in \mathsf{t}_{\gamma_j}\left(\eta_j + t\eta'_j \middle| l + |t|l\right)$ by Proposition (2.3.26). Thus, by induction, we get that:
$$\eta + t\eta' \in \mathfrak{T}_{\Gamma}\left(\omega + t\omega' | l + |t|l'\right).$$

Let now:
$$(b_j)_{j=0}^{n} \in \mathscr{Itineraries}\left(a \xrightarrow{\Gamma} b \middle| l\right).$$
For each j we have $b_{j+1}\eta_{j+1} \in \mathsf{t}_{\gamma_j}\left(b_j\eta_j \middle| G(\|b_j\| + l\lambda\left(\gamma_j\right), r, l)\right)$ by Proposition (2.3.27). Now, as before, we have $\|b_j\| \leqslant \|a\|_{\mathfrak{A}} + l\sum_{k=0}^{j}\lambda\left(\gamma_k\right) \leqslant \|a\|_{\mathfrak{A}} + l\lambda(\Gamma)$, and since $\lambda\left(\gamma_j\right) \leqslant \lambda(\Gamma)$, we have:
$$d_{j+1}\eta_{j+1} \in \mathsf{t}_{\gamma_j}\left(d_j\eta_j \middle| G(\|a\|_{\mathfrak{A}} + 2l\lambda(\Gamma), r, l)\right)$$
since $G(\cdot,r,l)$ is weakly increasing. This proves in turn that:
$$b\eta \in \mathfrak{T}_{\Gamma}\left(a\omega | G(\|a\|_{\mathfrak{A}} + 2l\lambda(\Gamma), r, l)\right).$$

Last, we address the property of target sets for treks and inner products. To ease notations, we set:

$$C = \left(8l\sqrt{2} + H(2l,2l) + 2H(l,l) + 2\sqrt{2}H(2l,1)\right),$$

which is the constant in Proposition (2.3.28). Moreover, we set $\Omega_j = (\mathscr{M}_j, \langle\cdot,\cdot\rangle_j, \mathsf{D}_j, \mathfrak{A}_j, \mathsf{L}_j)$ for all $j \in \{0,\ldots,n\}$. Moreover, we write $\gamma_j = (\mathfrak{D}_j, x_j, \pi_j, \rho_j, \text{anchors}\left(\gamma_j\right), \text{coanchors}\left(\gamma_j\right))$.

Let $b \in \mathfrak{T}_\Gamma\left(\langle\omega,\omega\rangle_{\mathscr{M}} | H(l,l)\right)$ and $(b_j)_{j=0}^n \in \mathscr{Itineraries}\left(\langle\omega,\omega\rangle_{\mathscr{M}} \xrightarrow{\Gamma} b \middle| H(l,l)\right)$.

Let us assume that for some $j \in \{1,\ldots,n-1\}$, we have:

$$\left\| b_j - \left\langle \xi_j, \xi_j \right\rangle_{\mathscr{N}} \right\|_{\mathfrak{A}_j} \leqslant (C + 4H(l,l)) \sum_{k=1}^{j} \lambda\left(\gamma_j\right). \tag{2.5.1}$$

By Definition (2.5.2), we have $b_{j+1} \in \mathfrak{t}_{\gamma_{j+1}}\left(b_j \middle| H(l,l)\right)$ and $\xi_{j+1} \in \mathfrak{t}_{\gamma_{j+1}}\left(\xi_j \middle| l\right)$. There is no expectation that $b_{j+1} \in \mathfrak{t}_{\gamma_{j+1}}\left(\left\langle \xi_j, \xi_j\right\rangle_{\mathscr{M}_j} \middle| H(l,l)\right)$. So we introduce $b'_{j+1} \in \mathfrak{t}_{\gamma_{j+1}}\left(\left\langle \xi_j, \xi_j\right\rangle_{\mathscr{M}_j} \middle| H(l,l)\right)$. By Proposition (2.3.25), we have:

$$\left\| b_{j+1} - b'_{j+1} \right\|_{\mathfrak{A}_{j+1}} \leqslant \| b_j - \left\langle \xi_j, \xi_j \right\rangle_{\mathscr{M}_j} \|_{\mathfrak{A}_j} + 4H(l,l)\lambda\left(\gamma_{j+1}\right).$$

On the other hand, by Proposition (2.3.28), we have:

$$\left\| b'_{j+1} - \left\langle \xi_{j+1}, \xi_{j+1} \right\rangle_{\mathscr{M}_+} \right\|_{\mathfrak{A}_{j+1}} \leqslant C\lambda\left(\gamma_{j+1}\right).$$

Therefore:

$$\left\| b_{j+1} - \left\langle \xi_{j+1}, \xi_{j+1} \right\rangle_{\mathscr{M}_{j+1}} \right\|_{\mathfrak{A}_{j+1}} \leqslant (C + 4H(l,l))\lambda\left(\gamma_{j+1}\right),$$

and thus using our induction hypothesis (2.5.1), we get:

$$\left\| b_{j+1} - \left\langle \xi_{j+1}, \xi_{j+1} \right\rangle_{\mathscr{M}_{j+1}} \right\|_{\mathfrak{A}_{j+1}} \leqslant (C + 4H(l,l)) \sum_{k=1}^{j+1} \lambda\left(\gamma_k\right),$$

which is our induction hypothesis (2.5.1) for $j+1$.

Now, by Proposition (2.3.28), we have:

$$\begin{aligned}\left\| b_1 - \left\langle \xi_1, \xi_1 \right\rangle_{\mathscr{M}_1} \right\|_{\mathfrak{A}_1} &\leqslant C\lambda\left(\gamma_1\right) \\ &\leqslant (C + 4H(l,l))\lambda\left(\gamma_1\right).\end{aligned}$$

Therefore, by induction, we have proven that:

$$\| b - \langle \eta, \eta \rangle_{\mathscr{N}} \|_{\mathfrak{B}} \leqslant (C + 4H(l,l))\lambda\left(\Gamma\right).$$

This concludes our proof. □

We also prove that target sets of modular treks are compact.

Proposition 2.5.8. *Let us assume Hypothesis (2.5.1). If $\omega \in \operatorname{dom}\left(\mathsf{D}_{\mathscr{M}}\right)$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$ then $\mathfrak{T}_\Gamma\left(\omega|l\right)$ is a nonempty and compact subset of $\mathscr{D}_r\left(\Omega_{\mathfrak{B}}\right)$ for $\|\cdot\|_{\mathscr{N}}$ (equivalently for $\mathsf{k}_{\Omega_{\mathfrak{B}}}$).*

Proof. We write $\Omega_j = \mathrm{codom}\left(\gamma^j\right)$ and $\Omega_j = (\mathscr{M}_j, \langle\cdot,\cdot\rangle_{\mathscr{M}_j}, \mathsf{D}_{\mathscr{M}_j}, \mathfrak{A}_j, \mathsf{L}_j)$ for all $j \in \{1,\ldots,n\}$.

We first note that a trivial induction prove that $\mathfrak{T}_\Gamma(\omega|l)$ is not empty using Proposition (2.3.23).

By construction, $\mathfrak{T}_\Gamma(\omega|l)$ is a subset of the $\|\cdot\|_{\mathscr{N}}$–compact set $\mathscr{D}_l(\Omega_{\mathfrak{B}})$. Thus it is sufficient to prove that it is closed for $\|\cdot\|_{\mathscr{N}}$.

Let $(\eta_k)_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{T}_\Gamma(\omega|l)$, converging to some $\eta \in \mathscr{N}$ for $\|\cdot\|_{\mathscr{N}}$.

Now, for each $k \in \mathbb{N}$, let $(\omega, \eta_k^1, \ldots, \eta_k^n)$ be an l-itinerary from ω to η_k. By Definition (2.2.8), each sequence $(\eta_k^j)_{k\in\mathbb{N}}$ lies in the compact set $\left\{\xi \in \mathscr{M}_j : \mathsf{D}_{\mathscr{N}_j}(\xi) \leqslant l\right\}$ for all $j \in \{1,\ldots,n\}$. Thus by a trivial induction, there exists strictly increasing functions $f_j : \mathbb{N} \to \mathbb{N}$ for $j \in \{1,\ldots,n\}$ such that $(\eta^j_{f_1\circ\cdots\circ f_j(k)})_{k\in\mathbb{N}}$ converges to some $\eta^j \in \mathscr{M}_j$ for $\|\cdot\|_{\mathscr{M}_j}$, for all $j \in \{1,\ldots,n\}$. Let $g : k \in \mathbb{N} \mapsto f_1 \circ f_2 \circ \cdots \circ f_n(k)$, so that $(\eta^j_{g(k)})_{k\in\mathbb{N}}$ converges to η^j for all $j \in \{1,\ldots,n\}$.

Our goal is to prove that $(\omega, \eta^1, \ldots, \eta^{n-1}, \eta = \eta^n)$ is an l-itinerary along Γ.

To begin with, $\mathsf{D}_{\mathscr{M}_j}(\eta^j) \leqslant l$ since $\mathsf{D}_{\mathscr{M}_j}$ is lower semi-continuous for all $j \in \{1,\ldots,n\}$.

Second of all, by continuity, we also have for all $j \in \{1,\ldots,n\}$:

$$\mathsf{dn}_{\gamma_j}\left(\eta^j, \eta^{j+1}\right) = \lim_{k\to\infty} \mathsf{dn}_{\gamma_j}\left(\eta^j_k, \eta^{j+1}_k\right) \leqslant l\varrho\left(\gamma_j\right).$$

This concludes our proof. □

Proposition (2.5.8) shows that modular trek target sets are in the hyperspace of a compact metric space, namely a closed unit ball for some D-norm: the norm topology and the modular Monge-Kantorovich metric topology on these balls are indeed the same and compact. The proof of our main Theorem (2.5.11) relies on an important property of the topology induced by the Hausdorff distance over the hyperspace of all nonempty closed subsets of a compact space: it only depends on the topological equivalence class of the chosen metric. We recall this well-known fact and include a proof for the convenience of the reader.

Lemma 2.5.9. *Let X be a compact space with topology τ and let $\mathcal{F} = \{U^c : U \in \tau, U \neq X\}$ be the set of all nonempty closed subsets of X. The* Vietoris topology *is the smallest topology on $\mathcal{F}$ generated from the topological basis:*

$$\mathscr{O}(U, V_1, \ldots, V_n) = \left\{F \in \mathcal{F} : F \subseteq U \text{ and } \forall j \in \{1,\ldots,n\} \quad F \cap V_j \neq \emptyset\right\}$$

for all $n \in \mathbb{N}$ and $U, V_1, \ldots, V_n \in \tau$.

If d is a metric on X which induced τ, then the topology induced by $\mathsf{Haus}_{\mathrm{d}}$ is the Vietoris topology.

Consequently, if d_1 and d_2 are two metrics which induce the same topology on X then $\mathsf{Haus}_{\mathrm{d}_1}$ and $\mathsf{Haus}_{\mathrm{d}_2}$ induce the same topology on $\mathcal{F}$.

Proof. Let $F \in \mathcal{F}$ and $r > 0$. Since F is compact, there exists $x_1, \ldots, x_n \in F$ for some $n \in \mathbb{N}$ such that $F \subseteq \bigcup_{j=1}^n X\left(x_j, \frac{r}{2}\right)$ where the open ball in (X, d) of center any $y \in X$ and radius r is denoted by $X(y, r)$. For all $j \in \{1,\ldots,n\}$, we set $V_j = X\left(x_j, \frac{r}{2}\right)$.

Let $U = \bigcup_{j=1}^n V_j$. Note that by construction, $F \in \mathcal{O}(U, V_1, \ldots, V_n)$. Now let $G \in \mathcal{O}(U, V_1, \ldots, V_n)$. If $x \in G$, then $x \in U$ and thus $x \in V_j$ for some $j \in \{1,\ldots,n\}$, implying that $\mathrm{d}(x,F) < \frac{r}{2}$. If $x \in F$, then $x \in V_j$ for some $j \in \{1,\ldots,n\}$. Since $G \cap V_j \neq \emptyset$, there exists $y \in G \cap V_j$ and by definition of V_j, we conclude $\mathrm{d}(x,y) < r$. Hence $\mathsf{Haus}_{\mathrm{d}}(F,G) < r$. Thus $\mathcal{O}(U,V_1,\ldots,V_n) \subseteq \mathcal{F}(F,r)$.

Let now $U,V_1,\ldots,V_n \in \tau$ be given with $F \in \mathcal{O}(U,V_1,\ldots,V_n)$. Since $X \setminus U$ is closed and disjoint from F, we conclude that there exists $\varepsilon_0 > 0$ such that, for all $x \in F$ and $y \in X \setminus U$, we have $\mathrm{d}(x,y) \geqslant \varepsilon_0$.

Now, for each $j \in \{1,\ldots,n\}$, there exists $x_j \in F \cap V_j$ and there exists $\varepsilon_j > 0$ such that $X(x_j,\varepsilon_j) \subseteq V_j$. Let $\varepsilon = \min\{\varepsilon_j : j \in \{0,\ldots,n\}\}$.

Let $G \in \mathcal{F}(F,\varepsilon)$. Let $x \in G$. There exists $y \in F$ such that $\mathrm{d}(x,y) < \varepsilon$. Thus $x \in U$ since $\mathrm{d}(x,y) < \varepsilon_0$. Thus $G \subseteq U$.

Let $j \in \{1,\ldots,n\}$. There exists $y \in G$ such that $\mathrm{d}(x_j,y) < \varepsilon \leqslant \varepsilon_j$, and thus by construction, $y \in X(x_j,\varepsilon_j) \subseteq V_j$ and thus $G \cap V_j \neq \emptyset$. We thus have shown that $G \in \mathcal{O}(U,V_1,\ldots,V_n)$. Thus $\mathcal{F}(F,\varepsilon) \subseteq \mathcal{O}(U,V_1,\ldots,V_n)$.

This proves our lemma. ☐

We conclude our preliminary statements with a simple, useful lemma which we will use a few times in our proof of our main theorem.

Lemma 2.5.10. *Let (E,dist) be a compact metric space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of E converging to some singleton $\{a\}$ for $\mathsf{Haus}_{\mathsf{dist}}$.*

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E such that $x_n \in A_n$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ converges to a.

Proof. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have:

$$\mathsf{Haus}_{\mathsf{dist}}(A_n,\{a\}) < \varepsilon.$$

Thus $\mathsf{dist}(x_n,a) < \varepsilon$ for all $n \geqslant N$. ☐

We are now ready to prove our main theorem.

Theorem 2.5.11. *Let $\mathcal{C}$ be a nonempty class of (F,G,H)–metrized quantum vector bundles, with (F,G,H) an admissible triple, and let $\mathcal{B}$ be a class of modular bridges compatible with $\mathcal{C}$. Let $\Omega_{\mathfrak{A}} = (\mathscr{M},\langle\cdot,\cdot\rangle_{\mathscr{M}},\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}} = (\mathscr{N},\langle\cdot,\cdot\rangle_{\mathscr{N}},\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ be two metrized quantum vector bundles in $\mathcal{C}$. The following two assertions are equivalent:*

- *I. $\Lambda^{\mathsf{mod}}_{\mathcal{B}}(\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}) = 0$,*
- *II. $\Omega_{\mathfrak{A}}$ and $\Omega_{\mathfrak{B}}$ are fully quantum isometric, i.e. there exists a *-isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ and a linear continuous isomorphism $\Theta : \mathscr{M} \to \mathscr{N}$ such that:*
  1. *$\mathsf{L}_{\mathfrak{B}} \circ \theta = \mathsf{L}_{\mathfrak{A}}$,*
  2. *$\Theta(a\omega) = \theta(a)\Theta(\omega)$ for all $a \in \mathfrak{sa}(\mathfrak{A})$, $\omega \in \mathscr{M}$,*
  3. *$\mathsf{D}_{\mathscr{N}} \circ \Theta = \mathsf{D}_{\mathscr{M}}$,*
  4. *$\langle\Theta(\cdot),\Theta(\cdot)\rangle_{\mathscr{N}} = \theta \circ \langle\cdot,\cdot\rangle_{\mathscr{M}}$.*

Proof. For all $n \in \mathbb{N}$, let $\Gamma_n \in \mathscr{T}\!\mathit{reks}\left[\Omega_{\mathfrak{A}} \xrightarrow{\mathcal{T}} \Omega_{\mathfrak{B}}\right]$ be given such that $\lambda(\Gamma_n) \leqslant \frac{1}{n+1}$.

We prove our theorem in a series of claim.

Claim 2.5.12. *If $\omega \in \mathrm{dom}(\mathsf{D}_{\mathscr{M}})$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$, and if $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that the sequence:*

$$\left(\mathfrak{T}_{\Gamma_{f\circ g(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$$

converges to a singleton for the Hausdorff distance $\mathsf{Haus}_{\|\cdot\|_{\mathscr{M}}}$.

The sequence $\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$ is a sequence of closed subsets of the compact $\mathscr{D}_l(\Omega_{\mathfrak{B}})$ by Proposition (2.5.8). The hyperspace of all closed nonempty subsets of the compact set $(\mathscr{D}_l(\Omega_{\mathfrak{B}}), \mathsf{k}_{\Omega_{\mathfrak{B}}})$ is compact for the Hausdorff distance $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$.

Thus, $\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$ admits a convergent subsequence $\left(\mathfrak{T}_{\Gamma_{f\circ g(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$ converging for $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$; let $\mathfrak{L}$ be its limit.

By Assertion (2) of Proposition (2.5.7), we have $\mathrm{diam}\left(\mathfrak{L}, \mathsf{k}_{\Omega_{\mathfrak{B}}}\right) = 0$, i.e. it is a singleton.

Now, on the compact set $\mathscr{D}_r(\Omega_{\mathfrak{B}})$, both $\mathsf{k}_{\Omega_{\mathfrak{B}}}$ and $\|\cdot\|_{\mathscr{N}}$ are topologically equivalent by Proposition (2.2.24). Hence, $\left(\mathfrak{T}_{\Gamma_{f\circ g(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$ converges to $\mathfrak{L}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathscr{N}}}$ by Lemma (2.5.9).

Claim 2.5.13. *Let us simplify our notations for this claim. Let* $(A_n)_{n\in\mathbb{N}}$ *and* $(B_n)_{n\in\mathbb{N}}$ *be two sequences of nonempty closed subsets in* $\mathscr{D}_K(\Omega_{\mathfrak{B}})$ *for some* $K>0$ *such that, for all* $n\in\mathbb{N}$, *we have* $A_n\subseteq B_n$, *and moreover:*

$$\lim_{n\to\infty}\mathrm{diam}\left(B_n, \mathsf{k}_{\Omega_{\mathfrak{B}}}\right) = \lim_{n\to\infty\}}\mathrm{diam}\left(A_n, \mathsf{k}_{\Omega_{\mathfrak{A}}}\right) = 0.$$

Then $(A_n)_{n\in\mathbb{N}}$ *converges for* $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ *if and only if* $(B_n)_{n\in\mathbb{N}}$ *converges for* $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ *(noting the limit must be a singleton and it must be the same for both sequences).*

Assume first that $(A_n)_{n\in\mathbb{N}}$ converges for $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ — the limit being necessarily a singleton $\{\eta\}$, since the diameter of A_n converges to 0 as n goes to infinity.

Let $\varepsilon>0$. There exists $N\in\mathbb{N}$ such that for all $n\geqslant N$ we have $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}(A_n,\{\eta\}) < \frac{\varepsilon}{2}$. There exists $N'\in\mathbb{N}$ such that for all $n\geqslant N'$, we have $\mathrm{diam}\left(B_n, \mathsf{k}_{\Omega_{\mathfrak{B}}}\right) < \frac{\varepsilon}{2}$. Let $n\geqslant\max\{N,N'\}$. If $\omega\in B_n$ then there exists $\xi\in A_n$ such that $\mathsf{k}_{\Omega_{\mathfrak{B}}}(\omega,\xi) < \frac{\varepsilon}{2}$, and then we have $\mathsf{k}_{\Omega_{\mathfrak{B}}}(\xi,\eta) < \frac{\varepsilon}{2}$. Thus $\mathsf{k}_{\Omega_{\mathfrak{B}}}(\omega,\eta) < \varepsilon$. It then follows that $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}(B_n,\{\eta\}) < \varepsilon$. This proves that $(B_n)_{n\in\mathbb{N}}$ converges to $\{\eta\}$.

Assume second that $(B_n)_{n\in\mathbb{N}}$ converges for $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$, again necessarily to a singleton $\{\eta\}$. It is then immediate that $\mathsf{k}_{\Omega_{\mathfrak{B}}}(\omega,\eta)\leqslant\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}(B_n,\{\eta\})$ for all $\omega\in A_n$ and thus in particular, $(A_n)_{n\in\mathbb{N}}$ converges to $\{\eta\}$ as well.

Claim 2.5.14. *If* $\omega\in\mathrm{dom}(\mathsf{D}_{\mathscr{M}})$ *and* $l\geqslant\mathsf{D}_{\mathscr{M}}(\omega)$, *and if* $f:\mathbb{N}\to\mathbb{N}$ *is a strictly increasing function such that*

$$\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$$

converges to $\{\eta\}$ *for the Hausdorff distance* $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$, *then for all* $l'\geqslant\mathsf{D}_{\mathscr{M}}(\omega)$, *the sequence:*

$$\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l')\right)_{n\in\mathbb{N}}$$

converges to $\{\eta\}$ *for the Hausdorff distance* $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$.

We note that for all $l\geqslant l'\geqslant\mathsf{D}_{\mathscr{M}}(\omega)$, we have:

$$\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l')\subseteq\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l)$$

for all $n \in \mathbb{N}$. Moreover, $\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l'), \mathfrak{T}_{\Gamma_{f(n)}}(\omega|l) \subseteq \mathscr{D}_l(\Omega_{\mathfrak{B}})$ for all $n \in \mathbb{N}$. Last:

$$\lim_{n\to\infty} \operatorname{diam}\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l'), \mathsf{k}_{\Omega_{\mathfrak{B}}}\right) = \lim_{n\to\infty} \operatorname{diam}\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l), \mathsf{k}_{\Omega_{\mathfrak{B}}}\right) = 0$$

by Assertion (2) of Proposition (2.5.7). This allows us to conclude our claim using Claim (2.5.13).

Claim 2.5.15. *There exists $f : \mathbb{N} \to \mathbb{N}$ strictly increasing such that for all $\omega \in \operatorname{dom}(\mathsf{D}_{\mathscr{M}})$ and for all $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$, the sequence:*

$$\left(\mathfrak{T}_{\Gamma_{f(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$$

converges to a singleton $\theta(\omega)$ for the Hausdorff distance $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ (or equivalently for $\mathsf{Haus}_{\|\cdot\|_{\mathscr{N}}}$).

We use a diagonal argument and Claim (2.5.12). As a compact metric space, the closed unit ball of $\operatorname{dom}(\mathsf{D}_{\mathscr{M}})$ is separable; however we can be a bit more precise in our case. For each $n \in \mathbb{N}$, let:

$$\Gamma_n = \left(\gamma_j^n : j \in \{1, \ldots, K_n\}\right) \text{ for some } K_n \in \mathbb{N} \setminus \{0\}. \tag{2.5.2}$$

Since the imprint of γ_1^n is less than $\lambda(\Gamma_n)$, we note that anchors (γ_1^n) is a finite, $\frac{1}{n+1}$-dense subset of $(\mathscr{D}_1(\Omega_{\mathfrak{A}}), \mathsf{k}_{\Omega_{\mathfrak{A}}})$.

Let:

$$\mathscr{S}_1 = \bigcup_{n\in\mathbb{N}} \text{anchors}(\gamma_1^n).$$

By construction, the set $\mathscr{S}$ is dense in $\mathscr{D}_1(\Omega_{\mathfrak{A}})$ — as well as countable.

For each $N \in \mathbb{N}$, the set $\mathscr{S}_N = N \cdot \mathscr{S}_1$ is dense in $\mathscr{D}_N(\Omega_{\mathfrak{A}})$, since we note that the modular Monge-Kantorovich metric is homogeneous, namely $\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega, \eta) = N\mathsf{k}_{\Omega_{\mathfrak{A}}}(N^{-1}\omega, N^{-1}\eta)$ for all $\omega, \eta \in \mathscr{M}$.

Thus, $\mathscr{S} = \bigcup_{N\in\mathbb{N}} \mathscr{S}_N$ is countable and dense in $\operatorname{dom}(\mathsf{D}_{\mathscr{M}})$. Let us write $\mathscr{S}$ as $\{\omega_n : n \in \mathbb{N}\}$.

By Claim (2.5.12), there exists $g_0 : \mathbb{N} \to \mathbb{N}$ strictly increasing, such that the sequence:

$$\left(\mathfrak{T}_{\Gamma_{g_0(n)}}(\omega_0|\mathsf{D}_{\mathscr{M}}(\omega_0))\right)_{n\in\mathbb{N}}$$

converges to a singleton $\{\Theta(\omega_0)\}$.

Assume now that for some $k \in \mathbb{N}$, we have built $g_0 : \mathbb{N} \to \mathbb{N}, \ldots, g_k : \mathbb{N} \to \mathbb{N}$ strictly increasing functions such that for all $j \in \{0, \ldots, k\}$, the sequence:

$$\left(\mathfrak{T}_{\Gamma_{g_0\circ\ldots\circ g_j(n)}}(\omega_j|\mathsf{D}_{\mathscr{M}}(\omega_j))\right)_{n\in\mathbb{N}}$$

converges to a singleton $\{\Theta(\omega_j)\}$.

Applying our Claim (2.5.12) again, there exists g_{k+1} strictly increasing, such that $\left(\mathfrak{T}_{\Gamma_{g_0\circ\ldots\circ g_{k+1}(n)}}(\omega_{k+1}|\mathsf{D}_{\mathscr{M}}(\omega_{k+1}))\right)_{n\in\mathbb{N}}$ converges. Thus by induction, there exists strictly increasing functions g_k for all $k \in \mathbb{N}$ such that:

$$\left(\mathfrak{T}_{\Gamma_{g_0\circ\ldots\circ g_j(n)}}(\omega_j|\mathsf{D}_{\mathscr{M}}(\omega_j))\right)_{n\in\mathbb{N}}$$

converges to a singleton denoted by $\{\Theta(\omega_j)\}$ for all $j \in \mathbb{N}$.

Since subsequences of converging sequences have the same limit as the original sequence, we conclude that, if we set $f : n \in \mathbb{N} \to f(n) = g_0 \circ \cdots \circ g_n(n) \in \mathbb{N}$, then f is strictly increasing and for all $\omega \in \mathscr{S}$, the sequence:

$$\left(\mathfrak{T}_{\Gamma_{f(m)}}(\omega|\mathsf{D}_{\mathscr{M}}(\omega))\right)_{m\in\mathbb{N}}$$

converges to a singleton $\{\Theta(\omega)\}$.

By Claim (2.5.14), we note that for any $\omega \in \mathscr{S}$ and $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$, we also have:

$$\lim_{n\to\infty} \mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_{f(m)}}(\omega|l), \{\Theta(\omega)\}\right) = 0.$$

We now move to prove that Θ can be extended to $\mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$. Let $\omega \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$. There exists $N \in \mathbb{N}$ such that $\omega \in \mathscr{D}_N(\Omega_{\mathfrak{A}})$. We may as well assume that $N > 0$.

Let $\varepsilon > 0$. There exists $\omega_\varepsilon \in \mathscr{S}_N$ such that $\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega, \omega_\varepsilon) < \frac{\varepsilon\sqrt{2}}{12}$. Then by Proposition (2.5.7), we have, for all $n \in \mathbb{N}$:

$$\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_n}(\omega|N), \mathfrak{T}_{\Gamma_n}(\omega_\varepsilon|N)\right) \leqslant \sqrt{2}\left(\frac{\sqrt{2}\varepsilon}{12} + (4N + H(2N,1))\frac{1}{n+1}\right).$$

Let $N' \in \mathbb{N}$ be chosen so that $\frac{1}{n+1} \leqslant \frac{\sqrt{2}\varepsilon}{12(4N+H(2N,1))}$ for all $n \geqslant N'$.

Therefore, for all $n \geqslant N'$, we have:

$$\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_n}(\omega|N), \mathfrak{T}_{\Gamma_n}(\omega_\varepsilon|N)\right) \leqslant \frac{\varepsilon}{3}.$$

Since $\mathfrak{T}_{\Gamma_{f(n)}}(\omega_\varepsilon|N)$ converges for $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$, it is Cauchy, and thus there exists $N'' \in \mathbb{N}$ such that for all $p, q \geqslant N''$ we have:

$$\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_{f(p)}}(\omega_\varepsilon|N), \mathfrak{T}_{\Gamma_{f(q)}}(\omega_\varepsilon|N)\right) \leqslant \frac{\varepsilon}{3}.$$

Thus if $p, q \geqslant \max\{N', N''\}$, we have:

$$\begin{aligned}
\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_{f(p)}}(\omega|N), \mathfrak{T}_{\Gamma_{f(q)}}(\omega|N)\right) &\leqslant \mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_{f(p)}}(\omega|N), \mathfrak{T}_{\Gamma_{f(p)}}(\omega_\varepsilon|N)\right) \\
&\quad + \mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_{f(p)}}(\omega_\varepsilon|N), \mathfrak{T}_{\Gamma_{f(q)}}(\omega_\varepsilon|N)\right) \\
&\quad + \mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}\left(\mathfrak{T}_{\Gamma_{f(q)}}(\omega_\varepsilon|N), \mathfrak{T}_{\Gamma_{f(q)}}(\omega|N)\right) \\
&\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus the sequence $\left(\mathfrak{T}_{\Gamma_{f(m)}}(\omega|N)\right)_{m\in\mathbb{N}}$ is Cauchy for $\mathsf{k}_{\Omega_{\mathfrak{B}}}$ inside the hyperspace of closed subsets of the compact $\mathscr{D}_N(\Omega_{\mathfrak{B}})$. and thus converges by completeness.

Again by Proposition (2.5.7), the limit of $\left(\mathfrak{T}_{\Gamma_{f(m)}}(\omega|N)\right)_{m\in\mathbb{N}}$ for $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ is a singleton which we denote by $\{\Theta(\omega)\}$. Moreover, by Claim (2.5.14), the sequence $\left(\mathfrak{T}_{\Gamma_{f(m)}}(\omega|l)\right)_{m\in\mathbb{N}}$ converges in $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ to $\{\Theta(\omega)\}$ for any $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$. Last, since $\mathsf{k}_{\Omega_{\mathfrak{B}}}$ and $\|\cdot\|_{\mathscr{N}}$ are topologically equivalent on $\mathscr{D}_K(\Omega_{\mathfrak{B}})$ for any $K \geqslant 0$, the Hausdorff distances $\mathsf{Haus}_{\mathsf{k}_{\Omega_{\mathfrak{B}}}}$ and $\mathsf{Haus}_{\|\cdot\|_{\mathscr{N}}}$ are also topologically equivalent by Lemma (2.5.9), which concludes the proof of our claim.

Claim 2.5.16. *For all $\omega \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$ we have $\mathsf{D}_{\mathscr{N}}(\Theta(\omega)) \leqslant \mathsf{D}_{\mathscr{M}}(\omega)$.*

Let $\omega \in \mathrm{dom}\left(\mathsf{D}_{\mathscr{M}}\right)$ and let $l = \mathsf{D}_{\mathscr{M}}(\omega)$. By Claim (2.5.15) and Lemma (2.5.10), if we pick $\eta_n \in \mathfrak{T}_{\Gamma_{f(n)}}(\omega|l)$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \|\eta_n - \Theta(\omega)\|_{\mathscr{N}} = 0$. Since $\mathsf{D}_{\mathscr{N}}$ is lower semi-continuous (as $\mathscr{D}_1\left(\Omega_{\mathfrak{B}}\right)$ is compact, hence closed, for the norm $\|\cdot\|_{\mathscr{N}}$), we conclude that $\mathsf{D}_{\mathscr{N}}(\Theta(\omega)) \leqslant l = \mathsf{D}_{\mathscr{M}}(\omega)$.

Claim 2.5.17. *There exists a unital *-morphism $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \theta = \mathsf{L}_{\mathfrak{A}}$ and a strictly increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that:*

i. for all $a \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right)$ and for all $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the sequence:

$$\left(\mathfrak{T}_{\Gamma_{g(n)}}(a|l)\right)_{n\in\mathbb{N}}$$

converges to $\{\theta(a)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{B}}}$;

ii. for all $\omega \in \mathrm{dom}\left(\mathsf{D}_{\mathscr{M}}\right)$ and any $l \geqslant \mathsf{D}_{\mathscr{M}}(\omega)$, the sequence:

$$\left(\mathfrak{T}_{\Gamma_{g(n)}}(\omega|l)\right)_{n\in\mathbb{N}}$$

converges to $\{\Theta(\omega)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathscr{N}}}$.

Moreover $\mathsf{L}_{\mathfrak{B}} \circ \theta \leqslant \mathsf{L}_{\mathfrak{A}}$.

For all $n \in \mathbb{N}$, let $\Upsilon_n = (\Gamma_{f(n)})_\flat$. We note that $\lambda\left(\Upsilon_n\right) \leqslant \frac{1}{n+1}$ by construction. The construction of θ follows the same techniques as used in [31, Theorem 5.13], which provides us with a *-isomorphism θ and some strictly increasing function $f_1 : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $\omega \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right)$, the sequence

$$\left(\mathfrak{T}_{\Upsilon_{f_1(n)}}(a|l)\right)_{n\in\mathbb{N}}$$

converges to $\{\theta(a)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{B}}}$.

The rest of the claim follows if we set $g = f \circ f_1$.

Claim 2.5.18. *For all $\omega, \omega' \in \mathrm{dom}\left(\mathsf{D}_{\mathscr{M}}\right)$ we have:*

$$\theta \circ \left\langle \omega, \omega' \right\rangle_{\mathscr{M}} = \left\langle \Theta(\omega), \Theta(\omega') \right\rangle_{\mathscr{N}}.$$

In particular, $\|\Theta(\omega)\|_{\mathscr{N}} = \|\omega\|_{\mathscr{M}}$.

Let $\omega \in \mathrm{dom}\left(\mathsf{D}_{\mathscr{M}}\right)$ and $l = \mathsf{D}_{\mathscr{M}}(\omega)$. For each $n \in \mathbb{N}$ we pick $\eta_n \in \mathfrak{T}_{\Gamma_{g(n)}}(\omega|l)$ and $b_n \in \mathfrak{T}_{\Gamma_{g(n)}}\left(\langle \omega, \omega \rangle_{moduleM} | H(l,l)\right)$.

By Lemma (2.5.10) and Claim (2.5.17), we conclude that $\lim_{n\to\infty} b_n = \theta(\langle \omega, \omega \rangle_{\mathscr{M}})$ and $\lim_{n\to\infty} \eta_n = \Theta(\omega)$.

By Proposition (2.5.7), for all $n \in \mathbb{N}$, we have:

$$\begin{aligned}
&\|b_n - \langle \eta_n, \eta_n \rangle_{\mathscr{N}}\|_{\mathfrak{B}} \\
&\qquad \leqslant \lambda\left(\Gamma_{g(n)}\right)\left(8l\sqrt{2} + H(2l,2l) + 6H(l,l) + 2\sqrt{2}H(2l,1)\right) \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\langle \Theta(\omega), \Theta(\omega) \rangle_{\mathscr{N}} &= \langle \eta, \eta \rangle_{\mathscr{N}} \\
&= \lim_{n\to\infty} \langle \eta_n, \eta_n \rangle_{\mathscr{N}} \\
&= \lim_{n\to\infty} b_n = \theta(\langle \omega, \omega \rangle_{\mathscr{M}}).
\end{aligned} \tag{2.5.3}$$

Let now $\omega' \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$. We note that:

$$\left\langle \omega, \omega' \right\rangle_{\mathscr{M}} = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle \omega + i^k\omega', \omega + i^k \right\rangle_{\mathscr{M}}.$$

The same polarizing identities hold in $\mathscr{N}$. Thus, Equality (2.5.3) coupled with the above polarizing identities proves our claim.

Claim 2.5.19. *For all $\omega, \omega' \in \mathscr{M}$, $t \in \mathbb{R}$ and $a \in \mathfrak{A}$:*

$$\Theta(\omega + t\omega') = \Theta(\omega) + t\Theta(\omega')$$

and

$$\Theta(a\omega) = \theta(a)\Theta(\omega).$$

Consequently, Θ is uniformly continuous with from $(\mathrm{dom}\,(\mathsf{D}_{\mathscr{M}}), \|\cdot\|_{\mathscr{M}})$ to $(\mathrm{dom}\,(\mathsf{D}_{\mathscr{N}}), \|\cdot\|_{\mathscr{N}})$ and thus has a unique extension as a continuous module morphism, denoted in the same manner, from $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ to $(\mathscr{N}, \|\cdot\|_{\mathscr{N}})$.

Let $a \in \mathrm{dom}\,(\mathsf{L}_{\mathfrak{A}})$, $\omega \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{D}_{\mathscr{M}}(\omega)\}$. Let $b_n \in \mathfrak{T}_{\Gamma_{g(n)}}(a|l)$ and $\eta_n \in \mathfrak{T}_{\Gamma_{g(n)}}(\omega|l)$

By Proposition (2.5.7), we have:

$$b_n\eta_n \in \mathfrak{T}_{\Gamma_n}\left(a\omega \middle| G(\|a\|_{\mathfrak{A}} + 2l\lambda\left(\Gamma_{g(n)}\right), l, l)\right).$$

Thus $(b_n\eta_n)_{n\in\mathbb{N}}$ converges to $\Theta(a\omega)$ by Claim (2.5.15) and Lemma (2.5.10). For the same reasons, $(b_n)_{n\in\mathbb{N}}$ converges to $\theta(a)$ and $(\eta_n)_{n\in\mathbb{N}}$ converges to $\Theta(\omega)$. By continuity of the left module action in $\mathscr{N}$ and uniqueness of the limit:

$$\theta(a)\Theta(\omega) = \Theta(a\omega).$$

A similar reasoning applies to prove the linearity of Θ. Let $\omega, \omega' \in \mathrm{dom}\,(\mathsf{D}_{\mathscr{M}})$ and $t \in \mathbb{C}$. Let $l \geqslant \max\{\mathsf{D}_{\mathscr{M}}(\omega), \mathsf{D}_{\mathscr{M}}(\omega')$. For all $n \in \mathbb{N}$, we let $\eta_n \in \mathfrak{T}_{\Gamma_{g(n)}}(\omega|l)$ and $\eta' \in \mathfrak{T}_{\Gamma_{g(n)}}(\omega'|l)$. By Proposition (2.5.7) again, we have:

$$\eta_n + t\eta'_n \in \mathfrak{T}_{\Gamma_{g(n)}}(\omega + t\omega|l + |t|l).$$

By Lemma (2.5.10) and Claim (2.5.15), we conclude that:

$$\begin{aligned}\Theta(\omega + t\omega') &= \lim_{n\to\infty}(\eta_n + t\eta'_n)\\ &= \lim_{n\to\infty}\eta_n + t\lim_{n\to\infty}\eta'_n\\ &= \Theta(\omega) + t\Theta(\omega).\end{aligned}$$

Now, $\langle\Theta(\cdot), \Theta(\cdot)\rangle_{\mathscr{N}} = \theta \circ \langle\cdot,\cdot\rangle_{\mathscr{M}}$ so $\|\Theta(\cdot)\|_{\mathscr{N}} = \|\cdot\|_{\mathscr{M}}$. Thus Θ, being linear, is continuous and of norm 1. It thus can be extended to $\mathscr{M}$ by continuity as a linear map.

By continuity, we have that $\Theta(a\omega) = \theta(a)\Theta(\omega)$ for all $a \in \mathfrak{sa}\,(\mathfrak{A})$ and $\omega \in \mathscr{M}$. By linearity, it follows that (Θ, θ) is a module morphism, as desired. This completes our claim.

Claim 2.5.20. *The map Θ is a continuous module isomorphism of norm* 1*.*

Let $\Xi_n = \Gamma^*_{g(n)}$ for all $n \in \mathbb{N}$. By construction, $\lambda\left(\Xi_n\right) \leqslant \frac{1}{n+1}$. We therefore apply all the work we have done up to now with Ξ_n in place of Γ_n for all $n \in \mathbb{N}$. We thus obtain maps $h : \mathbb{N} \to \mathbb{N}$, $\vartheta : \mathfrak{B} \to \mathfrak{A}$ and $\Phi : \mathscr{N} \to \mathscr{M}$ such that h is strictly increasing function, (ϑ, Φ) is a module morphism of norm 1 with the additional property that $\mathsf{D}_{\mathscr{M}}(\Phi(\eta)) \leqslant \mathsf{D}_{\mathscr{N}}(\eta)$, and such that:

$$\lim_{n\to\infty} \mathsf{Haus}_{\|\cdot\|_{\mathscr{M}}} \left(\mathfrak{T}_{\Gamma^*_{g(h(n))}} (\eta | \mathsf{D}_{\mathscr{N}}(\eta)), \{\Phi(\eta)\} \right) = 0$$

and:

$$\lim_{n\to\infty} \mathsf{Haus}_{\|\cdot\|_{\mathfrak{B}}} \left(\mathfrak{T}_{\Gamma^*_{g(h(n))}} (b | \mathsf{L}_{\mathfrak{B}}(b)), \{\vartheta(b)\} \right) = 0$$

for all $\eta \in \operatorname{dom}\left(\mathsf{D}_{\mathscr{N}}\right)$ and $b \in \mathfrak{sa}\left(\mathfrak{B}\right)$.

Now, let $\omega \in \operatorname{dom}\left(\mathsf{D}_{\mathscr{M}}\right)$ and $l > \mathsf{D}_{\mathscr{M}}(\omega)$. We begin with a simple observation, owing to the symmetry in Definition (2.3.13) of the deck seminorm of a bridge, which in turns implies symmetry in the notion of itinerary:

$$\eta \in \mathfrak{T}_{\Gamma_{g(h(n))}} (\omega | l) \iff \omega \in \mathfrak{T}_{\Gamma^*_{g(h(n))}} (\eta | l)$$

for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have:

(1) $\frac{1}{n+1} \leqslant \frac{\varepsilon}{8l}$ so that $\max\left\{ \lambda\left(\Gamma_{g(h(n))}\right), \lambda\left(\Gamma^*_{g(h(n))}\right) \right\} \leqslant \frac{\varepsilon}{8l}$,

(2) $\mathsf{Haus}_{\|\cdot\|_{\mathscr{N}}} \left(\mathfrak{T}_{\Gamma_{g(h(n))}} (\omega | l), \{\Theta(\omega)\} \right) \leqslant \frac{\varepsilon}{2}$.

Let $\zeta \in \mathfrak{T}_{\Gamma_{g(h(n))}} (\omega | l)$, so that in particular $\|\zeta - \Theta(\omega)\| \leqslant \frac{\varepsilon}{2}$. By symmetry, $\omega \in \mathfrak{T}_{\Gamma^*_{g(h(n))}} (\zeta | l)$.

Now, let $\xi \in \mathfrak{T}_{\Gamma^*_{g(h(n))}} (\Theta(\omega) | l)$. We then compute, using Proposition (2.5.7):

$$\begin{aligned} \|\omega - \Phi\circ\Theta(\omega)\|_{\mathscr{M}} &\leqslant \|\omega - \xi\|_{\mathscr{M}} + \|\xi - \Phi\circ\Theta(\omega)\|_{\mathscr{M}} \\ &\leqslant 4l\frac{\varepsilon}{8l} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\omega = \Phi(\Theta(\omega))$. The same computation would establish that $\Phi \circ \Omega$ is the identity on $\operatorname{dom}\left(\mathsf{D}_{\mathscr{N}}\right)$ as well. By continuity, $\Phi = \Theta^{-1}$. For similar reasons, $\vartheta = \theta^{-1}$.

We last note that $\mathsf{D}_{\mathscr{M}} = \mathsf{D}_{\mathscr{M}} \circ \Theta \circ \Theta^{-1} \leqslant \mathsf{D}_{\mathscr{N}} \circ \Theta \leqslant \mathsf{D}_{\mathscr{M}}$. Thus $\mathsf{D}_{\mathscr{N}} \circ \Theta = \mathsf{D}_{\mathscr{M}}$. This concludes our proof. $\square$

We now turn to our first examples of convergence of metrized quantum vector bundles. We begin with free modules, which gives us a chance to compare the modular Gromov-Hausdorff propinquity with the quantum Gromov-Hausdorff propinquity when working with quasi-Leibniz quantum compact metric spaces and their associated metrized quantum vector bundles via Example (2.2.14).

2.6. **Convergence of Free modules.** We wish to answer the following natural question: if a sequence of quasi-Leibniz quantum compact metric spaces converge in the quantum propinquity, then, do free modules over them, seen as metrized quantum vector bundles via Example (2.2.15), converge for the modular propinquity? One would certainly hope that the answer is positive, and we now prove it. An important side-product of this section is that the quantum propinquity and the modular propinquity restricted to the class of quasi-Leibniz quantum compact metric spaces — using Example (2.2.14) — are in fact equivalent.

The key step in our work is to lift a bridge between quasi-Leibniz quantum compact metric spaces to a bridge between a pair of free modules, in the manner given by the next lemma. In this section, we will employ the notations of Examples (2.2.14) and (2.2.15).

Lemma 2.6.1. *If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are two quasi-Leibniz quantum compact metric spaces, $n \in \mathbb{N} \setminus \{0\}$ and γ is some bridge from $\mathfrak{A}$ to $\mathfrak{B}$, then there exists a modular bridge γ_{mod} from $\left(\mathfrak{A}^n, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}\right)$ to $\left(\mathfrak{B}^n, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}}\right)$ such that:*

$$\lambda(\gamma) \leqslant \lambda(\gamma_{\mathrm{mod}}) \leqslant 2n\lambda(\gamma).$$

Proof. Let $\gamma = (\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be a bridge from $\mathfrak{A}$ to $\mathfrak{B}$ of length λ.

Let:

$$J_{\mathfrak{A}} = \{a \in \mathfrak{sa}(\mathfrak{A}) : \max\{\|a\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(a)\} \leqslant 1\} \text{ and } J_{\mathfrak{B}} = \{b \in \mathfrak{sa}(\mathfrak{B}) : \max\{\|b\|_{\mathfrak{B}}, \mathsf{L}_{\mathfrak{B}}(b)\} \leqslant 1\}.$$

Let $J_1 = J_{\mathfrak{A}} \coprod J_{\mathfrak{B}}$ be the disjoint union of $J_{\mathfrak{A}}$ and $J_{\mathfrak{B}}$. Note that to avoid confusion if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) = (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, we can regard $J_{\mathfrak{A}}$ and $J_{\mathfrak{B}}$ as their corresponding subsets of J from now on.

If $j \in J_{\mathfrak{A}}$, we let $a_j = a$ and set $b_j = \frac{1}{1+2\lambda} c_j$ where we choose $c_j \in \mathfrak{t}_{\gamma}(j|1)$. Note that:

$$\|c_j\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{A}} + 2\lambda \leqslant 1 + 2\lambda$$

by Proposition (2.3.25). Thus $\|b_j\|_{\mathfrak{B}} \leqslant 1$ and of course $\mathsf{L}_{\mathfrak{B}}(b_j) \leqslant 1$.

If $j \in J_{\mathfrak{B}}$ then we set $b_j = j$ and $a_j = \frac{1}{1+2\lambda} c_j$ with $c_j \in \mathfrak{t}_{\gamma^*}(j|1)$.

We note that $\max\{\|a_j\|_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{A}}(a_j), \|b_j\|_{\mathfrak{B}}, \mathsf{L}_{\mathfrak{B}}(b_j)\} \leqslant 1$ for all $j \in J_1$ by Proposition (2.3.25). Moreover, by construction, for all $j \in J$, we note that $\mathsf{bn}_{\gamma}(a, b) \leqslant \lambda$.

We now let $J = (J_1)^n$. Let $j = (j_1, \ldots, j_n) \in J$. We write $\omega_j = \begin{pmatrix} a_{j_1} \\ \vdots \\ a_{j_n} \end{pmatrix}$ and $\eta_j = \begin{pmatrix} b_{j_1} \\ \vdots \\ b_{j_n} \end{pmatrix}$. We note that by construction, if $j = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathscr{D}_1\left(\mathsf{D}^n_{\mathfrak{A}}\right)$, then $j \in J$ and $\omega_j = j$ so $\{\omega_j : j \in J\} = \mathscr{D}_1\left(\mathsf{D}^n_{\mathfrak{A}}\right)$. The same reasoning applies in with $\mathfrak{B}$ in place of $\mathfrak{A}$ and η in place of ω as well.

We now set:

$$\gamma_{\mathrm{mod}} = \left(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J}\right).$$

By construction:

(1) $\varpi(\gamma_{\mathrm{mod}}) = 0$,
(2) $\varrho_\flat(\gamma_{\mathrm{mod}}) = \varrho(\gamma)$,
(3) $\varsigma(\gamma_{\mathrm{mod}}) = \varsigma(\gamma)$.

We are left to compute the modular reach of γ_{mod}.

Let $j,k \in J$. We write:

$$\omega_j = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \omega_k = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \text{ and } \eta_j = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \eta_k = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

We then have:

$$\begin{aligned}
\|\pi_{\mathfrak{B}}(a_j^* c_j)x - x\pi_{\mathfrak{B}}(b_j^* d_j)\|_{\mathfrak{D}} &\leqslant \|\pi_{\mathfrak{B}}(a_j^*)\pi_{\mathfrak{B}}(c_j)x - \pi_{\mathfrak{B}}(a_j^*)x\pi_{\mathfrak{B}}(d_j)\|_{\mathfrak{D}} \\
&\quad + \|\pi_{\mathfrak{B}}(a_j^*)x\pi_{\mathfrak{B}}(d_j) - x\pi_{\mathfrak{B}}(b_j)\pi_{\mathfrak{B}}(d_j)\|_{\mathfrak{D}} \\
&\leqslant \|a_j\|_{\mathfrak{A}} \|\pi_{\mathfrak{B}}(c_j)x - x\pi_{\mathfrak{B}}(d_j)\|_{\mathfrak{D}} \\
&\quad + \|\pi_{\mathfrak{B}}(a_j)x - x\pi_{\mathfrak{B}}(b_j)\|_{\mathfrak{D}} \|d_j\|_{\mathfrak{B}} \\
&= \mathsf{bn}_\gamma\left(c_j, d_j\right) + \mathsf{bn}_\gamma\left(a_j, b_j\right) \leqslant 2\lambda\left(\gamma\right) = 2\lambda.
\end{aligned}$$

Thus:

$$\begin{aligned}
&\left\| \pi_{\mathfrak{A}}\left(\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right\rangle_{\mathfrak{A}} \right) x - x\pi_{\mathfrak{B}}\left(\left\langle \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \right\rangle_{\mathfrak{B}} \right) \right\|_{\mathfrak{D}} \\
&\qquad = \left\| \sum_{j=1}^n \left(\pi_{\mathfrak{A}}(a_j c_j)x - \pi_{\mathfrak{B}}(b_j d_j) \right) \right\|_{\mathfrak{D}} \\
&\qquad \leqslant 2 \sum_{j=1}^n \lambda = 2n\lambda.
\end{aligned}$$

Thus the modular reach of γ_{mod} is no more than $2n\lambda$.

Therefore, the reach of γ_{mod} is $2n\lambda$ and thus so is its length. □

Theorem 2.6.2. *If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are F–quasi-Leibniz quantum compact metric spaces for some admissible function F, and if $n \in \mathbb{N} \setminus \{0\}$, then:*

$$\begin{aligned}
&\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \\
&\qquad \leqslant \Lambda^{\mathsf{mod}}\left((\mathfrak{A}^n, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{B}}) \right) \\
&\qquad\qquad \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})),
\end{aligned}$$

where, for any quasi-Leibniz quantum compact metric space $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$, we set:

(1) $\left\langle \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \right\rangle_{\mathfrak{A}} = \sum_{j=1}^n d_j e_j^*$ *for all* $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ *in* $\mathfrak{D}^n$,

(2) $\mathsf{D}^d_{\mathfrak{D}} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \max\left\{ \left\| \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \right\|_{\mathfrak{A}^n}, \mathsf{L}_{\mathfrak{D}}(\Re d_j), \mathsf{L}_{\mathfrak{D}}(\Im d_j) : j \in \{1, \ldots n\} \right\}$ *for all* $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \in \mathfrak{D}^n$.

Proof. Let Γ be a trek from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. Write:

$$\Gamma = (\mathfrak{A}_j, \mathsf{L}_j, \gamma^j, \mathfrak{A}_{j+1}, \mathsf{L}_{j+1})_{j\in\{1,\ldots,k}$$

for some bridges γ^j ($j \in \{1, \ldots, k\}$) and some $k \in \mathbb{N}$.

Now, for each $j \in \{1, \ldots, k\}$, let γ^j_{mod} be the modular bridge given by Lemma (2.6.1) applied to γ^j. It is then straightforward to check that $\Gamma_{\mathrm{mod}} = (\gamma^j_{\mathrm{mod}})_{j\in\{1,\ldots,k\}}$ is a modular trek from $\left(\mathfrak{A}^n, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}\right)$ to $\left(\mathfrak{B}^n, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{B}}\right)$ whose length satisfies:

$$\lambda(\Gamma) \leqslant \lambda(\Gamma_{\mathrm{mod}}) \leqslant 4\lambda(\Gamma).$$

We thus conclude, by definition, that:

$$\begin{aligned}\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) & \\ \leqslant \Lambda^{\mathrm{mod}}((\mathfrak{A}^n, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}^n_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}^n, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}^n_{\mathfrak{B}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{B}})) & \\ \leqslant 2n\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})). &\end{aligned}$$

This concludes our proof. ☐

A simple yet reassuring consequence of Theorem (2.6.2) is that we have not introduced any new topology on the class of quasi-Leibniz quantum compact metric spaces with the modular propinquity, via the canonical Hilbert module structure carried on by any C*-algebra.

Corollary 2.6.3. *For any two quasi-Leibniz quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$*, we have:*

$$\begin{aligned}\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) & \\ \leqslant \Lambda^{\mathrm{mod}}((\mathfrak{A}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}^1_{\mathfrak{A}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \langle\cdot,\cdot\rangle_{\mathfrak{B}}, \mathsf{D}^1_{\mathfrak{B}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) & \\ \leqslant 2\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})), &\end{aligned}$$

using the notations of Theorem (2.6.2).

Proof. This is the case $n = 1$ of Theorem (2.6.2). ☐

Free modules are the direct sums, in the sense of Hilbert modules, of the canonical module associated with a C*-algebra. The next section discuss the matter of the continuity of the direct sum between general metrized quantum vector bundles on certain well-behaved classes of quasi-Leibniz quantum compact metric spaces. We note that the D-norms constructed in this section, and the ones in the later section, differ in general: in this section, we constructed the D-norms from the underlying Lip-norms, while in the next section, we will be given D-norms on some modules and construct a new one on their direct sum.

2.7. **Iso-pivotal families and Direct sum of convergent modules.** Let $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ and $(\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}})$ be two left Hilbert $\mathfrak{A}$-module. The direct sum $\mathscr{M} \oplus \mathscr{N}$ is a left $\mathfrak{A}$-module in an obvious manner, and a canonical $\mathfrak{A}$-inner product on this direct sum is given by:

$$\langle(\omega, \eta), (\omega', \eta')\rangle_{\mathfrak{A}} = \langle\omega, \omega'\rangle_{\mathscr{M}} + \langle\eta, \eta'\rangle_{\mathscr{N}}$$

for all $\omega, \omega' \in \mathscr{M}$ and $\eta, \eta' \in \mathscr{N}$.

Let us now assume we are given two metrized quantum vector bundles $\Omega = (\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}}, \mathsf{D}_{\mathscr{M}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega' = (\mathscr{N}, \langle\cdot,\cdot\rangle_{\mathscr{N}}, \mathsf{D}_{\mathscr{N}}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$. For all $\omega \in \mathscr{M}$ and $\eta \in \mathscr{N}$, we set:

$$\mathsf{D}(\omega, \eta) = \max\{\mathsf{D}(\omega), \mathsf{D}(\eta)\}.$$

It is easy to check that $(\mathscr{M} \oplus \mathscr{N}, \langle\cdot,\cdot\rangle_{\mathfrak{A}}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a metrized quantum vector bundle as well. We will simply denote it by $\Omega \oplus \Omega'$.

We shall now prove a continuity result for direct sums of metrized quantum vector bundles, under a uniformity assumption. In general, the construction of the inner product on the direct sum of two modules mixes up, in the base algebra, the contributions of each module to the reach of a given bridge. This complication is, however, not expected to often occur in practice. When working with modules over quasi-Leibniz quantum compact metric spaces, we envisage that modular bridges will be constructed out of bridges between the base quantum metric spaces — indeed, this is what motivated our definition of the modular propinquity. Thus one may expect that the same bridge between the base quantum spaces may be reused for multiple modular bridges between different modules. This expectation is formalized in the following notion, which will serve as an hypothesis for our direct sum continuity result.

Definition 2.7.1. Let $\Omega_{j,k} = (\mathscr{M}_{j,k}, \langle\cdot,\cdot\rangle_{j,k}, \mathsf{D}_{j,k}, \mathfrak{A}_k, \mathsf{L}_k)$ be metrized quantum vector bundles for $j,k \in \{1,2\}$. The family $((\Omega_{1,1}, \Omega_{1,2}), (\Omega_{2,1}, \Omega_{2,2}))$ is *iso-pivotal* when for all $\varepsilon > 0$, there exist two modular treks Γ^1, from $\Omega_{1,1}$ to $\Omega_{1,2}$, and Γ^2, from $\Omega_{2,1}$ to $\Omega_{2,2}$, such that:

1. $\lambda\left(\Gamma^1\right) \leqslant \Lambda^{\mathsf{mod}}(\Omega_{1,1}, \Omega_{1,2}) + \varepsilon$,
2. $\lambda\left(\Gamma^2\right) \leqslant \Lambda^{\mathsf{mod}}(\Omega_{2,1}, \Omega_{2,2}) + \varepsilon$,
3. the basic treks $\Gamma^1_\flat$ and $\Gamma^2_\flat$ from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2)$ obtained from Γ^1 and Γ^2 are identical.

Informally, in an iso-pivotal family, one may find modular treks whose length is arbitrary close to the modular propinquity between each pair, and which differ only in the choice of the anchors and co-anchors. This notion can be extended in an obvious manner to classes of pairs of metrized quantum vector bundles over various base spaces.

With this concept, we have the following result:

Theorem 2.7.2. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. If $\Omega_{1,\mathfrak{A}}, \Omega_{2,\mathfrak{A}}$ are metrized quantum vector bundles over $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\Omega_{1,\mathfrak{B}}, \Omega_{2,\mathfrak{B}}$ are metrized quantum vector bundles over $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ such that $((\Omega_{1,\mathfrak{A}}, \Omega_{1,\mathfrak{B}}), (\Omega_{2,\mathfrak{A}}, \Omega_{2,\mathfrak{B}}))$ is iso-pivotal, then:*

$$\Lambda^{\mathsf{mod}}((\Omega_{1,\mathfrak{A}} \oplus \Omega_{2,\mathfrak{A}}), (\Omega_{1,\mathfrak{B}} \oplus \Omega_{2,\mathfrak{B}})) \leqslant \Lambda^{\mathsf{mod}}(\Omega_{1,\mathfrak{A}}, \Omega_{1,\mathfrak{B}}) + \Lambda^{\mathsf{mod}}(\Omega_{2,\mathfrak{A}}, \Omega_{2,\mathfrak{B}})).$$

Proof. We begin by setting our notations: let $\Omega_{j,k} = (\mathscr{M}_{j,k}, \langle\cdot,\cdot\rangle_{j,k}, \mathsf{D}_{j,k}, k)$ for $j \in \{1,2\}$ and $k \in \{(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\}$.

Let γ_1 be a modular bridge from $\Omega_{1,\mathfrak{A}}$ to $\Omega_{1,\mathfrak{B}}$ and γ_2 be a modular bridge from $\Omega_{2,\mathfrak{A}}$ to $\Omega_{2,\mathfrak{B}}$. We assume that γ_1 and γ_2 are given as:

$$\gamma_1 = (\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j \in J_1}, (\eta_j)_{j \in J_1})$$

and

$$\gamma_2 = (\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega'_j)_{j\in J_2}, (\eta'_j)_{j\in J_2}).$$

Of course, modular bridges between $\Omega_{1,\mathfrak{A}}$ and $\Omega_{1,\mathfrak{B}}$, and between $\Omega_{2,\mathfrak{A}}$ and $\Omega_{2,\mathfrak{B}}$, may not share basic bridge; however, we will conclude this theorem using the iso-pivotal hypothesis, and thus this choice of bridge will always be possible, and sufficient for our purpose.

We first show that we may as well assume $J_1 = J_2$. Pick $j_* \in J_1$ and $k_* \in J_2$. Set $J = J_1 \coprod J_2$. If $j \in J_1 \setminus J_2$, we set $\omega'_j = \omega_{k_*}$ and $\eta'_j = \eta_{k^*}$. If $j \in J_2 \setminus J_1$, we set $\omega_j = \omega_{j_*}$ and $\eta_j = \eta_{j_*}$.

With this procedure, we note that, for instance, $(\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j)_{j\in J}, (\eta_j)_{j\in J})$ has the same length and the same basic bridge as γ_1. The same holds for γ_2. Thus, without loss of generality, we let $J = J_1 = J_2$.

Let now:

$$\gamma_1 \vee \gamma_2 = (\mathfrak{D}, x, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, (\omega_j, \omega'_k)_{j,k\in J}, (\eta_j, \eta'_k)_{j,k\in J\}}).$$

Note that $\gamma_1 \vee \gamma_2$ has, once again, the same basic bridge as γ_1 and γ_2. It is thus straightforward that:

$$\varsigma(\gamma_1 \vee \gamma_2) = \varsigma(\gamma_1) = \varsigma(\gamma_2),$$

and:

$$\varrho_\flat(\gamma_1 \vee \gamma_2) = \varrho_\flat(\gamma_1) = \varrho_\flat(\gamma_2).$$

Let $\omega \in \mathscr{D}_1(\Omega_{1,\mathfrak{A}})$ and $\omega' \in \mathscr{D}_1(\Omega_{2,\mathfrak{A}})$. By Definition (2.3.15) of the imprint of a bridge, there exist $j,k \in \{1,\ldots,n\}$ such that $\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega,\omega_j) \leqslant \varpi(\gamma_1)$ and $\mathsf{k}_{\Omega_{\mathfrak{A}}}(\omega',\omega'_k) \leqslant \varpi(\gamma_2)$. Therefore:

$$\begin{aligned}
&\mathsf{k}_{\Omega_{1,\mathfrak{A}}\oplus\Omega_{2,\mathfrak{A}}}((\omega,\omega'),(\omega_j,\omega'_k)) \\
&\qquad = \sup\left\{ \|\langle(\omega,\omega') - (\omega_j,\omega_k),(\eta,\eta')\rangle_{\mathfrak{A}} : \mathsf{D}(\eta,\eta') \leqslant 1 \right\} \\
&\qquad \leqslant \sup\left\{ \|\langle\omega,\eta\rangle_{\mathscr{M}_{1,\mathfrak{A}}} - \langle\omega_j,\eta\rangle_{\mathscr{M}_{1,\mathfrak{A}}} : \mathsf{D}_{1,\mathfrak{A}}(\eta) \leqslant 1 \right\} \\
&\qquad\quad + \sup\left\{ \|\langle\omega',\eta'\rangle_{\mathscr{M}_{2,\mathfrak{A}}} - \langle\omega'_k,\eta'\rangle_{\mathscr{M}_{2,\mathfrak{A}}} : \mathsf{D}_{2,\mathfrak{A}}(\eta') \leqslant 1 \right\} \\
&\qquad \leqslant \mathsf{k}_{\Omega_{1,\mathfrak{A}}}(\omega,\omega_j) + \mathsf{k}_{\Omega_{2,\mathfrak{A}}}(\omega',\omega'_k)) \\
&\qquad \leqslant \varpi(\gamma) + \varpi(\gamma').
\end{aligned}$$

The same argument can be made in $\Omega_{2,\mathfrak{B}} \oplus \Omega_{2,\mathfrak{B}}$. Thus:

$$\varpi(\gamma_1 \vee \gamma_2) \leqslant \varpi(\gamma_1) + \varpi(\gamma_2).$$

Last, let $j,k \in J$. By Definition (2.3.14) of the modular reach, we have:

$$\left\| \langle\omega,\omega_j\rangle_{\mathscr{M}_{1,\mathfrak{A}}} x - x\langle\eta,\eta_j\rangle_{\mathscr{M}_{1,\mathfrak{B}}} \right\|_{\mathfrak{D}} \leqslant \varrho^\sharp(\gamma_1)$$

and

$$\left\| \langle\omega',\omega_k\rangle_{\mathscr{M}_{2,\mathfrak{A}}} x - x\langle\eta,\eta_k\rangle_{\mathscr{M}_{2.\mathfrak{B}}} \right\|_{\mathfrak{D}} \leqslant \varrho^\sharp(\gamma_2).$$

We thus compute:

$$\mathsf{dn}_{\gamma_1\vee\gamma_2}\left((\omega_j,\omega'_k),(\eta_j,\eta'_k)\right)$$

$$
\begin{aligned}
&= \max_{n,m\in J} \left\| \pi_{\mathfrak{A}}\left(\left\langle (\omega_j,\omega'_k),(\omega_n,\omega'_m)\right\rangle_{\mathfrak{A}}\right) x - x\pi_{\mathfrak{B}}\left(\left\langle (\eta_j,\eta'_k),(\eta_n,\eta'_m)\right\rangle_{\mathfrak{B}}\right)\right\|_{\mathfrak{D}} \\
&= \max_{n\in J} \left\| \pi_{\mathfrak{A}}\left(\left\langle \omega_j,\omega_n\right\rangle_{\mathscr{M}_{1,\mathfrak{A}}}\right) x - x\pi_{\mathfrak{B}}\left(\left\langle \eta_j,\eta_n\right\rangle_{\mathscr{M}_{1,\mathfrak{B}}}\right)\right\|_{\mathfrak{D}} \\
&\quad + \max_{m\in J} \left\| \pi_{\mathfrak{A}}\left(\left\langle \omega'_k,\omega'_m\right\rangle_{\mathscr{M}_{2,\mathfrak{A}}}\right) x - x\pi_{\mathfrak{B}}\left(\left\langle \eta'_k,\eta'_m\right\rangle_{\mathscr{M}_{2,\mathfrak{B}}}\right)\right\|_{\mathfrak{D}} \\
&\leqslant \mathsf{dn}_{\gamma_1}\left(()\omega_j,\eta_j\right) + \mathsf{dn}_{\gamma_2}\left(\omega_k,\eta_k\right) \\
&\leqslant \varrho^\sharp\left(\gamma_1\right) + \varrho^\sharp\left(\gamma_2\right).
\end{aligned}
$$

We have therefore proven:

$$\lambda\left(\gamma_1 \vee \gamma_2\right) \leqslant \lambda\left(\gamma_1\right) + \lambda\left(\gamma_2\right).$$

Now, let $\varepsilon > 0$. As we work with an iso-pivotal family, there exist two treks Γ^1, from $\Omega_{1,\mathfrak{A}}$ to $\Omega_{2,\mathfrak{B}}$, and Γ^2, from $\Omega_{2,\mathfrak{A}}$ to $\Omega_{2,\mathfrak{B}}$, such that at once:

(1) $\lambda\left(\Gamma^1\right) \leqslant \Lambda^{\mathsf{mod}}(\Omega_{1,\mathfrak{A}},\Omega_{1,\mathfrak{B}}) + \frac{\varepsilon}{2}$,
(2) $\lambda\left(\Gamma^2\right) \leqslant \Lambda^{\mathsf{mod}}(\Omega_{2,\mathfrak{A}},\Omega_{2,\mathfrak{B}}) + \frac{\varepsilon}{2}$,
(3) $\Gamma^1_\flat = \Gamma^2_\flat$.

We set $\Gamma \vee \Gamma' = (\gamma_1^j \vee \gamma_2^j)$; using our work in the first part of this proof and a trivial induction, we conclude that $\Gamma_1 \vee \Gamma_2$ is a modular trek from $\Omega_{1,\mathfrak{A}} \oplus \Omega_{2,\mathfrak{A}}$ to $\Omega_{1,\mathfrak{B}} \oplus \Omega_{2,\mathfrak{B}}$ such that:

$$\lambda\left(\Gamma_1 \vee \Gamma_2\right) \leqslant \lambda\left(\Gamma_1\right) + \lambda\left(\Gamma_2\right)\varepsilon \leqslant \Lambda^{\mathsf{mod}}(\Omega_{1,\mathfrak{A}},\Omega_{1,\mathfrak{B}}) + \Lambda^{\mathsf{mod}}(\Omega_{2,\mathfrak{A}},\Omega_{2,\mathfrak{B}})) + \varepsilon.$$

By Definition (2.4.6) of the modular propinquity, we thus conclude that for all $\varepsilon > 0$, the following holds:

$$\Lambda^{\mathsf{mod}}(\Omega_{1,\mathfrak{A}} \oplus \Omega_{2,\mathfrak{A}},\Omega_{1,\mathfrak{B}} \oplus \Omega_{2,\mathfrak{B}}) \leqslant \Lambda^{\mathsf{mod}}(\Omega_{1,\mathfrak{A}},\Omega_{1,\mathfrak{B}}) + \Lambda^{\mathsf{mod}}(\Omega_{2,\mathfrak{A}},\Omega_{2,\mathfrak{B}})) + \varepsilon.$$

Our theorem is now proven. □

We provide an extension of the concept of iso-pivotal class to classes of metrized quantum vector bundles, by slight abuse of terminology.

Definition 2.7.3. Let $\mathfrak{L}$ be a class of F–quasi-Leibniz quantum compact metric spaces. A class $\mathfrak{M}$ of (F,G,H)–metrized quantum vector bundles over elements of $\mathfrak{L}$ is *iso-pivotal* when, for any two $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ in $\mathcal{L}$, for every $\varepsilon > 0$, for any two $\Omega_{\mathfrak{A}},\Omega'_{\mathfrak{A}} \in \mathfrak{M}$ over $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ and $\Omega_{\mathfrak{B}},\Omega'_{\mathfrak{B}} \in \mathfrak{M}$ over $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$, the family $((\Omega_{\mathfrak{A}},\Omega_{\mathfrak{B}}),(\Omega'_{\mathfrak{A}},\Omega'_{\mathfrak{B}}))$ is iso-pivotal.

We thus can state:

Corollary 2.7.4. *The direct sum is continuous on any iso-pivotal class of metrized quantum vector bundles.*

Proof. This follows immediately from Theorem (2.7.2). □

To fully reflect the potential of the modular propinquity, we now move toward a much more involved example of convergence for modules. The next section deals with non-free, finitely generated projective modules over quantum 2-tori.

3. Heisenberg modules over the quantum 2-tori

Finitely generated projective modules over irrational rotation algebras can be described, up to module isomorphism, as either free — a case with which we dealt in the previous section — or constructed through a projective representation of $\mathbb{R}^2$, as shown in [37]. The latter type of modules were introduced by Connes [6] and provided the background for the study of noncommutative geometry of quantum tori. The construction of these modules was later extended by Rieffel to all quantum tori in [38], where they provide a large class of (though in general, not all) projective finitely generated modules over quantum tori. As projective representations of $\mathbb{R}^2$ are in fact obtained from representations of the Heisenberg group, the modules constructed from these representations are called *Heisenberg modules* in [38], and we shall follow this terminology. In fact, the Heisenberg group action will prove essential to our construction.

Heisenberg modules over quantum 2-tori provide natural examples of metrized quantum vector bundles, whose D-norms are constructed from a noncommutative connection built from the action of the Heisenberg group. In particular, they are naturally endowed with a Hilbert module structure, and a connection was constructed in [6] from the infinitesimal representation of the Heisenberg Lie algebra. These connections were proven to solve the Yang-Mills problem for quantum tori [9].

Once we bring Heisenberg modules within the realm of our modular propinquity, we become capable of discussing the problem of convergence of such modules. Heisenberg modules are parametrized by the quantum torus acting on them and by a pair of integers p, q which, in particular, relate to a projective representation of the finite group of the form $\mathbb{Z}_q^2$ where $\mathbb{Z}_q = \mathbb{Z} \big/ q\mathbb{Z}$. We shall prove in this section, as our main application of the modular propinquity for this paper, that for a fixed pair p, q of integers, and thus in particular, for a fixed projective representation of some $\mathbb{Z}_q$, the family of Heisenberg modules over varying quantum tori, form a continuous family for the modular propinquity.

We begin our section with some background on quantum 2-tori and Heisenberg modules. We then define our D-norms candidates, and establish their basic properties. The first difficulty we address is to prove that our D-norm candidates have compact balls in the $C^\ast$-Hilbert norm. Then, we prove that our D-norms form a continuous family of norms. This step involves proving that the norms of the Heisenberg modules also form a continuous field of norms.

We then use all these ingredients to prove our main result on the continuity of families of Heisenberg modules for the modular propinquity,

3.1. **Background on Quantum 2-tori and Heisenberg modules.**

Quantum 2-tori are the twisted convolution C*-algebras of $\mathbb{Z}^2$. The projective finitely generated modules over quantum tori have been extensively studied, and next to the free modules, the most important class of projective, finitely generated modules over a quantum torus are the Heisenberg modules. This subsection introduces these modules, as well as the notations we will use throughout this section regarding quantum tori.

Twisted group C*-algebras are defined by twisting the convolution product over a locally compact group by a representative of a continuous 2-cocycle of the group.

Notation 3.1.1. For any $\theta \in \mathbb{R}$, we define the skew bicharacter of $\mathbb{R}^2$:

$$\mathrm{e}_\theta : ((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto \exp\left(i\pi\theta(x_1 y_2 - x_2 y_1)\right). \tag{3.1.1}$$

By [20], any 2-cocycle of $\mathbb{Z}^2$ is cohomologous to the restriction of a skew bicharacter e_θ to $\mathbb{Z}^2 \times \mathbb{Z}^2$ for some $\theta \in \mathbb{R}$. We shall use the same notation for e_θ and its restriction to $\mathbb{Z}^2$.

Moreover, for any $\theta, \vartheta \in \mathbb{R}$, the skew bicharacters e_θ and e_ϑ of $\mathbb{Z}^2$ are cohomologous if and only if $\theta \equiv \vartheta \mod 1$. We note that, as skew bicharacters of $\mathbb{R}^2$, they are cohomologous if and only if $\theta = \vartheta$.

We define the twisted convolution products on $\ell^1(\mathbb{Z}^2)$, where we use the following notation.

Notation 3.1.2. For any (nonempty) set E and any $p \in [1, \infty)$, the set $\ell^p(E)$ is the set of all absolutely p-summable complex valued functions over E, endowed with the norm:

$$\|\xi\|_{\ell^p(E)} = \left(\sum_{x \in E} |\xi(x)|^p\right)^{\frac{1}{p}}$$

for all $\xi \in \ell^p(E)$.

We write δ_n the function which is 1 at n and 0 otherwise; this function is an element of $\ell^p(E)$ for all p.

Moreover, if $p = 2$ then $(\ell^2(E), \|\cdot\|_{\ell^2(E)})$ is a Hilbert space, where the inner product $\langle \xi, \eta \rangle_{\ell^2(E)} = \sum_{x \in E} \xi(x)\overline{\eta(x)}$ for all $\xi, \eta \in \ell^2(E)$.

We now define:

Definition 3.1.3. Let $\theta \in \mathbb{R}$ and e_θ be defined by Expression (3.1.1). The *twisted convolution product* $*_\theta$ is defined for all $f, g \in \ell^1(\mathbb{Z}^2)$ and for all $n \in \mathbb{Z}^2$:

$$f *_\theta g(n) = \sum_{m \in \mathbb{Z}^2} f(m) g(n - m) \mathrm{e}_\theta(m, n).$$

The *adjoint* of any $f \in \ell^1(\mathbb{Z}^2)$ is defined for all $n \in \mathbb{Z}^2$ by:

$$f^*(n) = \overline{f(-n)}.$$

One checks easily that $\left(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*\right)$ is a *-algebra. In particular, the adjoint operation is an isometry of $\left(\ell^1(\mathbb{Z}^2), \|\cdot\|_{\ell^1(\mathbb{Z}^2)}\right)$. We now wish to construct its enveloping C*-algebra. To do so, we shall choose a natural faithful *-representation of $\left(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*\right)$ on $\ell^2(\mathbb{Z}^2)$. This representation was a key ingredient in the construction of bridges between quantum tori in our work in [25] on convergence of quantum tori for the quantum propinquity, and thus will re-appear in a similar role in this section.

Notation 3.1.4. If $T : E \to F$ is a continuous linear map between two normed spaces, we write its norm as $|||T|||_F^E$. When $E = F$, we simply write $|||T|||_F$.

Theorem 3.1.5 ([51]). *Let $\theta \in \mathbb{R}$. We define, for any $n \in \mathbb{Z}^2$ and $\xi \in \ell^2(\mathbb{Z}^2)$, the function:*

$$U_\theta^n \xi : m \in \mathbb{Z}^2 \mapsto \mathrm{e}_\theta(m, n)\xi(m - n).$$

The map $n \in \mathbb{Z}^2 \mapsto U_\theta^n$ is a unitary e_θ-projective representation of $\mathbb{Z}^2$, i.e. $U_\theta^n U_\theta^m = \mathrm{e}_\theta(n, m)U_\theta^{n+m}$ for all $n, m \in \mathbb{Z}^2$.

If, for all $f \in \left(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*\right)$, we define:

$$\pi_\theta(f) = \sum_{n\in\mathbb{Z}^2} f(n)U_\theta^n$$

which is a bounded operator on $\ell^2(\mathbb{Z}^2)$ with:

$$|||\pi_\theta(f)|||_{\ell^2(\mathbb{Z}^2)} \leqslant \|f\|_{\ell^1(\mathbb{Z}^2)},$$

*then π_θ is a faithful *-representation of $(\ell^1(\mathbb{Z}^2), *_\theta, *)$.*

Proof. It is a standard computation to check that $n \in \mathbb{Z}^2 \mapsto U_\theta^n$ is a e_θ-projective unitary representation of $\mathbb{Z}^2$, and that for all $f, g \in \ell^1(\mathbb{Z}^2)$:

$$\pi_\theta(f *_\theta g) = \pi_\theta(f)\pi_\theta(g),$$

and $\pi_\theta(f^*) = (\pi_\theta(f))^*$. As proven in [51], this representation is also faithful. ☐

Thus, we may define a C*-norm on $\ell^1(\mathbb{Z}^2)$ by setting:

$$\|f\|_{\mathcal{A}_\theta} = |||\pi_\theta(f)|||_{\ell^2(\mathbb{Z}^2)}$$

for all $f \in \ell^1(\mathbb{Z}^2)$. We thus can define quantum 2-tori.

Definition 3.1.6. The *quantum* 2*-torus* $\mathcal{A}_\theta$ is the completion of $\left(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*\right)$ for the norm $|||\pi_\theta(\cdot)|||_{\ell^2(\mathbb{Z}^2)}$.

As per our general convention, the norm on $\mathcal{A}_\theta$ is denoted by $\|\cdot\|_{\mathcal{A}_\theta}$ for all $\theta \in \mathbb{R}$.

Remark 3.1.7. Let $\theta \in \mathbb{R}$. By construction, $\ell^1(\mathbb{Z}^2)$ is identified with a dense *-subalgebra of $\mathcal{A}_\theta$, and we shall employ this identification all throughout this paper. With this identification, we also note that for all $f \in \ell^1(\mathbb{Z}^2)$ we have $\|f\|_{\mathcal{A}_\theta} \leqslant \|f\|_{\ell^1(\mathbb{Z}^2)}$, a fact which we will use repeatedly in the next section.

We take one derogation to the convention of using the same symbol for an element of $\ell^1(\mathbb{Z}^2)$ and its counter part in a given quantum torus, because the following notation is at once common and convenient.

Notation 3.1.8. Let $\theta \in \mathbb{R}$. The element $\delta_{1,0}$ is denoted by u_θ and the element $\delta_{0,1}$ is denoted by v_θ when regarded as elements of $\mathcal{A}_\theta$.

We now introduce a canonical action of $\mathbb{T}^2$ on quantum 2-tori. The dual action of $\mathbb{T}^2$ on quantum 2-tori provides, by transport of structure, the geometry of the quantum 2-tori. We shall discuss this matter at greater length in our next section. Our focus, of course, will be on the metric structure of quantum tori induced by the dual action and continuous length functions on $\mathbb{T}^2$, as first constructed in [40] as the prototype for compact quantum metric spaces.

Theorem-Definition 3.1.9. [51] *For all $z = (z_1, z_2) \in \mathbb{T}^2$ there exists a unique *-automorphism β^z_θ of $\mathcal{A}_\theta$ such that, for any $f \in \ell^1(\mathbb{Z}^2)$ and $(n,m) \in \mathbb{Z}^2$, we have:*

$$\beta^z_\theta f(n,m) = z_1^n z_2^m f(n,m).$$

The map $z \in \mathbb{T}^2 \mapsto \beta^z_\theta$ is a strongly continuous action of $\mathbb{T}^2$ on $\mathcal{A}_\theta$ called the dual action*. Moreover:*

$$\left\{ a \in \mathcal{A}_\theta : \forall z \in \mathbb{T}^2 \quad \beta^z(a) = a \right\} = \mathbb{C}1_{\mathcal{A}_\theta}.$$

An action of a group on a unital C*-algebra whose fixed point algebra is reduced to the scalar multiples of the unit, such as the dual action of $\mathbb{T}^2$ on $\mathcal{A}_\theta$, is called *ergodic*. In [6], this dual action, combined with the fact that $\mathbb{T}^2$ is a Lie group, was used to define a quantized differential calculus on quantum tori [8] which proved to be the start of noncommutative geometry. In [40], the dual action, combined with a choice of a continuous length function on $\mathbb{T}^2$, provided the first example of an L-seminorm and started the program within which the current paper participate.

We now turn to the class of modules to which we shall apply our new modular propinquity. We construct these modules following [6] using the universal property of quantum 2-tori, which we now recall.

Proposition 3.1.10 ([51])**.** *Let $\theta \in \mathbb{R}$. If U, V are two unitary operators on some Hilbert space $\mathscr{H}$ such that $UV = \exp(2i\pi\theta)VU$ for some $\theta \in [0,1)$, then there exists a *-morphism $\varpi : \mathcal{A}_\theta \to \mathfrak{B}(\mathscr{H})$ such that $\varpi(u_\theta) = U$ and $\varpi_\theta(v_\theta) = V$. The range of ϖ is $C^*(U,V)$.*

We note that one may construct an enveloping C*-algebra of $(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*)$ with the universal property described in Proposition (3.1.10) by choosing as a C*-norm of some element f, the supremum of the norm of $\pi(f)$ where π ranges over *all* *-representations of $(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*)$. The completion of $(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*)$ with this norm would be the *full* twisted convolution C*-algebra of $\mathbb{Z}^2$ for e_θ. However, as $\mathbb{Z}^2$ is Abelian, it is an amenable group, which implies that the full norm is in fact equal to the C*-norm obtained from our special representation π_θ. Thus amenability is in essence behind Proposition (3.1.10).

Another way to state Proposition (3.1.10) is that, for any $\theta \in \mathbb{R}$, if ς is some projective representation of $\mathbb{Z}^2$ on some Hilbert space $\mathscr{H}$ for some multiplier of $\mathbb{Z}^2$ cohomologous to e_θ, then $\mathscr{H}$ is a module over $\mathcal{A}_\theta$. Indeed, Proposition (3.1.10) gives us a *-morphism ϖ from $\mathcal{A}_\theta$ to the C*-algebra $\mathfrak{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$, with $\varpi(u_\theta) = \varsigma^{1,0}$ and $\varpi(v_\theta) = \varsigma^{0,1}$. Thus $\mathscr{H}$ is a $\mathcal{A}_\theta$ module.

With this observation in mind, we now turn to the construction of some particular projective representations of $\mathbb{Z}^2$. The idea, found in [6] and explicited in [38], is to take the tensor product of a projective representation of $\mathbb{R}^2$, restricted to $\mathbb{Z}^2$, and a finite dimensional projective representation of $\mathbb{Z}_q$ for some $q \in \mathbb{N} \setminus \{0\}$. By adjusting the choice of the multipliers associated with each projective representation, we get the desired module structure.

Projective representations of $\mathbb{R}^2$ are naturally related to the representations of the Heisenberg group, and we will make important use of this fact in our work. We thus begin with setting our notations for the Heisenberg group.

Convention 3.1.11. The vector space $\mathbb{C}^d$ is endowed by default with its standard inner product $\langle (z_1,\ldots,z_d),(y_1,\ldots,y_d)\rangle_{\mathbb{C}^d} = \sum_{j=1}^d z_j\overline{y_j}$, whose associated norm is denoted by $\|\cdot\|_{\mathbb{C}^d}$.

Notation 3.1.12. The Heisenberg group is the Lie group given by:

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,u \in \mathbb{R} \right\}.$$

We shall identify $\mathbb{H}_3$ with $\mathbb{R}^3$ via the natural map $(x,y,u) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, which is a Lie group isomorphism once we equip $\mathbb{R}^3$ with the multiplication:

$$(x_1,y_1,u_1)(x_2,y_2,u_2) = (x_1+x_2, y_1+y_2, u_1+u_2+x_1y_2)$$

for all $(x_1,y_1,u_1),(x_2,y_2,u_2) \in \mathbb{R}^3$.

The importance of the Heisenberg group for quantum mechanics [10] may be gleaned by looking at its Lie algebra, which is given by:

$$\mathbb{h} = \left\{ \begin{pmatrix} 0 & x & u \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x,y,u \in \mathbb{R} \right\}$$

which is a 2-nilpotent Lie algebra. We easily compute that for all $x,y,u \in \mathbb{R}^3$:

$$\exp \begin{pmatrix} 0 & x & u \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & u+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1.2}$$

This expression for the exponential will be important for our construction. Note that the exponential map is both injective and surjective.

We now set:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We easily check that $[P,Q] = T = -[Q,P]$ while other other commutators between P, Q and T are null, and $\mathrm{span}_{\mathbb{C}}\{P,Q,T\} = \mathbb{h}$.

We note that in particular, T is central, and thus the relations defining $\mathbb{h}$ from the basis $\{P,Q,T\}$ are the structural equations of quantum mechanics — the canonical commutation relation, as proposed by Heisenberg, in order to express the uncertainty principle between two conjugate observables. We refer to [10] for a detailed analysis of the Heisenberg group and its connections to the Moyal product, pseudo-differential calculus, and more fascinating topics.

Thus the study of the irreducible representations of $\mathbb{H}_3$ provide the irreducible representations of the canonical commutation relations. We first note that:

$$\mathbb{H}_3 \Big/ \{(0,0,u) : u \in \mathbb{R}\} = \mathbb{R}^2$$

is Abelian, and thus we get a collection of trivial, one-dimensional representations of $\mathbb{H}_3$ by simply lifting the irreducible representations of $\mathbb{R}^2$.

The nontrivial irreducible representations of the Heisenberg group are, up to unitary equivalence, given by one of the following:

$$\alpha_{\eth,1}^{x,y,u}\xi : s \in \mathbb{R} \mapsto \exp(2i\pi(\eth u + sx))\xi(s + \eth y) \tag{3.1.3}$$

form some $\eth \in \mathbb{R} \setminus \{0\}$. We note that they all are infinite dimensional.

Let $\eth \in \mathbb{R} \setminus \{0\}$. For all $(x, y) \in \mathbb{R}^2$ and for all $\xi \in L^2(\mathbb{R})$, set:

$$\begin{aligned}\sigma_{\eth,1}^{x,y}\xi &= \alpha_{\eth,1}^{\exp_{\mathbb{H}_3}(xP+yQ)}\xi \\ &= \alpha_{\eth,1}^{x,y,\frac{xy}{2}}\xi : s \in \mathbb{R} \mapsto \exp(i\pi\eth xy + 2i\pi sx)\xi(s + \eth y).\end{aligned}$$

The map $\sigma_{\eth,1}^{x,y}$ is a unitary on $L^2(\mathbb{R})$ for all $(x, y) \in \mathbb{R}^2$. Moreover, for all (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$, we note that:

$$\sigma_{\eth,1}^{x_1,y_1}\sigma_{\eth,1}^{x_2,y_2} = \mathrm{e}_{\eth}((x_1, y_1), (x_2, y_2))\sigma_{\eth,1}^{x_1+x_2,y_1+y_2},$$

i.e. $\sigma_{\eth,1}$ is a projective representation of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ for the bicharacter $\mathrm{e}_{\eth}$, namely the *Schrödinger representation* of "Plank constant" $\eth$. Moreover, every nontrivial irreducible unitary projective representation of $\mathbb{R}^2$ is unitarily equivalent to one of $\sigma_{1,\eth}$ for some $\eth \neq 0$ (by nontrivial, we mean associated with a nontrivial cocycle).

We introduce one more notation which will prove very useful in defining our D-norm on Heisenberg modules. If $d \in \mathbb{N}$ with $d > 0$, we define the following unitarry operators on $L^2(\mathbb{R}) \otimes \mathbb{C}^d$:

$$\alpha_{\eth,d}^{x,y,u} = \alpha_{\eth,1}^{x,y,u} \otimes \mathrm{id} \text{ and } \sigma_{\eth,d}^{x,y} = \sigma_{\eth,1}^{x,y} \otimes \mathrm{id}$$

for all $x, y, u \in \mathbb{R}$, where id is the identity map on $\mathbb{C}^d$. We trivially check that $\alpha_{\eth,d}$ is a unitary representation of $\mathbb{H}_3$ on $L^2(\mathbb{R}) \otimes \mathbb{C}^d$, while $\sigma_{\eth,d}$ is a $\mathrm{e}_{\eth}$-projective representation of $\mathbb{R}^2$ on $L^2(\mathbb{R}) \otimes \mathbb{C}^d$. Moreover, we also check immediately that $\alpha_{\eth,d}^{x,y,0} = \sigma_{\eth,d}^{x,y}$ for all $x, y \in \mathbb{R}$.

We now turn to the projective representations of $\mathbb{Z}_q^2$, where $q \in \mathbb{N} \setminus \{0\}$. We first note that, for any $p \in \mathbb{Z}$, the skew bicharacter $\mathrm{e}_{\frac{p}{q}}$ of $\mathbb{Z}^2$ induces a skew bicharacter of $\mathbb{Z}_q^2$ — which we keep denoting by $\mathrm{e}_{\frac{p}{q}}$. By [20], any multiplier of $\mathbb{Z}_q^2$ is cohomologous to $\mathrm{e}_{\frac{p}{q}}$ for some $p \in \mathbb{N}$.

For our purpose, we will thus get, up to unitary equivalence, every possible finite dimensional unitary projective representations of the groups $\mathbb{Z}_q^2$ for arbitrary $q \in \mathbb{N} \setminus \{0\}$ by considering the following family.

Notation 3.1.13. Let $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$. Let $n \in \mathbb{Z} \mapsto [n] \in \mathbb{Z}_q$ be the canonical surjection. Let:

$$u_{p,q} = \begin{pmatrix} 0 & \dots & & & 1 \\ 1 & 0 & \dots & & \\ \ddots & \ddots & & & \\ & & & 1 & 0 \end{pmatrix} \text{ and } v_{p,q} = \begin{pmatrix} 1 & & & & \\ & z & & & \\ & & z^2 & & \\ & & & \ddots & \\ & & & & z^{q-1} \end{pmatrix},$$

with $z = \exp\left(\frac{2i\pi p}{q}\right)$. Since $u_{p,q}^q = v_{p,q}^q = 1$, the map:

$$\rho_{p,q,1} : (z,w) \in \mathbb{Z}_q^2 \mapsto \rho_{p,q,1}^{z,w} = u_{p,q}^n v_{p,q}^m \text{ where } [n] = z \text{ and } [m] = w$$

is well-defined. An easy computation shows that $\rho_{p,q,1}$ is a projective action of $\mathbb{Z}_q^2$.

For all $d \in q\mathbb{N}$, $d > 0$, we now set:

$$\rho_{p,q,d}^{n,m} = \rho_{p,q,1}^{n,m} \otimes \mathrm{id}_{\frac{d}{q}}$$

where $\mathrm{id}_{\frac{d}{q}}$ is the identity map on $\mathbb{C}^{\frac{d}{q}}$.

We remark that $\rho_{p,q,d}$ acts on $\mathbb{C}^d$, i.e. we parametrized ρ by the dimension of the space on which it acts rather than the multiplicity of $\rho_{p,q,1}$, as it will make our notations much simpler.

If p and q are relatively prime, the representation $\rho_{p,q,1}$ is irreducible, with range the entire algebra of $q \times q$ matrices — it is in fact, the only irreducible $\mathrm{e}_{\frac{p}{q}}$-projective representation of $\mathbb{Z}_q^2$ up to unitary equivalence. Thus in general, any finite dimensional unitary representation of $\mathbb{Z}_q^2$ is unitarily equivalent to some $\rho_{l,r,d}$ for some $l \in \mathbb{Z}$, $r \in \mathbb{N} \setminus \{0\}$, $d \in r\mathbb{N} \setminus \{0\}$, with $l = 0$ and $r = 1$ or l, r relatively prime.

In order to construct the inner product on the Heisenberg modules, we shall need to first work on a space of well-behaved functions inside the Hilbert space $\ell^2(\mathbb{Z}^2)$ on which quantum tori will act. This space will consist of the Schwarz functions.

Definition 3.1.14. Let E be a finite dimensional vector space. A function $f : \mathbb{R} \to E$ is a *E-valued Schwarz function over* $\mathbb{R}$ when it is infinitely differentiable on $\mathbb{R}$ and, for all $j \in \mathbb{N}$ and all polynomial $p \in \mathbb{R}[X]$, we have:

$$\lim_{t \to \pm\infty} \left\| p(t) f^{(j)}(t) \right\|_E = 0.$$

The space of all E-valued Schwarz functions over $\mathbb{R}$ is denoted by $\mathcal{S}(E)$.

We note that if $f \in \mathcal{S}(E)$ for some finite dimensional space E, then in particular, $f \in L^p(\mathbb{R})$ for all $p \in [1, \infty]$, since for any $j \in \mathbb{N}$, there exists $M > 0$ such that $\|f(s)\|_E \leqslant \frac{M}{1+|s|^j}$ for all $s \in \mathbb{R}$. Indeed, by assumption, $\lim_{s \to \pm\infty} \|(1+s^j) f(s)\|_E = 0$. Thus for some $K > 0$ we have $\|f(s)\| \leqslant \frac{1}{1+|s|^j}$ for all $s \in \mathbb{R}$ with $|s| > K$. On the other hand, since f is continuous on $[-K, K]$, it is a bounded. Let $M = \max\{|f(s)| : s \in [-K, K]\}$. As $s \in [-K, K] \mapsto \frac{1}{1+|s|^j}$ is continuous as well as strictly positive, it is bounded below by some $m > 0$. Thus $\|f(s)\|_E \leqslant \frac{\max\{1, Mm^{-1}\}}{1+|s|^j}$ for all $s \in \mathbb{R}$.

We now implement the scheme which we described a few paragraphs above to construct modules over quantum tori. We refer to the mentioned works of Connes and Rieffel for the details and justification behind the following construction.

Theorem-Definition 3.1.15 ([6], [37], [9]). *Let $\theta \in \mathbb{R}$ and $q \in \mathbb{N} \setminus \{0\}$. Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$, and let $d \in q\mathbb{N} \setminus \{0\}$. The* Heisenberg module $\mathscr{H}_\theta^{p,q,d}$ *is the module over $\mathcal{A}_\theta$ defined as follows.*

Let $\rho_{p,q,d}$ be the projective action of $\mathbb{Z}_q^2$ with cocycle $\mathrm{e}_{\frac{p}{q}}$, consisting of the sum of $\frac{d}{q}$ copies of the unique, up to unitary equivalence, irreducible representation with the same cocycle. Up to unitary conjugation, we assume that $\rho_{p,q,d}$ acts on $\mathbb{C}^d$.

Let:

$$\eth = \theta - \frac{p}{q}.$$

Let $\alpha_{\eth,1}$ be the action of the Heisenberg group $\mathbb{H}_3$ on $L^2(\mathbb{R})$ given by Expression (3.1.3).

For $(n,m) \in \mathbb{Z}^2$, denoting the class of n and m in $\mathbb{Z} \big/ q\mathbb{Z}$, respectively, by $[n]$ and $[m]$, we set:

$$\varpi^{n,m}_{p,q,\eth,d} = \alpha^{n,m,0}_{\eth,1} \otimes \rho^{[n],[m]}_{p,q}.$$

For all $n,m \in \mathbb{Z}$, the map $\varpi^{n,m}_{p,q,\eth,d}$ is a unitary of $L^2(\mathbb{R}) \otimes \mathbb{C}^d$, and moreover $\varpi_{p,q,\eth,d}$ is an e_θ-projective representation of $\mathbb{Z}^2$.

By universality, the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^d$ is a module over $\mathcal{A}_\theta$, with, in particular, for all $f \in \ell^1(\mathbb{Z}^2)$ and $\xi \in L^2(\mathbb{R},\mathbb{C}^d) = L^2(\mathbb{R}) \otimes \mathbb{C}^d$:

$$f\xi = \sum_{n,m \in \mathbb{Z}} f(n,m)\varpi^{n,m}_{p,q,\eth,d}\xi.$$

Let $\mathscr{S}^{p,q,d}_\theta = \mathcal{S}(\mathbb{C}^d) \subseteq L^2(\mathbb{R}) \otimes \mathbb{C}^d$. For all $\xi,\omega \in \mathscr{S}^{p,q,d}_\theta$, define $\langle \xi,\omega \rangle_{\mathscr{H}^{p,q,d}_\theta}$ as the function in $\ell^1(\mathbb{Z}^2)$ given by:

$$\langle \xi,\omega \rangle_{\mathscr{H}^{p,q,d}_\theta} : (n,m) \in \mathbb{Z}^2 \longmapsto \left\langle \varpi^{n,m}_{p,q,\eth,d}\xi, \omega \right\rangle_{L^2(\mathbb{R})\otimes E}.$$

The Heisenberg module *$\mathscr{H}^{p,q,d}_\theta$ is the completion of $\mathscr{S}^{p,q,d}_\theta$ for the norm associated with the $\mathcal{A}_\theta$-inner product $\langle \cdot,\cdot \rangle_{\mathscr{H}^{p,q,d}_\theta}$.*

We note that $\mathscr{S}^{p,q,d}_\theta$ is not closed under the action of $\mathcal{A}_\theta$ but it is closed under the action of the subalgebra:

$$\{ f \in \ell^1(\mathbb{Z}^2) : \forall p \in \mathbb{R}[X,Y] \quad \lim_{n,m \to \pm\infty} p(n,m) f(n,m) = 0 \}$$

of $(\ell^1(\mathbb{Z}^2), *_\theta, \cdot^*)$, often referred to as the smooth quantum torus. We will not use this observation later on, though it is notable that the completion of $\mathscr{S}^{p,q,d}_\theta$ is indeed a $\mathcal{A}_\theta$-module.

We make some formulas explicit for clarity. We continue with the notations in Theorem-Definition (3.1.15). We let:

$$W_1 = \rho^{1,0}_{p,q,d} \text{ and } W_2 = \rho^{0,1}_{p,q,d}.$$

We thus have, in particular:

$$W_1 W_2 = \exp\left(\frac{2ip\pi}{q}\right) W_2 W_1.$$

We also denote by u_θ and v_θ the elements $\delta_{1,0}$ and $\delta_{0,1}$ of $\mathcal{A}_\theta$.

We then note that for all $\xi \in \mathscr{S}^\rho_\theta$:

$$u_\theta \xi : s \in \mathbb{R} \mapsto W_1 \xi(s + \eth)$$

and

$$v_\theta\xi : s \in \mathbb{R} \mapsto W_2 \exp(2i\pi s)\xi(s).$$

We observe that indeed, $\langle \cdot, \cdot \rangle_{\mathscr{H}_\theta^{p,q,d}}$ is hermitian. Indeed for all $n, m \in \mathbb{Z}^2$:

$$\begin{aligned}
\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}}(n,m) &= \left\langle \varpi_{p,q,\eth,d}^{n,m}\xi, \omega \right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \\
&= \left\langle \xi, \varpi_{p,q,\eth,d}^{-n,-m}\omega \right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \\
&= \overline{\langle \omega, \xi \rangle_{\mathscr{H}_\theta^{p,q,d}}}(-n,-m) \\
&= \left(\langle \omega, \xi \rangle_{\mathscr{H}_\theta^{p,q,d}} \right)^* .
\end{aligned} \tag{3.1.4}$$

Moreover, for all $n, m \in \mathbb{Z}$ and $\xi, \omega \in \mathscr{S}_\theta^\rho$, we have:

$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}}(n,m) = \int_{\mathbb{R}} \langle W_1^n W_2^m \xi(s + \eth m), \omega(s) \rangle_{\mathbb{C}^d} \exp(2i\pi n s)\, ds.$$

Now, $\mathcal{A}_\theta$ carries a unique tracial state tr, given for all $f \in \ell^1(\mathbb{Z}^2)$ by $\mathrm{tr}(f) = f(0)$. In particular:

$$\mathrm{tr}\left(\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} \right) = \int_{\mathbb{R}} \langle \xi(s), \omega(s) \rangle_{\mathbb{C}^d}\, ds,$$

so $\|\xi\|_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \leqslant \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}$. In particular, $\mathscr{H}_\theta^{p,q,d}$ is a dense subspace of $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

Last, we note that if $\rho_{p,q,d} = \rho_{l,r,m}$ for some $p, l \in \mathbb{Z}$, $q, r \in \mathbb{N} \setminus \{0\}$, $d \in q\mathbb{N}$ and $m \in r\mathbb{N}$, with $d, m > 0$, we may not conclude that $p = l$, $q = r$ and $d = m$ unless we assume that p, q are relatively prime and l, r are relatively prime, or $p = l = 0$ and $q = r = 1$. Consequently, p, q, d is not uniquely determined by the isomorphism class of $\mathscr{H}_\theta^{p,q,d}$ unless we assume relative primality of p and q. This will not be an issue for our work.

In this section, we introduced many representations of various groups, and the notations will be important all throughout this paper. We believe it may be convenient for the reader if we summarize these notations.

Notation 3.1.16. The following table summarizes all the group (projective) representations and related objects which will be used in this work.

Notation	Parameter(s)	Meaning
e_θ	$\theta \in \mathbb{R}$	the skew bicharacter defined by Expr. (3.1.1) on $\mathbb{R}^2$, identified with its quotient map on $\mathbb{Z}^2$, and even $\mathbb{Z}_q^2$ when $q\theta \in \mathbb{Z}$
$\rho_{p,q,d}$	$p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}$; $d \in q\mathbb{N} \setminus \{0\}$	direct sum of $\frac{d}{q} \in \mathbb{N}$ copies of the $\mathrm{e}_{\frac{p}{q}}$-projective representation $\rho_{p,q,1}$ of $\mathbb{Z}_q^2$ given in Notation (3.1.13)
$\alpha_{\eth,1}$	$\eth \in \mathbb{R} \setminus \{0\}$	$\alpha_{\eth,1}^{x,y,u}\xi : s \mapsto \exp(2i\pi(\eth u + xs))\xi(s + \eth y)$ for all $\xi \in \mathcal{S}(\mathbb{C})$ and $(x,y,u) \in \mathbb{H}_3$, see Notation (3.1.12)
$\alpha_{\eth,d}$	$\eth \in \mathbb{R} \setminus \{0\}$ and $d \in \mathbb{N}, d > 0$	$\alpha_{\eth,1} \otimes \mathrm{id}$ where id is the identity of $\mathbb{C}^d$, see Notation (3.1.12)
$\sigma_{\eth,d}$	$\eth \in \mathbb{R} \setminus \{0\}$ and $d \in \mathbb{N}, d > 0$	$\sigma_{\eth,d}^{x,y} = \alpha_{\eth,d}^{x,y,\frac{xy}{2}}$ for all $(x,y) \in \mathbb{R}^2$, see Notation (3.1.12)
$\varpi_{p,q,\eth,d}$	$p,q \in \mathbb{N}$; $d \in q\mathbb{N} > 0$, $\eth \in \mathbb{R} \setminus \{0\}$	$\sigma_{\eth,1} \otimes \rho_{p,q,d}$ as in Theorem (3.1.15)
β_θ	$\theta \in \mathbb{R}$	the dual action of $\mathbb{T}^2$ on $\mathcal{A}_\theta = C^*(\mathbb{Z}^2, \mathrm{e}_\theta)$ as in Theorem (3.1.9)
$U_\theta : n,m \mapsto U_\theta^{n,m}$	$\theta \in \mathbb{R}$	the e_θ-projective representation of $\mathbb{Z}^2$ on $\ell^2(\mathbb{Z}^2)$ defined in Theorem (3.1.5)
π_θ	$\theta \in \mathbb{R}$	the representation of $\mathcal{A}_\theta$ on $\ell^2(\mathbb{Z}^2)$ extending the integrated version of U_θ as in Theorem (3.1.5)

The purpose of this section is to show that for a fixed choice of $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$, and $d \in q\mathbb{N}$, $d > 0$, the function $\theta \in \mathbb{R} \mapsto \mathscr{H}_\theta^{p,q,d}$ is continuous for the modular propinquity. A first step in this direction is to check that the collection of Heisenberg modules form a continuous family of normed spaces.

3.2. **A continuous fields of C^*-Hilbert norms.** All Heisenberg modules are completions of $\mathcal{S}(\mathbb{C}^d)$ for some $d \in \mathbb{N}$, $d > 0$. For a fixed d, it thus becomes possible to ask whether the various C^*-Hilbert norms $\|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}$, as θ varies in $\mathbb{R}$, form a continuous family.

To this end, we establish a succession of lemmas whose primary goal is to provide us with estimates on the Heisenberg modules' C^*-Hilbert norms in terms of the norm of $\ell^1(\mathbb{Z}^2)$. While the Heisenberg modules' C^*-Hilbert norms are in general delicate to work with as they involve the no-less abstract quantum tori norms, the $\ell^1(\mathbb{Z}^2)$ norm, which dominates all of the quantum tori norms, is much more amenable to computations. For our purpose, we will take full advantage of the regularity of Schwarz functions, which will enable us to apply various analytic tools to derive our desired result.

The first step is a lemma which provides a first upper bound to the $\ell^1(\mathbb{Z}^2)$ norm of the difference between certain Heisenberg module inner products.

Lemma 3.2.1. *If $\theta, \vartheta \in \mathbb{R}$ and $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$, $d \in q\mathbb{N} \setminus \{0\}$, and if ω, η and ξ are C^2 functions from $\mathbb{R}$ to $\mathbb{C}^d$ such that for all $f \in \{\omega, \eta, \xi\}$:*

(1) all of f, f' and f'' are integrable on $\mathbb{R}$,
(2) $\lim_{t\to\pm\infty} f(t) = \lim_{t\to\pm\infty} f'(t) = \lim_{t\to\pm\infty} f''(t) = 0$,

then, writing $\eth_\theta = \theta - \frac{p}{q}$ and $\eth_\vartheta = \vartheta - \frac{p}{q}$, we have:

$$\begin{aligned}
&\left\| \langle \omega, \eta \rangle_{\mathscr{H}_\theta^{p,q,d}} - \langle \xi, \eta \rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right\|_{\ell^1(\mathbb{Z}^2)} \\
&\quad \leqslant \sum_{n\in\mathbb{Z}} \frac{1}{4\pi^2 n^2} \left(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \left\| \omega''(t+\eth_\theta m) - \xi''(t+\eth_\vartheta m) \right\|_{\mathbb{C}^d} \|\eta(t)\|_{\mathbb{C}^d}\, dt + \right. \\
&\qquad\quad + \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \left\| \omega'(t+\eth_\theta m) - \xi'(t+\eth_\vartheta m) \right\|_{\mathbb{C}^d} \|\eta'(t)\|_{\mathbb{C}^d}\, dt \\
&\qquad\qquad \left. + \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \|\omega(t+\eth_\theta m) - \xi(t+\eth_\vartheta m)\|_{\mathbb{C}^d} \|\eta''(t)\|_{\mathbb{C}^d}\, dt \right).
\end{aligned}$$

Proof. We begin with the observation that for all $(n,m) \in \mathbb{Z}^2$ we have:

$$\begin{aligned}
&\langle \omega, \eta \rangle_{\mathscr{H}_\theta^{p,q,d}}(n,m) - \langle \xi, \eta \rangle_{\mathscr{H}_\vartheta^{p,q,d}}(n,m) \\
&= \int_{\mathbb{R}} \left\langle \rho_{p,q,d}^{[n],[m]} \omega(t+\eth_\theta m), \eta(t) \right\rangle_{\mathbb{C}^d} \exp(2i\pi n t)\, dt \\
&\quad - \int_{\mathbb{R}} \left\langle \rho_{p,q,d}^{[n],[m]} \xi(t+\eth_\vartheta m), \eta(t) \right\rangle_{\mathbb{C}^d} \exp(2i\pi n t)\, dt \\
&= \int_{\mathbb{R}} \left(\left\langle \rho_{p,q,d}^{[n],[m]} \omega(t+\eth_\theta m), \eta(t) \right\rangle_{\mathbb{C}^d} - \left\langle \rho_{p,q,d}^{[n],[m]} \xi(t+\eth_\vartheta m), \eta(t) \right\rangle_{\mathbb{C}^d} \right) \exp(2i\pi n t)\, dt \\
&= \int_{\mathbb{R}} \left\langle \rho_{p,q,d}^{[n],[m]} \left(\omega(t+\eth_\theta m) - \xi(t+\eth_\vartheta m)\right), \eta(t) \right\rangle_{\mathbb{C}^d} \exp(2i\pi n t)\, dt.
\end{aligned}$$

For all $n, m \in \mathbb{Z}$, the function:

$$f_{n,m} : t \mapsto \left\langle \rho_{p,q,d}^{[n],[m]} \omega(t+\eth_\theta m) - \xi(t+\eth_\vartheta m), \eta(t) \right\rangle_{\mathbb{C}^d}$$

has a first and continuous second derivative which are integrable, and:

$$\lim_{t\to\pm\infty} f_{n,m}(t) = \lim_{t\to\pm\infty} f'_{n,m}(t) = \lim_{t\to\pm\infty} f''_{n,m}(t) = 0.$$

We consequently may apply integration by part and obtain, for all $m, n \in \mathbb{Z}$:

$$\int_{\mathbb{R}} \left\langle \rho_{p,q,d}^{[n],[m]} \omega(t+\eth_\theta m) - \xi(t+\eth_\vartheta m), \eta(t) \right\rangle_{\mathbb{C}^d} \exp(2i\pi n t)\, dt$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} f_{n,m}(t) \exp(2i\pi n t)\, dt \\
&= -\int_{\mathbb{R}} f'_{n,m}(t) \frac{\exp(2i\pi n t)}{2i\pi n}\, dt \\
&= -\int_{\mathbb{R}} f''_{n,m}(t) \frac{\exp(2i\pi n t)}{4\pi^2 n^2}\, dt.
\end{aligned}
$$

We compute trivially that for all $t \in \mathbb{R}$ and $m, n \in \mathbb{Z}$:

$$
\begin{aligned}
f''_{n,m}(t) = &\left\langle \rho_{p,q,d}^{[n],[m]} \left(\omega''(t + \eth_\theta m) - \xi''(t + \eth_\vartheta m)\right), \eta(t) \right\rangle_{\mathbb{C}^d} \\
&+ 2\left\langle \rho_{p,q,d}^{[n],[m]} \left(\omega'(t + \eth_\theta m) - \xi'(t + \eth_\vartheta m)\right), \eta'(t) \right\rangle_{\mathbb{C}^d} \\
&+ \left\langle \rho_{p,q,d}^{[n],[m]} \left(\omega(t + \eth_\theta m) - \xi(t + \eth_\vartheta m)\right), \eta''(t) \right\rangle_{\mathbb{C}^d}.
\end{aligned}
$$

Thus using Cauchy-Schwarz and since $\rho_{p,q,d}^{[n],[m]}$ is a unitary, we thus conclude:

$$
\begin{aligned}
&\left\| \langle \omega, \eta \rangle_{\mathscr{H}_\theta^{p,q,d}} - \langle \xi, \eta \rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right\|_{\ell^1(\mathbb{Z}^2)} \\
&= \sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}} \left\langle \rho_{p,q,d}^{[n],[m]} \left(\omega(t + \eth_\theta m) - \xi(t + \eth_\vartheta m)\right), \eta(t) \right\rangle_{\mathbb{C}^d} \exp(2i\pi n t)\, dt \right| \\
&\leqslant \sum_{m,n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\left|f''_{n,m}(t)\right|}{4\pi^2 n^2}\, dt \\
&\leqslant \sum_{m,n \in \mathbb{Z}} \frac{1}{4\pi^2 n^2} \left(\int_{\mathbb{R}} \|\omega''(t + \eth_\theta m) - \xi''(t + \eth_\vartheta m)\|_{\mathbb{C}^d} \|\eta(t)\|_{\mathbb{C}^d}\, dt \right. \\
&\quad + 2 \int_{\mathbb{R}} \|\omega'(t + \eth_\theta m) - \xi'(t + \eth_\vartheta m)\|_{\mathbb{C}^d} \|\eta'(t)\|_{\mathbb{C}^d}\, dt \\
&\quad \left. + \int_{\mathbb{R}} \|\omega(t + \eth_\theta m) - \xi(t + \eth_\vartheta m)\|_{\mathbb{C}^d} \|\eta''(t)\|_{\mathbb{C}^d}\, dt \right) \\
&= \sum_{n \in \mathbb{N}} \frac{1}{4\pi^2 n^2} \left[\int_{\mathbb{R}} \left(\sum_{m \in \mathbb{N}} \|\omega''(t + \eth_\theta m) - \xi''(t + \eth_\vartheta m)\|_{\mathbb{C}^d} \right) \|\eta(t)\|_{\mathbb{C}^d}\, dt \right. \\
&\quad + 2 \int_{\mathbb{R}} \left(\sum_{m \in \mathbb{N}} \|\omega'(t + \eth_\theta m) - \xi'(t + \eth_\vartheta m)\|_{\mathbb{C}^d} \right) \|\eta'(t)\|_{\mathbb{C}^d}\, dt \\
&\quad \left. + \int_{\mathbb{R}} \left(\sum_{m \in \mathbb{N}} \|\omega(t + \eth_\theta m) - \xi(t + \eth_\vartheta m)\|_{\mathbb{C}^d} \right) \|\eta''(t)\|_{\mathbb{C}^d}\, dt \right] \text{ by Tonelli's theorem.}
\end{aligned}
$$

This conclude our lemma. □

Our next lemma focuses on the type of estimates given in Lemma (3.2.1), and gives a sufficient condition for these upper bounds to converge to 0 when various parameters are allowed to converge to appropriate values.

Lemma 3.2.2. *Let $d \in \mathbb{N}$, $d > 0$. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$.*

If $(\omega_k)_{k \in \overline{\mathbb{N}}}$ and $(\eta_k)_{k \in \overline{\mathbb{N}}}$ are two families of C^2-functions from $\mathbb{R}$ to $\mathbb{C}^d$ and $(\eth_k)_{k \in \mathbb{N}}$ is a sequence of nonzero real numbers converging to some $\eth_\infty \neq 0$ such that:

(1) $(t,k) \in \mathbb{R}\times\overline{\mathbb{N}} \mapsto \omega_k(t)$ *and* $(t,k)\in\mathbb{R}\times\overline{\mathbb{N}}\mapsto \eta_k(t)$ *are jointly continuous,*
(2) there exists $M>0$ *such that for all* $k\in\overline{\mathbb{N}}$ *and* $t\in\mathbb{R}$*:*

$$\max\left\{\|\omega_k(t)\|_{\mathbb{C}^d}, \|\eta_k(t)\|_{\mathbb{C}^d}\right\} \leqslant \frac{M}{1+t^2},$$

then:

$$\lim_{k\to\infty}\sum_{n\in\mathbb{N}}\frac{1}{4\pi^2n^2}\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}\|\omega_k(t+\eth_k m)-\omega_\infty(t+\eth_\infty m)\|_{\mathbb{C}^d}\|\eta_k(t)\|_{\mathbb{C}^d}\,dt = 0. \tag{3.2.1}$$

Proof. First, we observe that Expression (3.2.1) is left unchanged if we replace $\eth_k$ with $-\eth_k$ for all $k\in\overline{\mathbb{N}}$, thanks to the summation over $m\in\mathbb{Z}$. Consequently, we may assume without loss of generality that $\eth_\infty>0$ and assume that $\eth_k>0$ for all $k\in\overline{\mathbb{N}}$ (since $(\eth_k)_{k\in\mathbb{N}}$ converges to $\eth_\infty\neq 0$, we must have that $\eth_k$ and $\eth_\infty$ have the same sign for k larger than some $K\in\mathbb{N}$; we thus can truncate our sequence to start at K and flip all the signs if necessary to work with positive values).

With this in mind, since $(\eth_k)_{k\in\mathbb{N}}$ is positive and converges to $\eth_\infty>0$, there exists $0<\eth_-<\eth_+$ such that for all $k\in\overline{\mathbb{N}}$, we have $\eth_k\in[\eth_-,\eth_+]$.

We shall employ the Lebesgue dominated convergence theorem. To this end, we introduce the following function to serve as our upper bound. For all $t\in\mathbb{R}, m\in\mathbb{Z}$ we set:

$$\mathrm{b}(t,m)=\begin{cases}\frac{M}{1+(t+m\eth_-)^2} \text{ if } m>0 \text{ and } m\geqslant\frac{-t}{\eth_-} \text{ or } m<0 \text{ and } m\leqslant\frac{-t}{\eth_-},\\ \frac{M}{1+(t+m\eth_+)^2} \text{ if } m>0 \text{ and } m\leqslant\frac{-t}{\eth_+} \text{ or } m<0 \text{ and } m\geqslant\frac{-t}{\eth_+},\\ M \text{ if } m=0 \text{ or } \frac{-t}{m}\in(\eth_-,\eth_+).\end{cases} \tag{3.2.2}$$

For a fixed $t\in\mathbb{R}$, we note that:

$$\mathrm{b}(t,m)\sim_{m\to\pm\infty}\frac{M}{\eth_-^2m^2},$$

so $\sum_{m\in\mathbb{Z}}\mathrm{b}(t,m)<\infty$. Moreover, by construction, for all $t\in\mathbb{R}, m\in\mathbb{Z}$ and $\eth\in[\eth_-,\eth_+]$, we have:

$$\frac{M}{1+(t+\eth m)^2}\leqslant \mathrm{b}(t,m).$$

Therefore, using our hypothesis, for all $t\in\mathbb{R}$, $m\in\mathbb{Z}$, $k\in\overline{\mathbb{N}}$ and $\eth\in[\eth_-,\eth_+]$:

$$\begin{aligned}\|\omega_k(t+m\eth)-\omega_\infty(t+m\eth_\infty)\|_{\mathbb{C}^d} &\leqslant \frac{M}{1+(t+m\eth)^2}+\frac{M}{1+(t+m\eth_\infty)^2}\\ &\leqslant 2\mathrm{b}(t,m).\end{aligned}$$

Thus for a fixed $t\in\mathbb{R}$, we may apply Lebesgue dominated convergence theorem to conclude:

$$\lim_{k\to\infty}\sum_{m\in\mathbb{Z}}\|\omega_k(t+m\eth_k)-\omega_\infty(t+m\eth_\infty)\|_{\mathbb{C}^d}=0, \tag{3.2.3}$$

since $(t,k)\in\mathbb{R}\times\overline{\mathbb{N}}\mapsto\omega_k(t)$ is jointly continuous.

We now make another observation. For any fixed $\eth>0$ and $k\in\overline{\mathbb{N}}$, The function:

$$t\in\mathbb{R}\mapsto\sum_{m\in\mathbb{Z}}\|\omega_k(t+\eth m)\|_{\mathbb{C}^d}$$

is $\eth$-periodic.

If $t \in [0, \eth_+]$, $k \in \overline{\mathbb{N}}$ and $\eth \in [\eth_-, \eth_+]$, then since:

$$\|\omega_k(t + \eth m)\|_{\mathbb{C}^d} \leqslant \sup_{x \in [0, \eth_+]} \mathrm{b}(x, m)$$

while, as can easily be checked:

$$\sup_{x \in [0, \eth_+]} \mathrm{b}(x, m) \sim_{m \to \pm\infty} \frac{M}{\eth_-^2 m^2},$$

we conclude that the series:

$$\left((t, k, \eth) \in \mathbb{R} \times \overline{\mathbb{N}} \times [\eth_-, \eth_+] \mapsto \sum \|\omega_k(t + \eth m)\|_{\mathbb{C}^d}\right)_{m \in \mathbb{Z}}$$

converges uniformly to its limit on $[0, \eth_+] \times \overline{\mathbb{N}} \times [\eth_-, \eth_+]$. In particular:

$$(t, k, \eth) \in [0, \eth_+] \times \overline{\mathbb{N}} \times [\eth_-, \eth_+] \mapsto \sum_{m \in \mathbb{Z}} \|\omega_k(t + \eth m)\|_{\mathbb{C}^d}$$

is continuous on a compact domain and so it is bounded. Let $C > 0$ such that for all $(t, k, \eth) \in [0, \eth_+] \times \overline{\mathbb{N}} \times [\eth_-, \eth_+]$, we have:

$$\sum_{m \in \mathbb{Z}} \|\omega_k(t + \eth m)\|_{\mathbb{C}^d} \leqslant C.$$

We conclude that $t \mapsto \sum_{m \in \mathbb{Z}} \|\omega_k(t - \eth_k m)\|_{\mathbb{C}^d}$ is bounded by C on $\mathbb{R}$, since it is an $\eth_k$-periodic function with $\eth_k \leqslant \eth_+$, for all $k \in \overline{\mathbb{N}}$.

We thus have that for all $t \in \mathbb{R}$ and $k \in \overline{\mathbb{N}}$:

$$\sum_{m \in \mathbb{Z}} \|\omega_k(t + m\eth_k) - \omega_\infty(t + m\eth_\infty)\|_{\mathbb{C}^d} \|\eta_k(t)\|_{\mathbb{C}^d} \leqslant 2C \|\eta_k(t)\|_{\mathbb{C}^d} \leqslant \frac{2CM}{1 + t^2}. \tag{3.2.4}$$

Now $t \in \mathbb{R} \mapsto \frac{2CM}{1+t^2}$ is integrable over $\mathbb{R}$. Once again, we apply Lebesgue dominated convergence theorem, and we conclude from Expression (3.2.3) that:

$$\lim_{k \to \infty} \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \|\omega_k(t + m\eth_k) - \omega_\infty(t + m\eth_\infty)\|_{\mathbb{C}^d} \|\eta_k(t)\|_{\mathbb{C}^d} \, dt = 0. \tag{3.2.5}$$

Last, using Inequality (3.2.4) again, we note that for all $k \in \overline{\mathbb{N}}$:

$$\int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \|\omega_k(t + m\eth_k) - \omega_\infty(t + m\eth_\infty)\|_{\mathbb{C}^d} \|\eta_k(t)\|_{\mathbb{C}^d} \, dt \leqslant \int_{\mathbb{R}} \frac{2CM}{1 + t^2} \, dt$$

and thus for all $n \in \mathbb{Z}$ and $k \in \overline{\mathbb{N}}$:

$$\frac{1}{4\pi^2 n^2} \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \|\omega(t + m\eth_k) - \omega(t + m\eth_\infty)\|_{\mathbb{C}^d} \|\eta(t)\|_{\mathbb{C}^d} \, dt \leqslant \frac{\int_{\mathbb{R}} \frac{2CM}{1+t^2} \, dt}{4\pi^2 n^2},$$

with $\sum_{n \in \mathbb{Z}} \frac{\int_{\mathbb{R}} \frac{2CM}{1+t^2} \, dt}{4\pi^2 n^2} < \infty$; hence we may apply Lebesgue dominated convergence theorem once more to conclude from Expression (3.2.5):

$$\lim_{k \to \infty} \sum_{n \in \mathbb{Z}} \frac{1}{4\pi^2 n^2} \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \|\omega_k(t + m\eth_k) - \omega_\infty(t + m\eth_\infty)\|_{\mathbb{C}^d} \|\eta_k(t)\|_{\mathbb{C}^d} \, dt = 0.$$

This concludes our lemma. $\square$

Remark 3.2.3. One may check that Lemma (3.2.1) and Lemma (3.2.2) together prove that if $p, q \in \mathbb{N}$, $\xi, \omega \in \mathcal{S}(\mathbb{C}^d)$, for any $d \in q\mathbb{N}$ with $d > 0$, and if $\theta \in \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$, then $\langle \xi, \omega \rangle_{\mathscr{H}^{p,q,d}_\theta} \in \ell^1(\mathbb{Z}^2)$. It is a well-known fact (indeed a basic fact for the very construction of Heisenberg modules) though maybe not apparent from Theorem-Definition (3.1.15) without consulting such sources as [37].

We now bring together Lemma (3.2.1) and Lemma (3.2.2) to obtain a first result of continuity on the Heisenberg module inner products, albeit using the $\ell^1(\mathbb{Z}^2)$ norm. This is the core result of this section, and it is phrased at a somewhat higher level of generality that what is needed for the proof of continuity of the family of Heisenberg C^*-Hilbert norms. Indeed, this level of generality will prove useful twice later in this paper: when proving that the Heisenberg group representations $\alpha_{\eth,d}$ define *strongly continuous* actions on Heisenberg modules, and when establishing that our prospective D-norms on Heisenberg modules will also form a continuous family of norms.

Lemma 3.2.4. *Let $p, q \in \mathbb{N}$ with $q > 0$ and $d \in q\mathbb{N}$ with $d > 0$. If $(\xi_k)_{k\in\overline{\mathbb{N}}}$ is a family of $\mathbb{C}^d$-valued C^2-functions such that:*

(1) there exists $M > 0$ such that for all $k \in \overline{\mathbb{N}}$ and $t \in \mathbb{R}$:

$$\max\left\{\|\xi_k(t)\|_{\mathbb{C}^d}, \|\xi_k'(t)\|_{\mathbb{C}^d}, \|\xi_k''(t)\|_{\mathbb{C}^d}\right\} \leqslant \frac{M}{1+t^2},$$

(2) $(t,k) \in \mathbb{R} \times \overline{\mathbb{N}} \mapsto \xi_k(t)$ is continuous,

and if $(\theta_k)_{k\in\mathbb{N}}$ is a sequence converging to θ_∞ such that $\theta_k - \frac{p}{q} \neq 0$ for all $k \in \overline{\mathbb{N}}$, then we have:

$$\lim_{k\to\infty} \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} = 0.$$

Proof. To fix notations, for all $k \in \overline{\mathbb{N}}$, we set $\eth_k = \theta_k - \frac{p}{q}$. Note that $(\eth_k)_{k\in\mathbb{N}}$ is a sequence of nonzero real numbers converging to $\eth_\infty \neq 0$.

We shall prove our result from the following inequality.

$$(3.2.6) \quad \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)}$$
$$\leqslant \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_k, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} + \left\| \langle \xi_k, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)}.$$

We begin with the first term of the right hand side of Inequality (3.2.6). We observe, using Expression (3.1.4) and the fact that the adjoint operation is an isometry for $\|\cdot\|_{\ell^1(\mathbb{Z}^2)}$, that:

$$\left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_k, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} = \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)}.$$

By Lemma (3.2.1), we then have for all $k \in \mathbb{N}$:

$$(3.2.7) \quad \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)}$$

$$\leqslant \sum_{n\in\mathbb{Z}} \frac{1}{4\pi^2 n^2} \left(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \left\| \xi_k''(t+\eth_k m) - \xi_\infty''(t+\eth_\infty m) \right\|_{\mathbb{C}^d} \|\xi_k(t)\|_{\mathbb{C}^d}\, dt + \right.$$
$$+ \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \left\| \xi_k'(t+\eth_k m) - \xi_\infty'(t+\eth_\infty m) \right\|_{\mathbb{C}^d} \|\xi_k'(t)\|_{\mathbb{C}^d}\, dt$$
$$\left. + \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \|\xi_k(t+\eth_k m) - \xi_\infty(t+\eth_\infty m)\|_{\mathbb{C}^d} \|\xi_k''(t)\|_{\mathbb{C}^d}\, dt \right).$$

Our assumptions allow us to apply Lemma (3.2.2) to each term in the right hand side of Inequality (3.2.7) to conclude that:

$$\lim_{k\to\infty} \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_k, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} = 0.$$

We handle the second term of Inequality (3.2.6) in a similar manner. By Lemma (3.2.1), we have:

$$\left\| \langle \xi_k, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)}$$
$$\leqslant \sum_{n\in\mathbb{Z}} \frac{1}{4\pi^2 n^2} \left(\int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \left\| \xi_k''(t+\eth_\infty m) - \xi_\infty''(t+\eth_\infty m) \right\|_{\mathbb{C}^d} \|\xi_\infty(t)\|_{\mathbb{C}^d}\, dt + \right.$$
$$+ \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \left\| \xi_k'(t+\eth_\infty m) - \xi_\infty'(t+\eth_\infty m) \right\|_{\mathbb{C}^d} \|\xi_\infty'(t)\|_{\mathbb{C}^d}\, dt$$
$$\left. + \int_{\mathbb{R}} \sum_{m\in\mathbb{Z}} \|\xi_k(t+\eth_\infty m) - \xi_\infty(t+\eth_\infty m)\|_{\mathbb{C}^d} \|\xi_\infty''(t)\|_{\mathbb{C}^d}\, dt \right).$$

Once again, by the assumptions on $(\xi_k)_{k\in\overline{\mathbb{N}}}$ allow us to apply Lemma (3.2.2) to conclude:

$$\lim_{k\to\infty} \left\| \langle \xi_k, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} = 0,$$

and thus from Inequality (3.2.6), our lemma is proven. □

We now conclude this section with the proof that indeed, Heisenberg C^*-Hilbert norms form continuous families of norms for a fixed projective representation of some $\mathbb{Z}_q^2$.

Proposition 3.2.5. *Let $p, q \in \mathbb{N}$ and $d \in q\mathbb{N}$ with $d > 0$. Let $(\xi)_{k\in\overline{\mathbb{N}}}$ be a family in $\mathcal{S}(\mathbb{C}^d)$ such that $(k,t) \in \overline{\mathbb{N}} \times \mathbb{R} \mapsto \xi_k(t)$ is (jointly) continuous and there exists $M > 0$ such that $\|\xi_k^{(s)}(t)\|_{\mathbb{C}^d} \leqslant \frac{M}{1+t^2}$ for all $k \in \overline{\mathbb{N}}$, $t \in \mathbb{R}$ and $s \in \{0,1,2\}$.*

If $(\theta_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to θ_∞ and such that $\theta_k - \frac{p}{q} = 0$ for all $k \in \overline{\mathbb{N}}$, then:

$$\lim_{k\to\infty} \|\xi_k\|_{\mathscr{H}^{p,q,d}_{\theta_k}} = \|\xi_\infty\|_{\mathscr{H}^{p,q,d}_{\theta_\infty}}.$$

Proof. For each $k \in \mathbb{N} \cup \{\infty\}$, we set $\eth_k = \theta_k - \frac{p}{q} \neq 0$.

We first compute:

(3.2.8)

$$\begin{aligned}
\left| \|\xi_k\|^2_{\mathscr{H}^{p,q,d}_{\theta_k}} - \|\xi_\infty\|^2_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right| &= \left| \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} \right\|_{\mathcal{A}_{\theta_k}} - \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_\infty}} \right| \\
&\leqslant \left| \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} \right\|_{\mathcal{A}_{\theta_k}} - \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_k}} \right| \\
&\quad + \left| \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_k}} - \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_\infty}} \right| \\
&\leqslant \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_k}} \\
&\quad + \left| \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_k}} - \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_\infty}} \right| \\
&\leqslant \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} \\
&\quad + \left| \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_k}} - \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_\infty}} \right| .
\end{aligned}$$

We now apply Lemma (3.2.4) to conclude that:

$$\lim_{k\to\infty} \left\| \langle \xi_k, \xi_k \rangle_{\mathscr{H}^{p,q,d}_{\theta_k}} - \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\ell^1(\mathbb{Z}^2)} = 0.$$

Now, for any $f \in \ell^1(\mathbb{Z}^2)$, the function $\theta \in \mathbb{R} \mapsto \|f\|_{\mathcal{A}_\theta}$ is continuous by [39, Corollary 2.7]. Hence, using Remark (3.2.3):

$$\lim_{k\to\infty} \left| \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_k}} - \left\| \langle \xi_\infty, \xi_\infty \rangle_{\mathscr{H}^{p,q,d}_{\theta_\infty}} \right\|_{\mathcal{A}_{\theta_\infty}} \right| = 0.$$

Thus, we conclude from Inequality (3.2.8) that:

$$\lim_{k\to\infty} \|\xi_k\|^2_{\mathscr{H}^{p,q,d}_{\theta_k}} = \|\xi_\infty\|^2_{\mathscr{H}^{p,q,d}_{\theta_\infty}}$$

which, by continuity of the square root, proves our lemma. □

Corollary 3.2.6. *Let $p, q \in \mathbb{N}$ and $d \in q\mathbb{N}$ with $d > 0$. Let $\xi \in \mathcal{S}(\mathbb{C}^d)$. If $(\theta_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to θ_∞ and such that $\theta_k - \frac{p}{q} = 0$ for all $k \in \overline{\mathbb{N}}$, then:*

$$\lim_{k\to\infty} \|\xi\|_{\mathscr{H}^{p,q,d}_{\theta_k}} = \|\xi\|_{\mathscr{H}^{p,q,d}_{\theta_\infty}} .$$

Proof. We apply Proposition (3.2.5) to the family $k \in \overline{\mathbb{N}} \mapsto \xi$. We note that since ξ is a Schwarz function, our assumptions are met. □

We shall return to Lemma (3.2.4) and its applications in two subsequent sections. The first such occurrence is in fact in the next section, where we establish all the basic results we will need on the Heisenberg group actions on Heisenberg modules. Lemma (3.2.4) will be the key ingredient to proving these actions will be strongly continuous.

3.3. **The action of the Heisenberg group on Heisenberg modules.** Heisenberg modules may be endowed with a metrized quantum vector bundle structure over quantum 2-tori using a D-norm built from a Lie group action and inspired by the construction of [40], albeit involving a projective action of a locally compact group, which will not act via isometries of the D-norm. These changes will introduce new difficulties which we will handle in the next few sections. As a first step, we study the actions of the Heisenberg group on Heisenberg modules.

One motivation for the results in this section is to establish the properties which will meet the hypothesis of the main results in our next section, from which our D-norm will emerge. We also note that the actions $\alpha_{\eth,d}$, for all $\eth \in \mathbb{R}\setminus\{0\}$ and $d \in \mathbb{N}\setminus\{0\}$, is a strongly continuous action by isometries of $L^2(\mathbb{R})\otimes\mathbb{C}^d$, but we need these results to be proven for the Heisenberg C^*-Hilbert norms, which dominate the norm of $L^2(\mathbb{R})\otimes\mathbb{C}^d$.

We shall use the same hypotheses for a series of lemmas and our main definition in this section, and thus we group them in the following.

Hypothesis 3.3.1. Let $p \in \mathbb{Z}$, $q \in \mathbb{N}\setminus\{0\}$, and let $d \in q\mathbb{N}$ with $d > 0$. Let $\theta \in \mathbb{R}\setminus\left\{\frac{p}{q}\right\}$. We write $\eth = \theta - \frac{p}{q}$.

We shall employ the notations of Theorem-Definition (3.1.15) and of Notation (3.1.16).

We begin with two lemmas which will prove that $\mathbb{H}_3 2$ acts via isometries of the norm of the Heisenberg modules on the subspace of Schwarz functions — where we have an explicit formula for our inner product — and thus can indeed be extended to the entire module.

Lemma 3.3.2. *We assume Hypothesis (3.3.1). For all $(x,y,u)\in\mathbb{H}_3$, if $z_1 = \exp(-2i\pi\eth x)$ and $z_2 = \exp(2i\pi\eth y)$, and if $\xi,\omega \in \mathscr{S}^{p,q,d}_\theta$, then:*

$$\left\langle \alpha^{x,y,u}_{\eth,d}(\xi), \alpha^{x,y,u}_{\eth,d}(\omega)\right\rangle_{\mathscr{H}^{p,q,d}_\theta} = \beta^{z_1,z_2}_\theta\left(\langle\xi,\omega\rangle_{\mathscr{H}^{p,q,d}_\theta}\right).$$

Proof. Let $n,m\in\mathbb{Z}$. We compute:

$$\begin{aligned}
&\left\langle \varpi^{n,m}_{p,q,\eth,d}\alpha^{x,y,u}_{\eth,d}\xi, \alpha^{x,y,u}_{\eth,d}\omega\right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \\
&\quad= \left\langle \left(\sigma^{n,m}_{\eth,1}\alpha^{x,y,u}_{\eth,1}\otimes\rho^{[n],[m]}_{p,q,d}\right)\xi, \alpha^{x,y,u}_{\eth,d}\omega\right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \\
&\quad= \left\langle \left(\alpha^{(x,y,u)^{-1}}_{\eth,1}\alpha^{n,m,0}_{\eth,1}\alpha^{x,y,u}_{\eth,1}\otimes\rho^{[n],[m]}_{p,q,d}\right)\xi, \omega\right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \\
&\quad= \left\langle \exp(2i\pi\eth(xm-yn))\left(\sigma^{n,m}_{\eth,1}\otimes\rho^{[n],[m]}_{p,q,d}\right)\xi, \omega\right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \\
&\quad= z_1^n z_2^m\left\langle \varpi^{n,m}_{p,q,\eth,d}\xi,\omega\right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d}.
\end{aligned}$$

Therefore, using the fact that $\beta_\theta^{z_1,z_2}$ is a *-morphism, and writing $u_\theta = \delta_{1,0}$ and $v_\theta = \delta_{0,1}$, we conclude:

$$\begin{aligned}\left\langle \alpha_{\eth,d}^{x,y,u}(\xi), \alpha_{\eth,d}^{x,y,u}(\omega) \right\rangle_{\mathscr{H}_\theta^{p,q,d}} &= \sum_{n,m\in\mathbb{Z}} \left\langle \varpi_{p,q,\eth,d}^{n,m} \alpha_{\eth,1}^{x,y,u}\xi, \alpha_{\eth,1}^{x,y,u}\omega \right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} u_\theta^n v_\theta^m \\ &= \sum_{n,m\in\mathbb{Z}} z_1^n z_2^m \left\langle \varpi_{p,q,\eth,d}^{n,m}\xi, \omega \right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} u_\theta^n v_\theta^m \\ &= \sum_{n,m\in\mathbb{Z}} \left\langle \varpi_{p,q,\eth,d}^{n,m}\xi, \omega \right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} \beta_\theta^{z_1,z_2}\left(u_\theta^n v_\theta^m\right) \\ &= \beta_\theta^{z_1,z_2}\left(\sum_{n,m\in\mathbb{Z}} \left\langle \varpi_{p,q,\eth,d}^{n,m}\xi, \omega \right\rangle_{L^2(\mathbb{R})\otimes\mathbb{C}^d} u_\theta^n v_\theta^m \right) \\ &= \beta_\theta^{z_1,z_2}\left(\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} \right),\end{aligned}$$

as desired. ☐

To ease our notations in this section, we set:

Notation 3.3.3. For all $(x,y) \in \mathbb{R}^2$ and $\eth > 0$, we define:

$$v_\eth(x,y) = (\exp(-2i\pi\eth x), \exp(2i\pi\eth y)) \in \mathbb{T}^2.$$

We now show that the Heisenberg group acts by isometries for the C^*-Hilbert norm.

Lemma 3.3.4. *We assume Hypothesis (3.3.1). For all $(x,y,u) \in \mathbb{H}_3$, the map $\alpha_{\eth,d}^{x,y,u}$ is an isometry of $\left(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}\right)$.*

Proof. Let $(x,y,u) \in \mathbb{H}_3$ and $\xi \in \mathscr{S}_\theta^{p,q,d}$. We compute:

$$\begin{aligned}\left\| \alpha_\eth^{x,y,u}\xi \right\|_{\mathscr{H}_\theta^{p,q,d}}^2 &= \left\| \left\langle \alpha_{\eth,d}^{x,y,u}\xi, \alpha_{\eth,d}^{x,y,u}\xi \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \right\|_{\mathcal{A}_\theta} \\ &= \left\| \beta_\theta^{v_r(x,y)} \langle \xi, \xi \rangle_{\mathscr{H}_\theta^{p,q,d}} \right\|_{\mathcal{A}_\theta} \text{ by Lemma (3.3.2)}, \\ &= \left\| \langle \xi, \xi \rangle_{\mathscr{H}_\theta^{p,q,d}} \right\|_{\mathcal{A}_\theta} \\ &= \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}^2.\end{aligned}$$

This completes our proof. ☐

Notation 3.3.5. We use the notations of Hypothesis (3.3.1). The action $\alpha_{\eth,d}$ of $\mathbb{H}_3$ on $\mathscr{S}_\theta^{p,q,d}$ may thus be extended to $\mathscr{H}_\theta^{p,q,d}$ by extending by continuity $\alpha_{\eth,d}^{x,y,u}$ for all $(x,y,u) \in \mathbb{H}_3$; we shall keep the notation of this extension as $\alpha_{\eth,d}$. We note that it also acts via isometry on $\left(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}\right)$.

We also use the same notation for $\sigma_{\eth,d}$ extended as $\alpha_{\eth,d}^{\cdot,\cdot,0}$ to $\left(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}\right)$.

The actions of the Heisenberg group on Heisenberg modules is by morphism modules, in the sense of Definition (2.2.5). This result will play a role in the proof that our D-norm satisfies the modular version of the Leibniz inequality.

Lemma 3.3.6. *We assume Hypothesis (3.3.1). For all $a \in \mathcal{A}_\theta$, $\xi \in \mathscr{H}_\theta^{p,q,d}$ and $(x,y,u) \in \mathbb{H}_3$, then:*

$$\alpha_{\eth,d}^{x,y,u}(a\xi) = \beta_\theta^{v_\eth(x,y)}(a)\alpha_{\eth,d}^{x,y,u}(\xi).$$

Proof. Let $n, m \in \mathbb{Z}$ and $\xi \in \mathscr{S}_\theta^{p,q,d}$. We compute:

$$\begin{aligned}
\alpha_{\eth,d}^{x,y,u}(u_\theta^n v_\theta^m \xi) &= \alpha_{\eth,d}^{x,y,u}\varpi_{p,q,\eth,d}^{n,m}\xi \\
&= \left(\alpha_{\eth,d}^{x,y,u}\alpha_{\eth,d}^{n,m,0} \otimes \rho_{p,q,d}^{[n],[m]}\right)\xi \\
&= \exp(2i\pi(xm - yn))\left(\alpha_{\eth,d}^{n,m,0}\alpha_{\eth,d}^{x,y.u} \otimes \rho_{p,q,\eth,d}^{[n],[m]}\right)\xi \\
&= \exp(2i\pi(xm - yn))\varpi_{p,q,\eth,d}^{n,m}\alpha_{\eth,d}^{x,y,u}\xi \\
&= \beta_\theta^{v_r(x,y)}(u_\theta^n v_\theta^m)\alpha_{\eth,d}^{x,y,u}\xi.
\end{aligned}$$

Since β_θ is an action by *-morphisms, we conclude that for all $a \in \mathcal{A}_\theta$:

$$\alpha_{\eth,d}^{x,y,u}(a\xi) = \beta_\theta^{z_1,z_2}(a)\alpha_{\eth,d}^{x,y,u}(\xi) \tag{3.3.1}$$

as desired. The lemma is concluded by extending Equality (3.3.1) to $\mathscr{H}_\theta^{p,q,d}$ by continuity. ☐

An important corollary of Lemma (3.3.6) is as follows:

Corollary 3.3.7. *We assume Hypothesis (3.3.1). For all $a \in \mathcal{A}_\theta$, $\xi \in \mathscr{H}_\theta^{p,q,d}$ and $(x,y,u) \in \mathbb{H}_3$, we observe that:*

$$\left\|\alpha_{\eth,d}^{x,y,u}(a\xi)\right\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \|a\|_{\mathcal{A}_\theta}\|\xi\|_{\mathscr{H}_\theta^{p,q,d}}.$$

Proof. Let $a \in \mathcal{A}_\theta$, $\xi \in \mathscr{H}_\theta^{p,q,d}$ and $(x,y,u) \in \mathbb{H}_3$. We compute:

$$\begin{aligned}
\left\|\alpha_{\eth,d}^{x,y,u}(a\xi)\right\|_{\mathscr{H}_\theta^{p,q,d}} &= \left\|\beta_\theta^{v_r(x,y)}(a)\alpha_{\eth,d}^{x,y,u}\xi\right\|_{\mathscr{H}_\theta^{p,q,d}} \text{ by Lemma (3.3.6)}, \\
&\leqslant \|\beta_\theta^{v_r(x,y)}a\|_{\mathcal{A}_\theta}\|\xi\|_{\mathscr{H}_\theta^{p,q,d}} \text{ by Lemma (3.3.4)}.
\end{aligned}$$

This completes our proof. ☐

We have checked that the actions of the Heisenberg group on Heisenberg modules, which the latter were constructed from, act by isometric module morphisms on the entire module. Note that we already observed that Heisenberg modules can be regarded as dense subspaces of $L^2(\mathbb{R}) \otimes \mathbb{C}^d$ spaces on which the same action of the Heisenberg group is defined, strongly continuous and isometric; however we needed to ensure that these actions are well-behaved with respect to the inner product and norm of the Heisenberg modules.

In order to define our D-norms, we shall require one more important analytic property: we want our actions to be strongly continuous for the Heisenberg C^*-Hilbert norms. This is the subject of the next proposition. We actually include in the next proposition a somewhat more general hypothesis and estimate than needed for the strong continuity of our actions, as this stronger statement will play an important role in our study of the continuity properties of our D-norms later on.

Proposition 3.3.8. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N}$ with $d > 0$. Let $C > 0$ and $M > 0$ some constant. Let $0 < \eth_- < \eth_+$. There exists $K > 0$ such that for all $\xi \in \mathscr{S}(\mathbb{C}^d)$ satisfying:*

$$\max\left\{\|\xi(s)\|_{\mathbb{C}^d}, \|s\xi(s)\|_{\mathbb{C}^d}, \|\xi'(s)\|_{\mathbb{C}^d}, \|s\xi'(s)\|_{\mathbb{C}^d}, \|\xi''(s)\|_{\mathbb{C}^d}, \|s\xi''(s)\|_{\mathbb{C}^d}\right\} \leqslant \frac{M}{1+s^2}, \tag{3.3.2}$$

the following holds for all $s \in \mathbb{R}$, $\eth \in [\eth_-, \eth_+]$ and $(x,y,u) \in \mathbb{R}^3$ with $|x|+|y|+|u| \leqslant C$:

$$\max\left\{\left\|\alpha_{\eth,d}^{x,y,u}\xi^{(n)}(s) - \xi^{(n)}(s)\right\|_{\mathbb{C}^d} : n \in \{0,1,2\}\right\} \leqslant \frac{K(|x|+|y|+|u|)}{1+s^2}. \tag{3.3.3}$$

In particular, for all $\eth \neq 0$ and $\theta = \eth + \frac{p}{q}$:

$$\lim_{(x,y,u)\to 0}\left\|\alpha_{\eth,d}^{x,y,u}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}} = 0.$$

Proof. Let $\xi \in \mathcal{S}(\mathbb{C}^d)$ and $(x,y,u) \in \mathbb{R}^3$. We note that for all $s \in \mathbb{R}$, using the continuity if ξ, we of course have:

$$\begin{aligned}\alpha_{\eth,d}^{x,y,u}\xi - \xi(s) &= \exp(2i\pi(u+xs))\xi(s+\eth y) - \xi(s) \\ &\xrightarrow{(x,y,u)\to 0} 0.\end{aligned}$$

However, we wish to apply Lemma (3.2.4), so we seek a more precise estimate. To this end, let $f_s(t) = \alpha_{\eth,d}^{tx,ty,tu}\xi(s) = \exp(2i\pi(\eth tu + txs))\xi(s+\eth ty)$ for all $t,s \in \mathbb{R}$. We compute for all $t,s \in \mathbb{R}$:

$$f_s'(t) = \exp(2i\pi(\eth tu + txs))\left(2i\pi(\eth u + xs)\xi(s+\eth ty) + \eth y\xi'(s+\eth ty)\right).$$

Let $\|(x,y,u)\|_1 = |x|+|y|+|u|$ for all $(x,y,u) \in \mathbb{R}^2$, i.e. $\|\cdot\|_1$ is the usual 1-norm on $\mathbb{R}^3$. Let us now assume $\|(x,y,u)\|_1 \leqslant C$ — in particular, $|y| < C$. We observe that for all $s \in \mathbb{R}$, using the function b introduced in Expression (3.2.2) in the proof of Lemma (3.2.2):

$$\left\|\alpha_{\eth,d}^{x,y,u}\xi(s) - \xi(s)\right\|_{\mathbb{C}^d}$$

$$
\begin{aligned}
&= \|f_s(1) - f_s(0)\|_{\mathbb{C}^d} \\
&= \left\| \int_0^1 f_s'(t)\,dt \right\|_{\mathbb{C}^d} \\
&\leqslant \int_0^1 \left\| \exp(2i\pi(\eth tu + txs)) \left(2i\pi(\eth u + xs)\xi(s+\eth ty) + \eth y\xi'(s+\eth ty)\right) \right\|_{\mathbb{C}^d}\,dt \\
&= \int_0^1 \left\| 2i\pi(\eth u + xs)\xi(s+\eth ty) + \eth y\xi'(s+\eth ty) \right\|_{\mathbb{C}^d}\,dt \\
&\leqslant \int_0^1 \|(u,x,y)\|_1 \max \left\{ \begin{array}{l} \|2i\pi\eth\xi(s+\eth ty)\|_{\mathbb{C}^d}, \\ \|2i\pi s\xi(s+\eth ty)\|_{\mathbb{C}^d}, \\ \|\xi'(s+\eth ty)\|_{\mathbb{C}^d} \end{array} \right\} dt \\
&\leqslant 2\pi M \max\{1,\eth_+\} \|(x,y,u)\|_1 \int_0^1 \mathrm{b}(s,ty)\,dt \\
&\leqslant 2\pi M \max\{1,\eth_+\} \|(x,y,u)\|_1 \left(\sup_{y\in[-C,C]} \mathrm{b}(s,y) \right).
\end{aligned}
$$

Since:

$$
\lim_{s\to\pm\infty} (1+s^2) \sup_{y\in[-C,C]} \mathrm{b}(s,y) = \frac{M}{\eth_-^2}, \tag{3.3.4}
$$

we conclude that there exists $M_1 > 0$ such that, for all $s \in \mathbb{R} \setminus \{[-\mathbb{C}\eth_+ - 1, C\eth_- + 1]$, we have:

$$
\|\alpha_{\eth,d}^{x,y,u}\xi - \xi\|_{\mathbb{C}^d} \leqslant \frac{M_1\|(x,y,u)\|_1}{1+s^2}.
$$

We note that M_1 depends only on M, $\eth_+$ and C through Expression (3.3.4), and not on ξ.

Since $s \in \mathbb{R} \mapsto \frac{1}{1+s^2}$ is continuous and strictly positive, we may adjust M_1 to a larger value if necessary such that:

$$
\min_{s\in[-C\eth_+-1, C\eth_-+1]} \frac{M_1}{1+s^2} \geqslant 2\pi M \max\{1,\eth_+\}.
$$

Therefore, we have, for all $s \in \mathbb{R}$ and $(x,y,u) \in \mathbb{R}^3$ with $\|(x,y,u)\|_1 \leqslant C$:

$$
\|\alpha_{\eth,d}^{x,y,u}\xi(s) - \xi(s)\|_{\mathbb{C}^d} \leqslant \frac{M_1\|(x,y,u)\|_1}{1+s^2} \leqslant \frac{M_1 C}{1+s^2}.
$$

Now, all the above computations may be applied equally well to ξ' and ξ''. We conclude that indeed, Expression (3.3.3) holds as stated.

Let now $\xi \in \mathcal{S} \otimes \mathbb{C}^d$ be chosen. Since ξ is a Schwarz function, there exists $M > 0$ such that for all $s \in \mathbb{R}$, we have:

$$
\max \left\{ \|\xi(s)\|_{\mathbb{C}^d}, \|s\xi(s)\|_{\mathbb{C}^d}, \|\xi'(s)\|_{\mathbb{C}^d}, \|s\xi'(s)\|_{\mathbb{C}^d}, \|\xi''(s)\|_{\mathbb{C}^d}, \|s\xi''(s)\|_{\mathbb{C}^d} \right\} \leqslant \frac{M}{1+s^2}.
$$

Thus we can apply our previous work to conclude that Expression (3.3.3) holds for some $K > 0$, having chosen $C = 1$ for this last part of our proof.

Furthermore, we can apply now Lemma (3.2.4). For this part, we pick $\eth > 0$; we need not to worry about the uniformity in $\eth$ (we may as well assume $\eth_- = \eth_+ = \eth$ here). Thus, if $(x_n, y_n, u_n)_{n\in\mathbb{N}}$ converges to 0, Lemma (3.2.4) implies that:

$$\begin{cases} 0 \leqslant \|\alpha_{\eth,d}^{x_n,y_n,u_n}\xi - \xi\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \sqrt{\|\left\langle \alpha_{\eth,d}^{x_n,y_n,u_n}\xi - \xi, \alpha_{\eth,d}^{x_n,y_n,u_n}\xi - \xi\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\|_{\ell_1(\mathbb{Z}^2)}} \\ \lim_{n\to\infty} \sqrt{\|\left\langle \alpha_{\eth,d}^{x_n,y_n,u_n}\xi - \xi, \alpha_{\eth,d}^{x_n,y_n,u_n}\xi - \xi\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\|_{\ell_1(\mathbb{Z}^2)}} = 0 \end{cases}$$

which concludes the proof of our proposition for $\eth > 0$.

To prove our result for a general $\eth \neq 0$, we simply observe that for all $(x, y, u) \in \mathbb{R}^3$ we have $\alpha_{\eth,d}^{x,y,u} = \alpha_{-\eth,d}^{x,-y,-u}$ and thus our proposition is completely proven. □

We wish to use the actions of $\mathbb{H}_3$ on Heisenberg modules to define our D-norms. The next section presents a general source of possible D-norms from actions of Lie groups satisfying the properties we have established in this section.

3.4. **Seminorms from Lie group actions.** Connes introduced a quantized differential calculus on quantum tori in [6] using the dual action of the tori, using the Lie group structure of the tori. Moreover, he introduced a noncommutative connection on Heisenberg modules, and these connections proved to be solutions of the Yang-Mills problem for quantum 2-tori [9]. These connections were also useful in Rieffel's work on the classification of modules over quantum tori [37].

Moreover, ergodic actions of metric compact groups on C*-algebras were the first example of L-seminorms constructed by Rieffel in [40]. In this section, we begin investigating how to build D-norms from Lie group actions. We will employ as assumptions the properties which we derived for the action of the Heisenberg group on Heisenberg modules. Our construction, as we shall see, lies at the intersection of the purely metric picture of Rieffel and the differential picture of Connes, and is a noncommutative version of Example (2.2.10).

Our D-norm will be constructed using the following theorem.

Definition 3.4.1. Let α be a strongly continuous action of a Lie group G on a Banach space $\mathscr{E}$. Let $\mathfrak{w}$ be a nonzero subspace of the Lie algebra of G. An element $\xi \in \mathscr{E}$ is *α-differentiable with respect to* $\mathfrak{w}$ when for all $X \in \mathfrak{w}$, the limit:

$$X(\xi) = \lim_{t\to 0} \frac{\alpha^{\exp(tX)}\xi - \xi}{t}$$

exists.

In any vector space E, and for any function $f : E \to \mathbb{R}$, we denote as usual:

$$\limsup_{x\to 0} f(x) = \inf_{\delta>0} \sup \{f(x) : 0 < \|x\| \leqslant \delta\}.$$

Theorem 3.4.2. *Let α be a strongly continuous action by linear isometries of a Lie group G on a Banach space $\mathscr{E}$. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a nonzero subspace of $\mathfrak{g}$.*

Let $\mathscr{S} \subseteq \mathscr{E}$ be the subspace of $\mathscr{E}$ consisting of α-differentiable elements of $\mathscr{E}$ with respect to $\mathfrak{h}$. We note that $\mathscr{S}$ is dense in $\mathscr{E}$.

Let $\|\cdot\|$ be a norm on $\mathfrak{h}$. For all $\xi \in \mathscr{S}$, the norm of the linear map:

$$\nabla\xi : X \in \mathfrak{h} \mapsto \nabla_X \xi = X(\xi)$$

is denoted by $|||\nabla\xi|||$.

If $\xi \in \mathscr{S}$*, then, for any* $\delta > 0$*:*

$$\begin{aligned}
|||\nabla\xi||| &= \sup\left\{ \frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h}\setminus\{0\} \right\} \\
&= \sup\left\{ \frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h}\setminus\{0\}, \|X\| \leqslant \delta \right\} \\
&= \limsup_{X\to 0} \frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|}.
\end{aligned}$$

Proof. A smoothing argument [3] proves that the set:

$$\left\{ \xi \in \mathscr{E} : t > 0 \mapsto \frac{\alpha^{\exp(tX)}\xi - \xi}{t} \text{ has a limit at 0 for all } X \in \mathfrak{g} \right\}$$

is dense in $\mathscr{E}$. Therefore, since $\mathscr{S}$ contains this set, $\mathscr{S}$ is dense in $\mathscr{E}$ as well.

Fix $\xi \in \mathscr{S}$. Let $X \in \mathfrak{h}$. We define:

$$F : t \in \mathbb{R} \mapsto \alpha^{\exp(tX)}\xi.$$

The function F is continuously differentiable, and in particular, $F(0) = \xi$ and $F(1) = \alpha^{\exp(X)}\xi$.

Moreover, using the fact that $t \in \mathbb{R} \mapsto \exp(tX)$ is a continuous group homomorphism:

$$\begin{aligned}
F'(t) &= \lim_{s\to 0} \frac{\alpha^{\exp((t+s)X)}\xi - \alpha^{\exp(tX)}\xi}{h} \\
&= \lim_{s\to 0} \frac{\alpha^{\exp(tX)}\left(\alpha^{\exp(hX)}\xi - \xi\right)}{h} \\
&= \alpha^{\exp(tX)}\nabla_X\xi.
\end{aligned}$$

Thus:

$$\begin{aligned}
\alpha^{\exp(X)}\xi - \xi &= \int_0^1 F'(t)\,dt \\
&= \int_0^1 \alpha^{\exp(tX)}\left(\nabla_X\xi\right)\,dt
\end{aligned}$$

so that:

$$\begin{aligned}
\frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} &= \frac{\left\|\int_0^1 F'(t)\,dt\right\|_{\mathscr{E}}}{\|X\|} \\
&\leqslant \frac{1}{\|X\|}\int_0^1 \left\|\alpha^{\exp(X)}\left(\nabla_X \xi\right)\right\|_{\mathscr{E}}\,dt \\
&= \frac{1}{\|X\|}\int_0^1 \|\nabla_X \xi\|_{\mathscr{E}}\,dt \text{ since } \alpha^{\exp(tX)} \text{ is an isometry by hypothesis,} \\
&\leqslant \frac{1}{\|X\|}\int_0^1 |||\nabla\xi|||\,\|X\|\,dt \\
&= |||\nabla\xi|||.
\end{aligned}$$

This proves that:

$$\sup\left\{\frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h}\setminus\{0\}\right\} \leqslant |||\nabla\xi|||.$$

On the other hand, let us now fix some $\delta > 0$. let us now assume that $\|X\| = 1$. We first note that:

$$\begin{aligned}
\nabla_X\xi &= F'(0) \\
&= \lim_{t\to 0}\frac{F(t)-F(0)}{t} \text{ where lim is used for the topology of } (\mathscr{E}, \|\cdot\|_{\mathscr{E}}), \\
&= \lim_{t\to 0}\frac{\alpha^{\exp(tX)}\xi - \xi}{t\|X\|} \\
&= \lim_{t\to 0}\frac{\alpha^{\exp(tX)}\xi - \xi}{\|tX\|}.
\end{aligned}$$

Thus for all $X \in \mathfrak{h}$ with $\|X\| = 1$, since $\|tX\| \leqslant \delta$ for all $t \in \mathbb{R}$ with $|t| < \delta$:

$$\|\nabla_X\xi\| \leqslant \sup\left\{\frac{\left\|\alpha^{\exp(Y)}\xi - \xi\right\|_{\mathscr{E}}}{\|Y\|} : Y \in \mathfrak{h}\setminus\{0\}, \|Y\| \leqslant \delta\right\}$$

and thus:

$$\begin{aligned}
|||\nabla\xi||| &\leqslant \sup\left\{\frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h}\setminus\{0\}, \|X\| \leqslant \delta\right\} \\
&\leqslant \sup\left\{\frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h}\setminus\{0\}\right\}.
\end{aligned}$$

We have thus concluded our argument, as the function:

$$\delta \in (0,\infty) \mapsto \sup\left\{\frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|} : X \in \mathfrak{h}\setminus\{0\}, \|X\| \leqslant \delta\right\}$$

has been shown to be constant. □

We now make an important and non-trivial observation. Among the seminorms constructed via Theorem (3.4.2), we find L-seminorms built originally by Rieffel [40] and which play the fundamental role of providing a quantum metric to quantum tori. Thus Theorem (3.4.2) provide a reasonable generalization of Rieffel's approach to L-seminorms to the noncompact framework.

Corollary 3.4.3. *Let α be a strongly continuous action by linear isometries of a compact connected Lie group G on a Banach space $\mathscr{E}$. As a compact Lie group, G admits an* Ad*-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. Let $\|\cdot\|$ be the norm associated with $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. For any $g \in G$, since G is connected and compact, we may define $\ell(g)$ as the distance from 1_G to g for the Riemannian metric induced by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.*

If $\xi \in \mathscr{S}$ then:

$$\sup\left\{\frac{\|\alpha^g\xi - \xi\|_{\mathscr{E}}}{\ell(g)} : g \in G \setminus \{1_G\}\right\} = |||\nabla\xi|||.$$

Proof. As G is a compact group, it admits a right Haar probability measure μ. Let $\langle\cdot,\cdot\rangle$ be any inner product on $\mathfrak{g}$. If we set, for all $X, Y \in \mathfrak{g}$:

$$\langle X, Y\rangle_G = \int_G \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y\rangle \, d\mu(g)$$

then one easily verifies that $\langle\cdot,\cdot\rangle_G$ is an Ad-invariant inner product on $\mathfrak{g}$.

Now, we endow G with the Riemannian metric induced by left translation of the inner product $\langle\cdot,\cdot\rangle_G$. As this metric is induced by an Ad-invariant inner product, it is in fact right invariant as well.

In particular, G, as a connected compact Riemannian manifold, is geodesically complete by Hopf-Rinow theorem. As a first application, we let $\ell(g)$ be the distance from 1_G to g in G for this Riemannian metric, for all $g \in G$. As a second application, we note that the Riemannian exponential map of G for our metric is indeed surjective.

It is now possible to check that the exponential map for the Lie group G and the exponential map for the Riemannian metric coincide. This is done by checking that the Riemannian exponential map defines a 1-parameter subgroup of G.

With this in mind, we conclude that for all $X \in \mathfrak{g}$, we have:

$$\ell(\exp(X)) = \inf\{\|Y\| : \exp(X) = \exp(Y)\}.$$

We note that the Lie exponential map is certainly not injective, at least as long as G is of dimension at least one, though this does not affect our conclusion.

Moreover, since G is a compact connected Lie group, exp is surjective since the Riemannian exponential is surjective. Thus, our corollary is proven using Theorem (3.4.2). ☐

Now, Rieffel proved in [40] that the obvious necessary condition for a seminorm of the type given in Corollary (3.4.3) to be a L-seminorm is, remarkably, sufficient as well. This fact is highly non-trivial as well, and we record it here as it will be the source of quantum metrics we put on quantum tori.

Theorem 3.4.4 ([40, Theorem 1.9])**.** *Let β be a strongly continuous group action by *-automorphisms of a compact group G on a unital C*-algebra $\mathfrak{A}$. Let ℓ be a continuous* *length function on G. For all $a \in \mathfrak{A}$, we define:*

$$\mathsf{L}(a) = \sup\left\{ \frac{\|\beta^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{e\} \right\},$$

allowing for this quantity to be infinite. Then the following are equivalent:

(1) $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space (which is necessarily Leibniz),*
(2) $\{a \in \mathfrak{A} : \forall g \in G \quad \beta^g(a) = a\} = \mathbb{C}1_{\mathfrak{A}}$.

We pause for two important observations. First of all, Rieffel's theorem does not place much requirement on the length function used: certainly it need not be from any sort of Lie structure, and in fact, the acting group G need not be a Lie group. None the less, if G is a Lie group and if we choose a bi-invariant Riemannian metric on G, then Corollary (3.4.3) applies and Rieffel's metric is also given by the differential calculus naturally induced by G on $\mathfrak{A}$. If G is a torus, i.e. a compact Abelian Lie group, then all choices of norms on the Lie algebra of G will provide bi-invariant metrics via translations in the usual manner, and thus in that case, the Lipschitz approach and the differential approaches coincide. This will be our setup in our main result.

Second of all, we note that the proof of Theorem (3.4.4) involves explicitly the fact that the spectral subspaces of the action β are finite dimensional under the condition of ergodicity [14]. This result is not trivial, and worse yet for our purpose, does not carry to locally compact group. In fact, besides the trivial representation, no irreducible representation of the Heisenberg group is finite dimensional — so we are as far as we can to apply the idea in [40]. In this paper, we shall focus on the Heisenberg modules, and we will prove in this case that the seminorms constructed in Theorem (3.4.2) have compact unit balls using quite different techniques.

The rest of this section introduces the general scheme to construct D-norms from Lie group actions which we will employ in this paper, and prove that this construction meets all our requirements except, maybe, for the compactness of the unit ball which, in the case of Heisenberg modules, will be the subject of our next section.

Theorem (3.4.2) thus suggests two expressions for L-seminorms and D-norms constructed from actions of Lie groups. However, only one of these expression will provide lower semicontinuous seminorms. The following proposition establishes that the inner quasi-Leibniz and modular quasi-Leibniz property hold for this choice.

Proposition 3.4.5. *Let β be the action of a compact connected Lie group G on a unital C*-algebra $\mathfrak{A}$ via *-automorphisms. Let α be the action by isometric $\mathbb{C}$-linear isomorphisms of a Lie group H on a Hilbert module $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ over $\mathfrak{A}$. We write $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras of G and H, and $\exp_G : \mathfrak{g} \to G$ and $\exp_H : \mathfrak{h} \to H$ be the respective Lie exponential maps of G and H.*

Let $\mathfrak{w}$ be a nonzero subspace of $\mathfrak{h}$. Let $\|\cdot\|_\flat$ be a norm on $\mathfrak{g}$ and $\|\cdot\|^\sharp$ be a norm on $\mathfrak{w} \subseteq \mathfrak{h}$.

We set for all $a \in \mathfrak{A}$:

$$\mathsf{L}(a) = \sup\left\{ \frac{\left\|\beta^{\exp(X)}a - a\right\|_{\mathfrak{A}}}{\|X\|_\flat} : X \in \mathfrak{g} \setminus \{0\} \right\},$$

and for all $\xi \in \mathscr{E}$:

$$\mathsf{D}(\xi) = \sup\left\{ \|\xi\|_{\mathscr{M}}, \frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{M}}}{\|X\|^\sharp} : X \in \mathfrak{w} \setminus \{0\} \right\}.$$

If there exist two linear maps $j : \mathfrak{w} \to \mathfrak{g}$ and $q : \mathfrak{g} \to \mathfrak{w}$ such that:

(1) *for all $\xi, \omega \in \mathscr{M}$ and $X \in \mathfrak{w}$:*

$$\beta^{\exp_G(X)}\langle \xi, \omega\rangle_{\mathscr{M}} = \left\langle \alpha^{\exp_H(j(X))}\xi, \alpha^{\exp_H(j(X))}\omega \right\rangle_{\mathscr{E}} \tag{3.4.1}$$

and:

$$\alpha^{\exp_H(X)}(a\xi) = \beta^{\exp_G(q(X))}(a)\alpha^{\exp_H(X)}\xi, \tag{3.4.2}$$

(2) *j is an isometry from $(\mathfrak{g}, \|\cdot\|_\flat)$ to $(\mathfrak{w}, \|\cdot\|^\sharp)$,*

(3) *q is a surjection of norm at most 1, i.e. $\|q(X)\|_\flat \leqslant \|X\|^\sharp$ for all $X \in \mathfrak{w}$,*

then:

(1) *L is a seminorm on a dense subspace of $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$, and moreover:*

$$\mathsf{L}(a) = 0 \iff \forall g \in G \quad \beta^g(a) = a,$$

(2) *D is a norm on a dense subspace of $(\mathscr{M}, \langle\cdot,\cdot\rangle_{\mathscr{M}})$ and $\mathsf{D}(\cdot) \geqslant \|\cdot\|_{\mathscr{M}}$,*

(3) *L and D are lower semicontinuous,*

(4) *for all $a \in \mathfrak{A}$ and $\xi \in \mathscr{M}$:*

$$\mathsf{D}(a\xi) \leqslant \|a\|_{\mathfrak{A}}\mathsf{D}(\xi) + \mathsf{L}(a)\|\xi\|_{\mathscr{M}},$$

(5) *for all $\xi, \omega \in \mathscr{M}$:*

$$\mathsf{L}\left(\langle \xi, \omega\rangle_{\mathscr{M}}\right) \leqslant \|\xi\|_{\mathscr{M}}\mathsf{D}(\omega) + \mathsf{D}(\xi)\|\omega\|_{\mathscr{M}}.$$

Proof. Let $\mathscr{S}_{\mathfrak{g}}(\mathfrak{A})$ be the subspace of $\mathfrak{A}$ consisting of all the β-differentiable elements with respect to $\mathfrak{g}$, and $\mathscr{S}_{\mathfrak{h}}(\mathscr{M})$ be the subspace of $\mathscr{M}$ consisting of all the α-differentiable elements of $\mathscr{M}$ with respect to $\mathfrak{w}$.

For any $a \in \mathscr{S}_{\mathfrak{g}}(\mathfrak{A})$, we define the linear map $\partial a : X \in \mathfrak{g} \mapsto X(a)$ whose norm is denoted by $|||\partial a|||_{\mathfrak{A}}^{\mathfrak{g}}$, where $\mathfrak{g}$ is endowed with $\|\cdot\|_\flat$. Since $\mathfrak{g}$ is finite dimensional, ∂a is continuous and thus has finite norm for all $a \in \mathscr{S}_{\mathfrak{g}}(\mathfrak{A})$.

For any $\xi \in \mathscr{S}_{\mathfrak{w}}(\mathscr{M})$, we also define $\nabla\xi : X \in \mathfrak{w} \mapsto X(\xi)$ whose norm is $|||\nabla\xi|||_{\mathscr{M}}^{\mathfrak{w}}$ where $\mathfrak{w}$ is endowed by $\|\cdot\|^\sharp$ — since $\mathfrak{w}$ is finite dimensional, the norm of $\nabla\xi$ is finite as well.

By Theorem (3.4.2), for all $a \in \mathscr{S}_{\mathfrak{g}}(\mathfrak{A})$ and for all $\xi \in \mathscr{S}_{\mathfrak{w}}(\mathscr{M})$, then:

$$\mathsf{L}(a) = |||\partial a|||_{\mathfrak{A}}^{\mathfrak{g}} < \infty \text{ and } \mathsf{D}(\xi) = |||\nabla\xi|||_{\mathscr{M}}^{\mathfrak{w}} < \infty.$$

Since $\mathscr{S}_{\mathfrak{g}}(\mathfrak{A})$ and $\mathscr{S}_{\mathfrak{w}}(\mathscr{E})$ are dense, we conclude that the domains of L and D are indeed dense.

Since $\mathsf{D}(\cdot) \geqslant \|\cdot\|_{\mathscr{M}}$ by construction, D is in particular a norm on its domain.

Moreover if $\mathsf{L}(a) = 0$ for some $a \in \mathfrak{A}$, we immediately conclude that $\beta^g a = a$ for all $g \in G$ since the exponential map of G is surjective.

The function $\xi \in \mathscr{M} \mapsto \frac{\alpha^{\exp(X)}\xi - \xi}{\|X\|^\sharp}$ is continuous for all $X \in \mathfrak{w} \setminus \{0\}$ and thus D is lower semi-continuous as the pointwise supremum of continuous functions. The same reasoning and conclusion applies to L.

We are left to prove the two forms of the Leibniz inequalities. We take the quickest path, which is a direct computation.

Let $\xi, \omega \in \mathscr{M}$. We compute:

$$\begin{aligned}
\mathsf{L}\left(\langle \xi, \omega\rangle_{\mathscr{E}}\right) &= \sup\left\{ \frac{\left\|\beta^{\exp(X)}\langle \xi,\omega\rangle_{\mathscr{E}} - \langle \xi,\omega\rangle_{\mathscr{E}}\right\|_{\mathfrak{A}}}{\|X\|_\flat} : X \in \mathfrak{g}\setminus\{0\}\right\}\\
&= \sup\left\{ \frac{\left\|\left\langle \alpha^{\exp(j(X))}\xi, \alpha^{\exp(j(X))}\omega\right\rangle_{\mathscr{E}} - \langle \xi,\omega\rangle_{\mathscr{E}}\right\|_{\mathfrak{A}}}{\|j(X)\|^\sharp} : X \in \mathfrak{g}\setminus\{0\}\right\}\\
&\leqslant \sup\left\{ \frac{\left\|\left\langle \alpha^{\exp(X)}\xi, \alpha^{\exp(X)}\omega\right\rangle_{\mathscr{E}} - \langle \xi,\omega\rangle_{\mathscr{E}}\right\|_{\mathfrak{A}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\}\\
&\leqslant \sup\left\{ \frac{\left\|\left\langle \alpha^{\exp(X)}\xi, \alpha^{\exp(X)}\omega\right\rangle_{\mathscr{E}} - \left\langle \alpha^{\exp(X)}\xi,\omega\right\rangle_{\mathscr{E}}\right\|_{\mathfrak{A}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\}\\
&\quad + \sup\left\{ \frac{\left\|\left\langle \alpha^{\exp(X)}\xi, \omega\right\rangle_{\mathscr{E}} - \langle \xi,\omega\rangle_{\mathscr{E}}\right\|_{\mathfrak{A}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\}\\
&\leqslant \sup\left\{ \frac{\left\|\alpha^{\exp(X)}\xi\right\|_{\mathscr{M}} \left\|\alpha^{\exp(X)}\omega - \omega\right\|_{\mathscr{E}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\}\\
&\quad + \sup\left\{ \frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\} \|\omega\|_{\mathscr{M}}\\
&\leqslant \|\xi\|_{\mathscr{M}} \sup\left\{ \frac{\left\|\alpha^{\exp(X)}\omega - \omega\right\|_{\mathscr{E}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\}\\
&\quad + \sup\left\{ \frac{\left\|\alpha^{\exp(X)}\xi - \xi\right\|_{\mathscr{E}}}{\|X\|^\sharp} : X \in \mathfrak{w}\setminus\{0\}\right\} \|\omega\|_{\mathscr{M}}\\
&= \|\xi\|_{\mathscr{M}} \mathsf{D}(\omega) + \mathsf{D}(\xi)\, \|\omega\|_{\mathscr{M}} .
\end{aligned}$$

Now, let $a \in \mathfrak{A}$ and $\xi \in \mathscr{M}$. We compute:

$$
\begin{aligned}
\sup&\left\{\frac{\left\|\alpha^{\exp(X)}\left(a\xi\right)-a\xi\right\|_{\mathscr{M}}}{\|X\|^{\sharp}} : X\in\mathfrak{w}\setminus\{0\}\right\}\\
&=\sup\left\{\frac{\left\|\beta^{\exp(q(X))}(a)\alpha^{\exp(X)}\left(\xi\right)-a\xi\right\|_{\mathscr{M}}}{\|X\|^{\sharp}} : X\in\mathfrak{w}\setminus\{0\}\right\}\\
&\leqslant\sup\left\{\frac{\left\|\beta^{\exp(q(X))}(a)\alpha^{\exp(X)}\left(\xi\right)-a\alpha^{\exp(X)}\xi\right\|_{\mathscr{M}}}{\|q(X)\|_{\flat}} : X\in\mathfrak{w}\setminus\ker q\right\}\\
&\quad+\sup\left\{\frac{\left\|a\alpha^{\exp(X)}\left(\xi\right)-a\xi\right\|_{\mathscr{M}}}{\|X\|^{\sharp}} : X\in\mathfrak{w}\setminus\{0\}\right\}\\
&\leqslant\sup\left\{\frac{\left\|\beta^{\exp(q(X))}(a)-a\right\|_{\mathscr{M}}}{\|X\|_{\flat}} : X\in\mathfrak{g}\setminus\{0\}\right\}\|\xi\|_{\mathscr{M}}+\|a\|_{\mathfrak{A}}\mathsf{D}(\xi)\\
&=\mathsf{L}(a)\|\xi\|_{\mathscr{M}}+\|a\|_{\mathfrak{A}}\mathsf{D}(\xi),
\end{aligned}
$$

as desired. □

Thus, Proposition (3.4.5) shows that if we follow the scheme suggested by Theorem (3.4.2), then we obtain potential D-norms on modules. The missing property is the compactness of the closed unit ball for the D-norm candidate.

We conclude our section by connecting our metric framework with the noncommutative differential framework of connections on modules. Let us use the notations of Proposition (3.4.5). A direct computation shows that for all $X\in\mathfrak{w}$, the following holds:

$$\nabla_X(a\xi)=q(X)a\cdot\xi+a\nabla_X\xi \tag{3.4.3}$$

while for all $X\in\mathfrak{g}$, we also have:

$$X(\langle\xi,\omega\rangle_{\mathscr{M}})=\langle j(X)\xi,\omega\rangle_{\mathscr{M}}+\langle\xi,j(X)\omega\rangle_{\mathscr{M}}. \tag{3.4.4}$$

We also denote $\mathfrak{A}\otimes\mathfrak{g}^*$ by Ω_1 and the space of β-differentiable elements of $\mathfrak{A}$ by $\mathfrak{A}_1$. We define $\partial:\mathfrak{A}_1\to\Omega_1$ by setting, for all $a\in\mathfrak{A}_1$:

$$\partial a : X\in\mathfrak{g}\mapsto X(a).$$

We observe trivially that Ω_1 is an $\mathfrak{A}$-$\mathfrak{A}$-bimodule and that ∂ is a derivation, i.e. $\partial(ab)=a\partial(b)+\partial(a)b$ for all $a,b\in\mathfrak{A}_1$.

We first note that to get an interesting connection, we want q to be injective, i.e. $\mathfrak{g}$ and $\mathfrak{w}$ to be isomorphic. It is always possible to increase the dimension of $\mathfrak{g}$ (the Lie algebra structure is actually not involved in the computations to follow, so this is always possible), but this would amount to define $\partial_X=0$ for all vector X not in $\mathfrak{g}$, and this is rather awkward and artificial.

Since, for the differential picture, the norms $\|\cdot\|_\flat$ and $\|\cdot\|^\sharp$ do not play a role in the construction of the connection, we will for now identify $\mathfrak{g}$ and $\mathfrak{w}$ and j and q with the identity map.

With this assumption, Expressions (3.4.3) translates to the operator $\nabla : \mathscr{M} \rightarrow \mathscr{M} \otimes \mathfrak{g}^*$, defined by:

$$\nabla(\xi) : X \in \mathfrak{g} \mapsto \nabla_X \xi$$

for all α-differentiable $\xi \in \mathscr{M}$ with respect to $\mathfrak{g}$, to be a noncommutative connection. We indeed easily check that for all $a \in \mathfrak{A}$ and $\xi \in \mathscr{M}$:

$$\nabla(a\xi) = a\nabla(\xi) + \partial(a)\xi.$$

Expression (3.4.4) means that the connection ∇ is hermitian, i.e. it is compatible with the noncommutative equivalent of a metric on the quantum vector bundle $\mathscr{M}$. It is tempting to call ∇ a Levi-Civita connection, although we do not address here the computation of the torsion of ∇. Nonetheless, we see that our structure provides a noncommutative Riemannian geometry. This is the structure which inspired our definition of metrized quantum vector bundle, and we now can see how it is implemented through our main example.

In summary, we have constructed a natural D-norm candidate on modules carrying certain Lie group actions. The key difficulty, of course, regards the compactness of the unit ball of such a D-norm.

3.5. **A D-norm from a connection on Heisenberg modules.** We now define our D-norms on Heisenberg modules. Our method employs the idea of Theorem (3.4.2) and Proposition (3.4.5), where the actions of the Heisenberg group on Heisenberg modules defines a norm which restricts to the operator norm of a connection constructed via the associated action of the Heisenberg Lie algebra.

As noted at the end of the previous section, we want to only work with a subspace of the Heisenberg Lie algebra to build our D-norm and its associated connection, since the central element of the Heisenberg Lie algebra does not act, so to speak, as a derivation — it simply acts by multiplication by a scalar. We follow a pattern which is common in the literature on the Heisenberg group: we only consider the action of the subspace $\operatorname{span}\{P, Q\}$ in the Lie algebra $\mathfrak{H}$.

We thus endow $\operatorname{span}\{P, Q\}$ with a norm. If we were to construct a metric on the Heisenberg group using this data — by defining the length of a curve whose tangent vector at (almost) every point lies in $\operatorname{span}\{P, Q\}$ in the usual manner by integrating the norm of the tangent vector along the curve, and then defining the distance between two points as the infimum of the length of all so-called horizontal curves — we would actually obtain a sub-Finslerian metric (if our choice of norm comes from a Hilbert space structure, we would have a sub-Riemannian structure and our construction would give rise to a Carnot-Carathédory distance on the Heisenberg group).

However, as discussed, we do not transport the Carnot-Carathédory metric from the Heisenberg group via its action in this paper. We prefer to carry the norm of the subspace $\operatorname{span}\{P, Q\}$ of the Heisenberg Lie algebra to our modules. This approach means that we work with a connection, and seems more natural. In essence, the Carnot-Caratheodory is the metric obtained on the group while our D-norms are the quantum metrics obtained on our modules; as the acting group is not compact, we have no reason to expect them to agree.

With this in mind, we now introduce:

Definition 3.5.1. Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N}$ with $d > 0$. Let $\theta \in \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. We endow the Heisenberg module $\mathscr{H}_\theta^{p,q,d}$ with the norm:

$$\mathsf{D}_\theta^{\|\cdot\|,p,q,d}(\xi) = \sup\left\{\|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth,d}^{\exp_{\mathbb{H}_3}(xP+yQ)}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\}\right\}$$

where $\eth = \theta - \frac{p}{q}$.

We now lighten our notation for the rest of our paper.

Convention 3.5.2. We endow $\mathbb{R}^2$ with a fixed norm $\|\cdot\|$ for the rest of this paper. We shall denote $\mathsf{D}_\theta^{\|\cdot\|,p,q,d}$ simply by $\mathsf{D}_\theta^{p,q,d}$, as the norm on $\mathbb{R}^2$ will not be understood. We emphasize that $\|\cdot\|$ is *independent* of any of the parameters p, q, d and θ.

The norm $\|\cdot\|$ on $\mathbb{R}^2$ provides us with a continuous length function on $\mathcal{A}_\theta$ for all $\theta \in \mathbb{R}$. This length function arises from the invariant Finslerian metric induced by $\|\cdot\|$. A direct computation simply shows that:

$$\ell(\exp(ix), \exp(iy)) = \inf\{\|(x + 2n\pi, y + 2m\pi)\| : n, m \in \mathbb{Z}^2\}.$$

For all $\theta \in \mathbb{R}$, we denote by L_θ the L-seminorm on $\mathcal{A}_\theta$ associated with the action β_θ on $\mathcal{A}_\theta$ and the length function ℓ via [40, Theorem 1.9]. We note that since $\mathbb{T}^2$ is compact and Abelian, Corollary (3.4.3) implies that for all $a \in \mathcal{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{\frac{\|\beta_\theta^{\exp_{\mathbb{T}^2}(x,y)}\xi - \xi\|_{\mathcal{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\}\right\}$$

and L_θ agrees with the operator norm of derivative for the natural differential calculus defined by β_θ on β_θ-differentiable elements. We refer to the previous section for a discussion of these matters.

We begin by listing various equivalent expressions for our D-norm candidates, as we shall use whichever may prove useful in this paper.

Remark 3.5.3. We recall from Notation (3.1.12) that:

$$\exp_{\mathbb{H}_3}(xP + yQ) = \left(x, y, \frac{1}{2}xy\right)$$

for all $x, y \in \mathbb{R}$.

For all $p,q \in \mathbb{N}$, $d \in q\mathbb{N}$ with $d > 0$, $\theta \in \mathbb{R} \setminus \{pq^{-1}\}$ and $\xi \in \mathscr{H}_\theta^{p,q,d}$, the following identities hold:

$$\begin{aligned}\mathsf{D}_\theta^{p,q,d}(\xi) &= \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\alpha_{\eth,d}^{x,y,\frac{1}{2}xy}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\} \\ &= \sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\sigma_{\eth,d}^{x,y}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.\end{aligned}$$

Proposition 3.5.4. *Let $p,q \in \mathbb{N}$ and $d \in q\mathbb{N}$ with $d > 0$. Let $\theta \in \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$.*

We endow $\operatorname{span}\{P,Q\}$ with the norm $2\pi|\eth|\|\cdot\|$. We also define, for all $(x,y) \in \mathbb{R}^2$ and $\xi \in \mathscr{S}_\theta^{p,q,d}$:

$$\begin{aligned}\nabla_{x,y}^{\eth}\xi &= \lim_{t\to 0} \frac{\alpha_{\eth,d}^{\exp_{\mathbb{H}_3}(t(xP+yQ))}\xi - \xi}{t} \\ &= \lim_{t\to 0} \frac{\alpha_{\eth,d}^{tx,ty,\frac{1}{2}t^2xy}\xi - \xi}{t}.\end{aligned}$$

To ease notation, let $|||\cdot|||_{2\pi|\eth|}$ denote the operator norm for linear maps from $(\mathbb{R}^2, 2\pi|\eth|\|\cdot\|)$ to $(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}})$.

We record:

1. *$\mathsf{D}_\theta^{p,q,d}$ is a norm on a dense subspace of $\mathscr{H}_\theta^{p,q,d}$,*
2. *For all $\xi \in \mathscr{S}_\theta^{p,q,d}$ and for all $\delta > 0$, the following expressions hold:*
$$\begin{aligned}\mathsf{D}_\theta^{p,q,d}(\xi) &= \max\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, |||\nabla_\eth\xi|||_{2\pi|\eth|} \right\} \\ &= \sup\left\{ \frac{\|\sigma_{\eth,d}^{x,y}\xi - \xi\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2, 0 < \|(x,y)\| < \delta \right\} \\ &= \limsup_{(x,y)\to 0} \frac{\left\|\sigma_{\eth,d}^{x,y}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|}.\end{aligned}$$
3. *If $a \in \mathcal{A}_\theta$ and $\xi \in \mathscr{H}_\theta^{p,q,d}$ then:*
$$\mathsf{D}_\theta^{p,q,d}(a\xi) \leqslant \|a\|_{\mathcal{A}_\theta}\mathsf{D}_\theta^{p,q,d}(\xi) + \mathsf{L}_\theta(a)\|\xi\|_{\mathscr{H}_\theta^{p,q,d}}.$$
4. *If $\xi, \omega \in \mathscr{H}_\theta^{p,q,d}$ then:*
$$\mathsf{L}_\theta\left(\langle\xi,\omega\rangle_{\mathscr{H}_\theta^{p,q,d}}\right) \leqslant \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}\mathsf{D}_\theta^{p,q,d}(\omega) + \mathsf{D}_\theta^{p,q,d}(\xi)\|\omega\|_{\mathscr{H}_\theta^{p,q,d}}.$$

Proof. The Lie algebra of $\mathbb{T}^2$ is $\mathbb{R}^2$ with the exponential map given as:

$$\exp_{\mathbb{T}^2} : (x,y) \in \mathbb{R}^2 \mapsto (\exp(ix), \exp(iy)).$$

Now, the map $q : (x,y) \in \mathbb{R}^2 \mapsto (2i\pi\eth x, 2i\pi\eth y)$ satisfies, according to Lemma (3.3.2), the relation:

$$\beta_\theta^{\exp_{\mathbb{T}^2}(q(x,y))}\langle\xi,\omega\rangle_{\mathscr{H}_\theta^{p,q,d}} = \left\langle \alpha_{\eth,d}^{\exp_{\mathbb{H}_3}(x,y,0)}\xi, \alpha_{\eth,d}^{\exp_{\mathbb{H}_3}(x,y,0)}\omega \right\rangle_{\mathscr{H}_\theta^{p,q,d}}.$$

and, according to Lemma (3.3.6), the relation:

$$\alpha_{\eth,d}^{\exp_{\mathbb{H}_3}(x,y,0)}(a\xi) = \beta_\theta^{\exp_{\mathbb{T}^2}(q(x,y))}(a)\alpha_{\eth,d}^{\exp_{\mathbb{H}_3}(x,y,0)}(\xi).$$

In order to apply Proposition (3.4.5), since q is indeed a linear isomorphism, we endow $\operatorname{span}\{P,Q\}$ with the norm:

$$\|xP + yQ\|_* = 2\pi|\eth|\|(x,y)\|.$$

We now are in the setting of Proposition (3.4.5), which allows us to conclude all but Assertion (2) in our proposition. Assertion (2), in turn, follows from Theorem (3.4.2), with our choice of norm. ☐

We now turn to the remaining, main issue of the compactness of the closed unit balls for our D-norm candidates. The strategy we employ relies on a particular source of finite rank operators naturally associated with the Schödinger representations of $\mathbb{R}^2$ via the Weyl calculus.

Our first step is to introduce the convolution-like operators at the core of our analysis.

Lemma 3.5.5. *Assume Hypothesis (3.3.1). If $f \in L^1(\mathbb{R}^2)$ and:*

$$\sigma_{\eth,d}^f = \iint_{\mathbb{R}^2} f(x,y)\alpha_{\eth,d}^{x,y,\frac{xy}{2}}\,dxdy$$

then $\sigma_{\eth,d}^f$ is a well-defined operator on $\mathscr{H}_\theta^{p,q,d}$ and $\left|\left|\left|\sigma_{\eth,d}^f\right|\right|\right|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \|f\|_{L^1(\mathbb{R}^2)}$.

Proof. Let $\xi \in \mathscr{H}_\theta^{p,q,d}$. Using Lemma (3.3.4), i.e. the fact that $\alpha_{\eth,d}^{x,y,u}$ is an isometry of $\mathscr{H}_\theta^{p,q,d}$ for all $(x,y,u) \in \mathbb{H}_3$, we simply compute:

$$\begin{aligned}\iint_{\mathbb{R}^2}\left\|f(x,y)\alpha_{\eth,d}^{x,y,\frac{xy}{2}}(\xi)\right\|_{\mathscr{H}_\theta^{p,q,d}}dxdy &= \iint_{\mathbb{R}^2}|f(x,y)|\left\|\alpha_{\eth,d}^{x,y,\frac{xy}{2}}(\xi)\right\|_{\mathscr{H}_\theta^{p,q,d}}dxdy\\ &= \iint_{\mathbb{R}^2}|f(x,y)|\,\|\xi\|_{\mathscr{H}_\theta^{p,q,d}}\,dxdy\\ &= \|f\|_{L^1(\mathbb{R}^2)}\|\xi\|_{\mathscr{H}_\theta^{p,q,d}}.\end{aligned}$$

Thus $\sigma_{\eth,d}^f$ is well-defined, and moreover:

$$\left\|\sigma_{\eth,d}^f(\xi)\right\|_{\mathscr{H}_\theta^{p,q,d}} = \left\|\iint_{\mathbb{R}^2} f(x,y)\sigma_{\eth,d}^{x,y}(\xi)\,dxdy\right\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \|f\|_{L^1(\mathbb{R}^2)}\|\xi\|_{\mathscr{H}_\theta^{p,q,d}}.$$

This completes our proof. ☐

We now prove the first of two core lemmas of this section, which provides us with a mean to approximate elements in Heisenberg modules using our convolution-type operators, in a manner which is uniform in our prospective D-norms. This lemma is an adjustment of [49] to our context.

Lemma 3.5.6. *Assume Hypothesis (3.3.1). Let $\varepsilon > 0$. If $f : \mathbb{R}^2 \to [0,\infty)$ is measurable and satisfies:*

(1) $\int_{\mathbb{R}^2} f = 1$,
(2) $\iint_{\mathbb{R}^2} f(x,y)\|(x,y)\|\, dxdy \leqslant \frac{\varepsilon}{2\pi|\eth|}$,

then for all $\xi \in \mathscr{H}_\theta^{p,q,d}$:

$$\left\|\xi - \sigma^f_{\eth,d}\xi\right\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \varepsilon \mathsf{D}_\theta^{p,q,d}(\xi).$$

Proof. If $\xi \in \mathscr{H}_\theta^{p,q,d}$, then:

$$\begin{aligned}
\left\|\xi - \sigma^f_{\eth,d}\xi\right\|_{\mathscr{H}_\theta^{p,q,d}} &= \left\|\iint_{\mathbb{R}^2} f(x,y)\xi\, dxdy - \iint_{\mathbb{R}^2} f(x,y)\alpha_{\eth,d}^{x,y,\frac{xy}{2}}\xi\, dxdy\right\|_{\mathscr{H}_\theta^{p,q,d}} \\
&\leqslant \iint_{\mathbb{R}^2} f(x,y)\|\xi - \alpha_{\eth,d}^{x,y,\frac{xy}{2}}\xi\|_{\mathscr{H}_\theta^{p,q,d}}\, dxdy \\
&\leqslant \iint_{\mathbb{R}^2} f(x,y)2\pi|\eth|\|(x,y)\|\frac{\|\xi - \alpha_{\eth,d}^{x,y,\frac{xy}{2}}\xi\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|}\, dxdy \\
&\leqslant \iint_{\mathbb{R}^2} f(x,y)2\pi|\eth|\|(x,y)\|\mathsf{D}_\theta^{\rho}(\xi)\, dxdy \\
&= \mathsf{D}_\theta^{p,q,d}(\xi)\left(2\pi|\eth|\frac{\varepsilon}{2\pi|\eth|}\right) \\
&= \varepsilon\mathsf{D}_\theta^{\rho}(\xi),
\end{aligned}$$

as desired. □

We now ensure that we indeed have an ample source of functions which meet the hypothesis of Lemma (3.5.6).

Notation 3.5.7. If (E,d) is a metric space then the closed ball $\{x \in E : d(x_0,x) \leqslant r\}$ of center $x_0 \in E$ and radius $r \geqslant 0$ is denoted by $E[x_0,r]$.

The following lemma is valid for any norm on $\mathbb{R}^2$; we shall work within our context with the fixed norm $\|\cdot\|$.

Lemma 3.5.8. *For all $n \in \mathbb{N}$, let $\psi_n : \mathbb{R}^2 \to [0,\infty)$ be an integrable function supported on $\mathbb{R}^2\left[0,\frac{1}{n+1}\right]$ and with $\int_{\mathbb{R}^2}\psi_n = 1$.*

If $f : \mathbb{R}^2 \to [0,\infty)$ is integrable on some ball centered at 0 in $(\mathbb{R}^2,\|\cdot\|)$, and f continuous at 0, then:

$$\lim_{n\to\infty}\iint_{\mathbb{R}^2}\psi_n(x,y)f(x,y)\, dxdy = f(0).$$

Proof. Let $\delta > 0$ such that f is integrable on $\mathbb{R}^2[0,\delta]$.

Let $\varepsilon > 0$. Since f is continuous at 0, there exists $\delta_c > 0$ such that $|f(x) - f(0)| \leqslant \varepsilon$ for all $x \in \mathbb{R}^2[0,\delta_c]$.

Let $N \in \mathbb{N}$ be chosen so that $\frac{1}{N+1} \leqslant \min\{\delta,\delta_c\}$. For all $n > N$, we first note that since ψ_n is supported on a subset of $\mathbb{R}^2[0,\delta]$, the function $\psi_n f$ is integrable on $\mathbb{R}^2$. Moreover for all $n \geqslant N$:

$$
\begin{aligned}
\left|\iint_{\mathbb{R}^2} \psi_n(x,y) f(x,y)\, dxdy - f(0)\right| &\leqslant \int_{\mathbb{R}^2} |\psi_n(x,y)(f(x,y)-f(0))|\, dxdy \\
&= \iint_{\mathbb{R}^2[0,n^{-1}]} |\psi_n(x,y)||f(x,y)-f(0)|\, dxdy \\
&\leqslant \iint_{\mathbb{R}^2[0,n^{-1}]} \psi_n(x,y)\varepsilon\, dxdy \leqslant \varepsilon.
\end{aligned}
$$

Thus we have shown that $\lim_{n\to\infty} \int_{\mathbb{R}^2} \psi_n(x,y) f(x,y)\, dxdy = f(0)$. □

As a quick digression, which will turn out to be useful in our last section, we note that while in general, the action of the Heisenberg groups on Heisenberg modules is not by isometry of our D-norms, we can use our approximation operators as near isometries:

Lemma 3.5.9. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $d \in \mathbb{N}\setminus\{0\}$. For all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $f : \mathbb{R}^2 \to \mathbb{R}_+$ is an integrable function supported on a $\mathbb{R}^2[0,\delta]$, and if $\eth \neq 0$, then for all $\xi \in \mathcal{S}(\mathbb{C}^d)$, we have:*

$$
\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}\left(\sigma^{f}_{\eth,d}\xi\right) \leqslant (1+\varepsilon)\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\xi).
$$

Proof. Let $\eth \in \mathbb{R}\setminus\{0\}$ and let $\theta = \frac{p}{q}+\eth$. We first record that for all $(x,y,u),(z,w,v) \in \mathbb{H}_3$:

$$
\alpha^{x,y,u}_{\eth,d}\alpha^{z,w,v}_{\eth,d} = \exp\left(2i\pi\eth(xw-zy)\right)\alpha^{z,w,v}_{\eth,d}\alpha^{x,y,u}_{\eth,d}.
$$

Next, we denote by $\|\cdot\|_2$ the standard Euclidean norm on $\mathbb{R}^2$ and, since $\mathbb{R}^2$ is finite dimensional, we can find $k > 0$ such that $\|\cdot\|_2 \leqslant k\|\cdot\|$. For all $x,y,z,w \in \mathbb{R}$, we then compute:

$$
\begin{aligned}
|\exp\left(2i\pi\eth(xw-zy)\right) - 1| &= 2|\sin(\eth\pi(xw-zy))| \\
&\leqslant 2\pi|\eth||xw-yz| \\
&\leqslant 2\pi|\eth|\|(x,y)\|_2\|(z,w)\|_2 \\
&\leqslant 2\pi k|\eth|\|(x,y)\|\|(z,w)\|_2.
\end{aligned}
$$

Let $\varepsilon > 0$. Let f be an integrable function supported on $\mathbb{R}^2[0,\delta]$.
For all $(z,w) \in \mathbb{R}^2[0,\delta]$ and $(x,y) \in \mathbb{R}^2$ with $\|(x,y)\| \leqslant \delta = \frac{\varepsilon}{k}$, we compute:

$$
\begin{aligned}
&\left\|\alpha^{x,y,\frac{xy}{2}}_{\eth,d}\left(\iint_K f(z,w)\alpha^{z,w,\frac{zw}{2}}_{\eth,d}\xi\, dxdy\right) - \iint f(z,w)\alpha^{z,w,\frac{zw}{2}}_{\eth,d}\xi\, dxdy\right\|_{\mathscr{H}^{p,q,d}_{\theta}} \\
&\quad\leqslant \iint_K f(z,w)\left|\exp\left(2i\pi\eth(xy-zw)\right)-1\right|\left\|\alpha^{z,w,\frac{zw}{2}}_{\eth,d}\alpha^{x,y,\frac{xy}{2}}_{\eth,d}\xi\right\|_{\mathscr{H}^{p,q,d}_{\theta}} dzdw \\
&\quad\quad+ \iint_K f(z,w)\left\|\alpha^{z,w,\frac{zw}{2}}_{\eth,d}\left(\alpha^{x,y,\frac{xy}{2}}_{\eth,d}\xi - \xi\right)\right\|_{\mathscr{H}^{p,q,d}_{\theta}} dzdw \\
&\quad\leqslant 2\pi k|\eth|\delta\|(x,y)\|\iint_K f\|\xi\|_{\mathscr{H}^{p,q,d}_{\theta}} + \iint_K f 2\pi|\eth|\|(x,y)\|\mathsf{D}^{p,q,d}_{\theta}(\xi) \\
&\quad\leqslant (k\delta+1)\, 2\pi|\eth|\|(x,y)\|\mathsf{D}^{p,q,d}_{\theta}(\xi) \\
&\quad\leqslant (\varepsilon+1)2\pi|\eth|\|(x,y)\|\mathsf{D}^{p,q,d}_{\theta}(\xi).
\end{aligned}
$$

By Definition (3.5.1), we now have:

$$\begin{aligned}\mathsf{D}(\sigma^f_{\eth,d}\xi) &= \sup\left\{ \frac{\left\| \alpha^{x,y,\frac{xy}{2}}_{\eth,d} \sigma^f_{\eth,d}\xi - \sigma^f_{\eth,d}\xi \right\|_{\mathscr{H}^{p,q,d}_\theta}}{2\pi|\eth|\|(x,y)\|} : x,y \in \mathbb{R}\setminus\{0\} \right\} \\ &\leqslant (\varepsilon+1)\mathsf{D}^{p,q,d}_\theta(\xi).\end{aligned}$$

Our lemma is now proven. ☐

We are now ready to prove the second core lemma of this section. We begin with an explanation of the ideas and reasons behind this lemma.

By a compact operator on a Banach space $(E, \|\cdot\|_{\mathbb{C}^d})$, we mean as usual an operator which maps bounded subsets of E to totally bounded subsets of E.

The map $f \in L^1(\mathbb{R}^2) \mapsto \sigma^f_{\eth,d}$ is a *-representation of the twisted convolution algebra $L^1(\mathbb{R}^2)$ for the convolution product defined for all $f, g \in L^1(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ by:

$$f *_\eth g(x) = \int_{\mathbb{R}^2} f(y)g(x-y)\mathrm{e}_\eth(y, x-y)\,dy$$

and the involution:

$$f \in L^1(\mathbb{R}^2) \mapsto f^* = x \in \mathbb{R}^2 \mapsto \overline{f(-x)},$$

as can be directly checked, or is established in [10]. It is an important, well-known fact [10, Theorem 1.30] that this representation is valued in the algebra of compact operators on $L^2(\mathbb{R})\otimes\mathbb{C}^d$, and is faithful; the completion of $(L^1(\mathbb{R}^2), *_\eth, *)$ for the norm $f \in \ell^1(\mathbb{Z}^2) \mapsto \|f\|_{C^*(\mathbb{R}^2,\mathrm{e}_\eth)} = \left|\left|\left|\sigma^f_{\eth,1}\right|\right|\right|_{L^2(\mathbb{R})}$ is the entire algebra of compact operators.

The fact that $\sigma^f_{\eth,d}$ is compact as an operator of $L^2(\mathbb{R})\otimes\mathbb{C}^d$ does not immediately imply that it is compact for the Banach space $\left(\mathscr{H}^{p,q,d}_\theta, \|\cdot\|_{\mathscr{H}^{p,q,d}_\theta}\right)$ since in general, we only know that $\|\cdot\|_{L^2(R)} \leqslant \|\cdot\|_{\mathscr{H}^{p,q,d}_\theta}$. We thus must prove compactness of these operators for our C^*-Hilbert norm. However, we can extract the essential tools for our work from the expansive work on Laguerre expansion of functions and the study of the Moyal plane. We will prove that, at least when f is a radial function, then we can approximate $\sigma^f_{\eth,d}$ by finite rank operators, in norm. To this end, we need a supply of finite rank operators, which provide a mean to approximate any $\sigma^f_{\eth,d}$ for f radial. The theory of the quantum harmonic oscillator provides us with a well-suited family of finite rank projections, obtained as $\sigma^\psi_{\eth,d}$ for ψ a properly scaled Laguerre function [10, Ch. 1, sec. 9].

To obtain the desired approximation result, however, we need to approximate our radial functions in the norm of $L^1(\mathbb{R}^2)$ using functions obtained from Laguerre functions. As Laguerre functions form an orthonormal basis for some L^2 space, we certainly do have a Laguerre expansion which converges in some L^2 norm, but convergence in $L^1(\mathbb{R}^2)$ is highly not trivial.

The work of Sundaram Thangaveru in [50] comes to our rescue, however, by proving that we may obtain the desired convergence if we replace the Laguerre expansion series by the sequence of its Césaro averages. We now formalize our discussion in the next key lemma.

Lemma 3.5.10. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is a function such that $r \in \mathbb{R} \mapsto rf(r)$ is Lebesgue integrable, and if we set:*

$$f^\circ : (x,y) \in \mathbb{R}^2 \mapsto f\left(\sqrt{x^2+y^2}\right),$$

then the operator $\sigma_{\eth,d}^{f^\circ}$ is a compact operator for the Banach space $\left(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}\right)$.

Proof. Our goal is to write $\sigma_{\eth,d}^{f^\circ}$ as a limit, in the operator norm, of finite rank operators. To this end, let us first assume that $\eth > 0$ and for all $n \in \mathbb{N}$, we let $\psi_\eth^n$ be the n^{th} Laguerre function defined for all $r \in [0,\infty)$ by:

$$\psi_\eth^n(r) = \eth \exp\left(-\frac{\pi\eth r^2}{2}\right) \mathrm{L}_n\left(\pi\eth r^2\right),$$

where L_n is the n^{th} Laguerre polynomials, given for all $x \in \mathbb{R}$ by:

$$\mathrm{L}_n(x) = \sum_{j=1}^{n} \frac{(-1)^j}{j!}\binom{n}{n-j} x^j.$$

Note that these functions are given in [50, (6.1.17)] for $\eth = \frac{1}{\pi}$. An observation which will be important for us in later proofs is that $\psi_\eth^n = \eth\psi_1^n(\sqrt{\eth}\cdot)$, i.e. we can obtain all the Laguerre functions we are considering via a simple rescaling.

By slight abuse of notation, we denote by $L^p(\mathbb{R}_+, rdr)$ the p-Lebesgue space for the measure defined, for all measurable $f : [0,\infty] \to [0,\infty)$, by $\int_0^\infty f(r)\,rdr$. In particular, note that the inner product of $L^2(\mathbb{R}_+, rdr)$ is given for any two $f, g \in L^2(\mathbb{R}, rdr)$, by:

$$\langle f, g\rangle_{L^2(\mathbb{R}_+,rdr)} = \int_0^\infty f(r)\overline{g}(r)\,rdr.$$

With all these notations set, we define, for each $n \in \mathbb{N}\setminus\{0\}$, the n^{th} Césaro sum of the series given by the Laguerre expansion of f:

$$\mathrm{C}_\eth^n(f) = \sum_{j=0}^{n} \frac{n+1-j}{n+1}\left\langle f\,\psi_\eth^j, \psi_\eth^j\right\rangle_{L^2(\mathbb{R}_+,rdr)} \psi_\eth^j.$$

Then by the work of S. Thangavelu in [50, Theorem 6.2.1] — where our $\psi_\eth^j$ is a rescaled version of the function denoted by ψ_j^0 in [50, Chapter 6] and we use the Césaro sums for "$\delta = 1$" in his notations — we conclude:

$$\lim_{n\to\infty} \|\mathrm{C}_\eth^n f - f\|_{L^1(\mathbb{R}_+,rdr)} = 0.$$

Now, a quick computation shows that for all $n \in \mathbb{N}\setminus\{0\}$:

$$\left\|(\mathrm{C}_\eth^j(f))^\circ - f^\circ\right\|_{L^1(\mathbb{R}^2)} = \left\|\mathrm{C}_\eth^j(f) - f\right\|_{L^1(\mathbb{R}_+,rdr)},$$

and therefore:

$$\lim_{n\to\infty} \|(\mathrm{C}^n_{\eth}(f))^\circ - f^\circ\|_{L^1(\mathbb{R}^2)} = 0$$

where of course, $L^1(\mathbb{R}^2)$ stands for the 1-Lebesgue space with respect to the usual Lebesgue measure on $\mathbb{R}^2$.

By Lemma (3.5.5), writing $\kappa_n = (\mathrm{C}^n_{\eth}(f))^\circ$ for all $n \in \mathbb{N}$, we then conclude:

$$\lim_{n\to\infty} \left|\left|\left|\sigma^{\kappa_n}_{\eth,d} - \sigma^{f^\circ}_{\eth,d}\right|\right|\right|_{\mathscr{H}^{p,q,d}_\theta} = 0.$$

By construction, $\sigma^{\kappa_n}_{\eth,d}$ is finite rank. Indeed, the operator $\sigma^{\kappa_n}_{\eth,d}$ is a linear combination of the operators $\sigma^{(\psi^j_\eth)^\circ}_{\eth,d}$ with $j \in \{0,\ldots,n\}$. The operators $\sigma^{(\psi^j_\eth)^\circ}_{\eth,d}$ are, in turn, projections on $\mathbb{C}\mathcal{H}^j_\eth \otimes \mathbb{C}^d \subseteq L^2(\mathbb{R}) \otimes \mathbb{C}^d$, where $\mathcal{H}^n_\eth$ is the Hermite function:

$$\mathcal{H}^j_\eth : t \in \mathbb{R} \mapsto \frac{(2\eth)^{\frac{1}{4}}}{\sqrt{j!2^j}} \exp\left(-\frac{t^2\sqrt{2\pi\eth}}{2}\right) \mathrm{H}_j\left(t\sqrt{2\pi\eth}\right)$$

where H_j is the j^{th} Hermite polynomial, given for instance by:

$$\mathrm{H}_j : t \in \mathbb{R} \mapsto (-1)^j \exp(t^2) \frac{\mathrm{d}^j}{\mathrm{d}t^j} \exp(-t^2).$$

Indeed, by [10, p. 65], the operators $\sigma^{(\psi^j_\eth)^\circ}_{\eth,1}$ are projections on $\mathbb{C}\mathcal{H}_j \subseteq L^2(\mathbb{R})$ for all $j \in \mathbb{N}$. We note that reassuringly, we will not need the explicit form of the Hermite polynomials or the Laguerre polynomials in our work.

Thus the image of the unit ball $\mathscr{H}^{p,q,d}_\theta[0,1]$ of $\left(\mathscr{H}^{p,q,d}_\theta, \|\cdot\|_{\mathscr{H}^{p,q,d}_\theta}\right)$ by $\sigma^{\kappa_n}_{\eth,d}$ is totally bounded in $\left(\mathscr{H}^{p,q,d}_\theta, \|\cdot\|_{\mathscr{H}^{p,q,d}_\theta}\right)$ for all $n \in \mathbb{N}$, as a bounded subset of a finite dimensional space (as all norms are equivalent in finite dimension, this observation does not depend on $\|\cdot\|_{\mathscr{H}^{p,q,d}_\theta}$).

Thus $\sigma^{f^\circ}_{\eth,d}$ is compact as the norm limit of compact operators.

We are left to treat the case when $\eth < 0$. We note that for all $(x,y,u) \in \mathbb{H}_3$, we have:

$$\alpha^{x,y,u}_{\eth,d} = \alpha^{x,-y,-u}_{-\eth,d}.$$

We thus proceed as above with $-\eth$ in place of $\eth$, and note that $\sigma^{\kappa_n}_{\eth,d} = -\sigma^{\kappa_n}_{-\eth,d}$ since κ_n is a radial function. The rest of the proof is left unchanged. □

With Lemma (3.5.10) and Lemma (3.5.6), we are now able to prove the desired property for our D-norms:

Lemma 3.5.11. *We assume Hypothesis (3.3.1). The set:*

$$\mathscr{D}_1\left(\mathsf{D}^{p,q,d}_\theta\right) = \left\{\xi \in \mathscr{H}^{p,q,d}_\theta : \mathsf{D}^{p,q,d}_\theta(\xi) \leqslant 1\right\}$$

is compact in $\left(\mathscr{H}^{p,q,d}_\theta, \|\cdot\|_{\mathscr{H}^{p,q,d}_\theta}\right)$.

Proof. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions from $[0,\infty)$ to $[0,\infty)$ such that for all $n \in \mathbb{N}$, the function ψ_n is supported on $\left[-\frac{1}{n+1}, \frac{1}{n+1}\right]$ and:

$$\int_0^\infty \psi_n(r)\, rdr = \frac{1}{2\pi}.$$

Thus, using the notations of Lemma (3.5.10), we note that:

$$\int_{\mathbb{R}^2} \psi_n^\circ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty \psi_n(r)\, rdrd\theta = \frac{2\pi}{2\pi} = 1.$$

Let $\varepsilon > 0$ be given. By Lemma (3.5.8), we have:

$$\lim_{n\to\infty} \iint_{\mathbb{R}^2} \psi_n^\circ(x,y) \|(x,y)\|\, dxdy = 0.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, the following inequality holds:

$$\iint_{\mathbb{R}^2} \psi_n^\circ(x,y)\|(x,y)\|\, dxdy < \frac{\varepsilon}{4\pi\eth}$$

We may thus apply Lemma (3.5.6) to conclude that for all $\xi \in \mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$ and $n \geqslant N$:

$$\left\|\xi - \sigma_{\eth,d}^{\psi_n^\circ}\xi\right\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \frac{\varepsilon}{2}.$$

Now, $\sigma_{\eth,d}^{\psi_n^\circ}$ is compact in $\left(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}\right)$ by Lemma (3.5.10), and $\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$ is bounded for $\|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}$ by construction. Thus the image of $\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$ by $\sigma_{\eth,d}^{\psi_n^\circ}$ is totally bounded in $\left(\mathscr{H}_\theta^{p,q,d}, \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}\right)$ for all $n \in \mathbb{N}$. In particular, there exists a $\frac{\varepsilon}{2}$-dense subset $\mathscr{B}_\varepsilon$ in $\sigma_{\eth,d}^{\psi_N^\circ}\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$.

Consequently, if $\xi \in \mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$, then there exists $\eta \in \mathscr{B}_\varepsilon$ such that:

$$\left\|\eta - \sigma_{\eth,d}^{\psi_N^\circ}\xi\right\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \frac{\varepsilon}{2}.$$

Thus $\|\xi - \eta\|_{\mathscr{H}_\theta^{p,q,d}} \leqslant \varepsilon$.

We thus conclude that $\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$ is totally bounded.

Moreover, for all $(x,y) \in \mathbb{R}^2$, the map $\xi \mapsto \frac{\|\alpha_{\eth,d}^{x,y,\frac{xy}{2}}\xi - \xi\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|}$ is continuous, and thus $\mathsf{D}_\theta^{p,q,d}$ is lower semi-continuous with respect to $\|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}$. Hence $\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right) = \left(\mathsf{D}_\theta^{p,q,d}\right)^{-1}((-\infty,1])$ is closed. Since $\mathscr{H}_\theta^{p,q,d}$ is complete and $\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$ is closed and totally bounded, it is in fact compact, as desired. □

We summarize the results of this section with the following theorem announcing that indeed, we have defined D-norms on Heisenberg modules, turning them into metrized quantum vector bundles over quantum 2-tori.

Theorem 3.5.12. *Let $\mathscr{H}_\theta^{p,q,d}$ be the Heisenberg module over $\mathcal{A}_\theta$ for some $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$. Let $\eth = \theta - \frac{p}{q}$ and assume $\eth \neq 0$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. If we set, for all $\xi \in \mathscr{H}_\theta^{p,q,d}$:*

$$\mathsf{D}_\theta^\rho(\xi) = \sup\left\{ \frac{\left\|\sigma_{\eth,d}^{x,y}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\},$$

and for all $a \in \mathcal{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)}a - a\right\|_{\mathcal{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

then $\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^\rho, \mathcal{A}_\theta, \mathsf{L}_\theta\right)$ is a Leibniz metrized quantum vector bundle.

Proof. Proposition (3.5.4) proves that $\mathsf{D}_\theta^{p,q,d}$ is a norm on a dense subspace of $\mathscr{H}_\theta^{p,q,d}$ which satisfies the inner and modular quasi-Leibniz inequalities and, by construction, $\mathsf{D}_\theta^{p,q,d} \geqslant \|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}$.

Lemma (3.5.11) moreover gives us that $\mathscr{D}_1\left(\mathsf{D}_\theta^{p,q,d}\right)$ is compact for $\|\cdot\|_{\mathscr{H}_\theta^{p,q,d}}$. □

We are now in a position to investigate the geometry of the space of Heisenberg modules over quantum 2-tori under the modular propinquity. There are many natural questions one immediately thinks about in this context, and we choose to focus on one of them: how do Heisenberg modules, for a fixed choice of p, q, d, vary when the base quantum torus is allowed to vary continuously?

To address this question, we first prove that our D-norms actually form a continuous field of norms. This is the key step in proving our continuity result for the modular propinquity.

3.6. **A continuous field of D-norms.** Our first step in establishing a continuity result for D-norms on Heisenberg modules is to reformulate the expression of our D-norms.

Lemma 3.6.1. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$. Let $r : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ be a continuous function.*

If $\xi \in \mathcal{S}(\mathbb{C}^d)$, then for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y)\| = 1$, $\eth \in \mathbb{R} \setminus \{0\}$, the function :

$$t \in (0,\infty) \mapsto \omega_{x,y,t,\eth} = \frac{\exp(i\pi\eth t^2 xy)\sigma_{\eth,d}^{tx,ty}\xi_{r(\eth)} - \xi_{r(\eth)}}{2\pi|\eth|t}$$

where $\xi_{r(\eth)} : t \in \mathbb{R} \mapsto \xi(r(\eth)t)$, can be extended by continuity at 0. Moreover, for all $\eth \in \mathbb{R} \setminus \{0\}$:

$$\mathsf{D}_{\eth+\frac{p}{q}}^{p,q,d}(\xi_{r(\eth)}) = \sup\left\{ \left\|\omega_{x,y,t,\eth}\right\|_{\mathscr{H}_\theta^{p,q,d}} : (x,y) \in \mathbb{R}^2, \|(x,y)\| = 1, t \in [0,1] \right\}$$

and

$$(x,y,t,\eth) \in \mathbb{R}^2[0,1] \times \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \mapsto \left\langle \omega_{x,y,t,\eth}, \omega_{x,y,t,\eth} \right\rangle_{\mathscr{H}_{\eth+\frac{p}{q}}^{p,q,d}}$$

is continuous to $\left(\ell^1(\mathbb{Z}^2), \|\cdot\|_{\ell^1(\mathbb{Z}^2)}\right)$.

Proof. We begin by setting a domain over which we shall study our functions ω. For our purpose, we choose some arbitrary $\eth_\infty \neq 0$ and then $0 < \eth_- < \eth_+$ such that $|\eth_\infty| \in (\eth_-, \eth_+)$. We set:

$$\Omega = \left\{(x,y,\eth) \in \mathbb{R}^3 : \|(x,y)\| = 1, |\eth| \in [\eth_-, \eth_+]\right\}$$

while:

$$\Sigma = \{(x,y,t,\eth) \in \mathbb{R}^4 : (x,y,\eth) \in \Omega, t \in [0,1]\}$$

and:

$$\Sigma_* = \{(x,y,t,\eth) \in \mathbb{R}^4 : (x,y,\eth) \in \Omega, t \in (0,1]\}.$$

Let $\xi \in \mathcal{S}(\mathbb{C}^d)$ and let $M_0 > 0$ be chosen so that for all $s \in \mathbb{R}$:

$$\max\{\|\xi^{(n)}(s)\|_{\mathbb{C}^d}, \|s\xi^{(n)}(s)\|_{\mathbb{C}^d} : n \in \{0,1,2,3,4\}\} \leqslant \frac{M_0}{1+s^2}.$$

Now, r is continuous on $[\eth_-, \eth_+]$, and thus there exists $R_-, R_+ > 0$ such that $R_- \leqslant r(\eth) \leqslant R_+$ for all $\eth \in [\eth_-, \eth_+]$. Thus for all $s \in \mathbb{R}$:

$$\max\{\|\xi^{(n)}(r(\eth)s)\|_{\mathbb{C}^d} : n \in \{0,1,2,3,4\}\} \leqslant \frac{M_0}{1+R_-^2 s^2}$$

and

$$\max\left\{\|s\xi^{(n)}(r(\eth)s)\|_{\mathbb{C}^d} : n \in \{0,1,2,3,4\}\right\} \leqslant \frac{M_0}{R_-(1+R_-^2 s^2)}.$$

By the same reasoning as we have already seen in our paper, we thus conclude that there exists $M > 0$ such that for all $\eth \in \mathbb{R}$ with $|\eth| \in [\eth_-, \eth_+]$ and for all $s \in \mathbb{R}$:

$$\max\{\|\xi_{r(\eth)}^{(n)}(s)\|_{\mathbb{C}^d}, \|s\xi_{r(\eth)}^{(n)}(s)\|_{\mathbb{C}^d} : n \in \{0,1,2,3,4\}\} \leqslant \frac{M}{1+s^2}.$$

We first extend $(x,y,t,\eth) \in \Sigma_* \mapsto \omega_{x,y,t,\eth}$ to Σ by continuity. For all $(x,y,t,\eth) \in \Sigma_*$, we observe that for all $s \in \mathbb{R}$:

$$\begin{aligned}
\omega_{x,y,t,\eth}(s) &= \frac{\exp(i\pi\left(t^2\eth xy + 2txs\right))\xi_{r(\eth)}(s+\eth ty) - \xi_{r(\eth)}(s)}{2\pi|\eth|t} \\
&= \frac{\exp(i\pi(t^2\eth xy + 2txs))\xi_{r(\eth)}(s+\eth ty) - \xi_{r(\eth)}(s+\eth ty)}{2\pi|\eth|t} \\
&\quad + \frac{\xi_{r(\eth)}(s+\eth ty) - \xi_{r(\eth)}(s)}{2\pi|\eth|t} \\
&= \xi_{r(\eth)}(s+\eth yt)\frac{\exp(i\pi(\eth t^2 xy + 2txs)) - 1}{2\pi|\eth|t} + \frac{\xi_{r(\eth)}(s+\eth ty) - \xi_{r(\eth)}(s)}{2\pi|\eth|t}.
\end{aligned}$$

Since ξ is a Schwarz function, thus differentiable, we have for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y)\| = 1$ and $\eth \in \mathbb{R} \setminus \{0\}$:

$$\lim_{t \to 0^+} \omega_{x,y,t,\eth}(s) = x\frac{is}{\eth}\xi_{r(\eth)}(s) + yr(\eth)\frac{\xi'_{r(\eth)}(s)}{2\pi}.$$

Thus we set, for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y)\| = 1$ and $\eth \in \mathbb{R} \setminus \{0\}$:

$$\omega_{x,y,0,\eth} : s \in \mathbb{R} \mapsto x\frac{is}{\eth}\xi_{r(\eth)}(s) + yr(\eth)\frac{\xi'_{r(\eth)}(s)}{2\pi}.$$

We observe that our statement thus far is about pointwise convergence of the family of functions $(\omega_{x,y,t,\eth})_{t>0}$ to $\omega_{x,y,0,\eth}$ for fixed $x,y \in \mathbb{R}^2$ with $\|(x,y)\| = 1$ and $\eth \neq 0$. This is different from the notion of convergence in the C^*-Hilbert norm. To obtain convergence for the Heisenberg C^*-Hilbert norm, and more information, we now proceed to establish some regularity properties for ω, in order to apply Lemma (3.2.4).

By Proposition (3.3.8), we already know that there exists $M_1 > 0$ such that for all $(x,y,t,\eth) \in \Sigma_*$ and $s \in \mathbb{R}$:

$$\|\sigma^{tx,ty,\frac{1}{2}t^2xy}\xi_{r(\eth)}(s) - \xi_{r(\eth)}(s)\|_{\mathbb{C}^d} \leqslant \frac{M_1 t\|(x,y,\frac{1}{2}txy)\|_1}{1+s^2},$$

where $\|(x,y,t)\|_1 = |x|+|y|+|t|$ is the usual 1-norm on $\mathbb{R}^3$.

The map $(x,y,t) \in \mathbb{R}^3 \mapsto (x,y,\frac{txy}{2})$ is continuous from $\mathbb{R}^3$ to itself. The set $K = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| = 1\} \times [0,1]$ is compact and thus there exists $M_2 > 0$ such that:

$$\sup\left\{\left\|\left(x,y,\frac{txy}{2}\right)\right\|_1 : (x,y,t) \in K\right\} = M_2.$$

Thus:

$$\begin{aligned}\|\omega_{x,y,t,\eth}(s)\|_{\mathbb{C}^d} &\leqslant \frac{t\|(x,y,\frac{1}{2}txy)\|_1}{2\pi|\eth|t}\frac{M_1}{1+s^2}\\ &\leqslant \frac{\frac{M_1M_2}{2\pi|\eth|}}{1+s^2} \leqslant \frac{\frac{M_1M_2}{2\pi|\eth|_-}}{1+s^2}.\end{aligned}$$

On the other hand, by assumption:

$$\|\omega_{x,y,0,\eth}(s)\|_{\mathbb{C}^d} \leqslant |x|\frac{M}{|\eth|(1+s^2)} + |y|\frac{R_+M}{2\pi(1+s^2)} \leqslant \left(\frac{1}{\eth_-} + \frac{R_+}{2\pi}\right)\frac{MM_2}{1+s^2}.$$

In summary, there exists $M_3 = \max\left\{\frac{M_1M_2}{\eth_-}, MM_2\left(\frac{1}{\eth_-} + \frac{R_+}{2\pi}\right)\right\} > 0$ such that for all $(x,y,\eth,t) \in \Sigma$ and all $s \in \mathbb{R}$:

$$\left\|\omega_{x,y,t,\eth}(s)\right\|_{\mathbb{C}^d} \leqslant \frac{M_3}{1+s^2}.$$

By construction, $(\omega_{x,y,t,\eth})_{t>0}$ converges pointwise to $\omega_{x,y,0,\eth}$ as $t \to 0$. We now prove that this convergence is indeed uniform.

We begin with the following computation for all $(x,y,t,\eth) \in \Sigma_*$ and for all $s \in \mathbb{R}$:

$$\|\omega_{x,y,t,\eth}(s) - \omega_{x,y,0,\eth}(s)\|_{\mathbb{C}^d}$$

$$
\begin{aligned}
&= \left| \frac{\exp(i\pi\eth t^2 xy)\sigma_{\eth,d}^{tx,ty}\xi_{r(\eth)}(s) - \xi_{r(\eth)}(s)}{2\pi t\eth} - \left(x\frac{is}{\eth}\xi_{r(\eth)}(s) + y\frac{r(\eth)}{2\pi}\xi'_{r(\eth)}(s) \right) \right| \\
&\leqslant \left| \frac{\exp(i\pi(\eth t^2 xy + 2stx)) - 1}{2\pi t\eth}\xi_{r(\eth)}(s+\eth ty) - x\frac{is}{\eth}\xi_{r(\eth)}(s+\eth ty) \right| \\
&\quad + \left| x\frac{is}{\eth}\xi_{r(\eth)}(s+\eth ty) - x\frac{is}{\eth}\xi_{r(\eth)}(s) \right| \\
&\quad + \frac{1}{2\pi}\left| \frac{\xi_{r(\eth)}(s+\eth ty) - \xi_{r(\eth)}(s)}{\eth t} - yr(\eth)\xi'_{r(\eth)}(s) \right|.
\end{aligned}
$$

We first note that for all $(x,y,t,\eth) \in \Sigma_*$, by the mean value theorem, if $g : t > 0 \mapsto \exp(i\pi(t^2 xy + 2txs))$, then there exists $t_c \in [0,t]$ such that:

$$
\begin{aligned}
&\left| \frac{\exp(i\pi(\eth t^2 xy + 2txs)) - 1}{2\pi t\eth}\xi_{r(\eth)}(s+t\eth y) - x\frac{is}{\eth}\xi_{r(\eth)}(s+t\eth y) \right| \\
&\qquad \leqslant \frac{M}{1+(s+\eth ty)^2}\left| \frac{\exp(i\pi(t^2\eth xy + 2xts)) - 1}{2\pi t\eth} - \frac{ixs}{\eth} \right| \\
&\qquad = \frac{M}{2\pi|\eth|\,(1+(s+\eth ty)^2)}\left| \frac{g(t)-g(0)}{t} - g'(0) \right| \\
&\qquad = \frac{M}{2\pi|\eth|\,(1+(s+\eth ty)^2)}\left| \frac{t}{2}g''(t_c) \right| \\
&\qquad = \frac{M|t|}{2\pi|\eth|\,(1+(s+\eth ty)^2)}\left| (i\pi\eth xy - \frac{\pi^2}{2}(2xs + 2t_c\eth xy)^2)\exp(2i\pi st_c x) \right| \\
&\qquad \leqslant \frac{M\pi|t|}{2\eth\,(1+(s+\eth ty)^2)}\left(|xs| + (1+|t|)\eth_+|xy|\right)^2 \\
&\qquad \leqslant \frac{M\pi|t|}{2\eth_-\,(1+(s+\eth ty)^2)}\left(|xs| + 2\eth_+|xy|\right)^2.
\end{aligned}
$$

Once again, the functions $(x,y,\eth,t) \in \Omega \times [0,1] \mapsto |x|$ and $(x,y,\eth,t) \in \Omega \times [0,1] \mapsto |ty\eth|$ are continuous on the compact $\Omega \times [0,1]$, so we conclude that there exists $M_4 > 0$ such that:

$$\max\left\{ |x|, |\eth ty| : (x,y,\eth,t) \in \Omega \times [0,1] \right\} \leqslant M_4.$$

Consequently, for all $s > M_4$ we have:

$$\frac{M\pi(|xs| + 2|\eth_+||xy|)^2}{2\eth_-(1+(s+\eth ty)^2)} \leqslant \frac{\pi M M_4^2 (s+2M_4)^2}{2\eth_-(1+(s-M_4)^2)}$$

while for all $s < -M_4$ we have:

$$\frac{M\pi(|xs| + |\eth_+||xy|)^2}{2\eth_-(1+(s+\eth ty)^2)} \leqslant \frac{\pi M M_4^2 (s+2M_4)^2}{2\eth_-(1+(s+M_4)^2)}$$

Thus, there exists $M_5 > 0$ such that if $|s| > M_4$ then:

$$\frac{M\pi(|xs| + |\eth_+||xy|)^2}{2\eth_-(1+(s+\eth ty)^2)} \leqslant M_5.$$

The function $(x,y,t,s) \in \mathbb{R}^4 \mapsto \frac{M\pi(|xs|+|t|\eth_+|xy|)^2}{2\eth_-(1+(s+\eth ty)^2)}$ is continuous, and thus it is bounded by some $M_6 > 0$ on the compact $\Sigma \times [-M_4, M_4]$.

Letting $M_7 = \max\{M_5, M_6\}$, we conclude that for all $(x,y,\eth,t) \in \Sigma_*$ and $s \in \mathbb{R}$, we have:

$$\left|\frac{\exp(2i\pi txs)-1}{\eth t}\xi_{r(\eth)}(s+t\eth y) - x\frac{2i\pi}{\eth}\xi_{r(\eth)}(s+t\eth y)\right| \leqslant M_7|t|.$$

Consequently, if $|t| < \delta_1 = \frac{\varepsilon}{3M_7}$, then:

$$\left|\frac{\exp(i\pi(t^2\eth xy+2txs))-1}{\eth t}\xi_{r(\eth)}(s+t\eth y) - x\frac{2i\pi}{\eth}\xi_{r(\eth)}(s+t\eth y)\right| < \frac{\varepsilon}{3}. \tag{3.6.1}$$

Now, since $s \in \mathbb{R} \mapsto s\xi(s)$ is uniformly continuous on $\mathbb{R}$ as Schwarz function, there exists $\delta_2 > 0$ such that for all $0 < r < \delta_2$ and for all $s \in \mathbb{R}$, we have $|(s+r)\xi(s+r) - s\xi(s)| < \frac{\varepsilon R_-\eth_-}{6M_4}$.

Moreover, since ξ is bounded on $\mathbb{R}$, we may choose $\delta_2 > 0$ small enough so that $\delta_2 \sup_{s\in\mathbb{R}} \|\xi(s)\|_{\mathbb{C}^d} < \frac{\varepsilon R_-\eth_-}{6M_4}$.

Let now $\delta_3 = \frac{\delta_2}{M_4R_+}$. If $|t| < \delta_3$ then $|r(\eth)\eth ty| < \delta_2$ and therefore:

$$\begin{aligned}
&\left\|x\frac{is}{\eth}\xi_{r(\eth)}(s+\eth ty) - x\frac{is}{\eth}\xi_{r(\eth)}(s)\right\|_{\mathbb{C}^d} \\
&\qquad\leqslant \frac{|x|}{\eth_-R_-}\left(\|r(\eth)(s+\eth ty)\xi(r(\eth)s+r(\eth)\eth ty) - r(\eth)s\xi(r(\eth)s)\|_{\mathbb{C}^d}\right. \\
&\qquad\quad \left.+|r(\eth)\eth ty|\|\xi(r(\eth)(s+\eth ty))\|_{\mathbb{C}^d}\right) \\
&\qquad\leqslant \frac{M_4}{R_-\eth_-}\left(\frac{\varepsilon R_+\eth_-}{6M_4} + \delta_2\sup_{s\in\mathbb{R}}\|\xi(s)\|_{\mathbb{C}^d}\right) \\
&\qquad\leqslant \frac{\varepsilon}{6}+\frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
\end{aligned}$$

We thus deduce that for all $(x,y,t,\eth) \in \Sigma$, if $|t| < \delta_3$, then:

$$\sup_{s\in\mathbb{R}} |x\frac{is}{\eth}\xi_{r(\eth)}(s+\eth ty) - x\frac{is}{\eth}\xi_{r(\eth)}(s)| < \frac{\varepsilon}{3}. \tag{3.6.2}$$

Last, since ξ' is also a Schwarz function and in particular, also uniformly continuous on $\mathbb{R}$, there exists $\delta_4 > 0$ such that $|\xi'(s+r) - \xi'(s)| < \frac{\varepsilon}{3R_+}$ for all $0 \leqslant r < \delta_4$. Thus for all $(x,y,t,\eth) \in \Sigma$ and $s \in \mathbb{R}$, if $|t| < \delta_5 = \frac{\delta_4}{R_+M_4}$, then:

$$\left|\left(\xi_{r(\eth)}(s+t\eth y) - \xi_{r(\eth)}(s)\right) - y\xi'_{r(\eth)}(s)\right| \leqslant |r(\eth)|\int_0^t |\xi'_{r(\eth)}(s+r\eth y) - \xi'_{r(\eth)}(s)|\,dr \leqslant t\frac{\varepsilon}{3}.$$

Thus for all $(x,y,t,\eth) \in \Sigma_*$ with $|t| < \delta_4$, we have:

$$\sup_{s\in\mathbb{R}}\left|\frac{\xi_{r(\eth)}(s+\eth ty) - \xi_{r(\eth)}(s)}{t} - \xi'_{r(\eth)}(s)\right| < \frac{\varepsilon}{3}.$$

In conclusion, for all $(x,y,t,\eth) \in \Sigma_*$ with $0 < t < \min\{\delta_1,\delta_3,\delta_5\}$ and for all $s \in \mathbb{R}$, we have established:

$$\sup_{s\in\mathbb{R}} |\omega_{x,y,t,\eth}(s) - \omega_{x,y,0,\eth}(s)| < \varepsilon.$$

In other words, setting for all $t \in (0,1]$:

$$\mathsf{f}_t : (x,y,\eth,s) \in \Omega \times \mathbb{R} \mapsto \omega_{x,y,t,\eth}(s)$$

converges uniformly on $\Omega \times \mathbb{R}$ to:

$$\mathsf{f}_0 : (x,y,\eth,s) \in \Omega \times \mathbb{R} \mapsto \omega_{x,y,0,\eth}(s)$$

when t goes to 0.

Since $(x,y,t,\eth,s) \in \Sigma_* \times \mathbb{R} \mapsto \mathsf{f}_t(x,y,\eth,s)$ and f_0 are both continuous, we deduce, in particular, that:

$$(x,y,t,\eth,s) \in \Sigma \times \mathbb{R} \mapsto \omega_{x,y,t,\eth}(s)$$

is (jointly) continuous.

The entire reasoning up to now may be applied equally well to $\xi^{(n)}$ for $n \in \{0,1,2\}$ — as one may check that $\omega^{(n)}$ is indeed obtained by substituting ξ with $\xi^{(n)}$.

Therefore, we are now able to apply Lemma (3.2.4) to conclude that:

$$(x,y,t,\theta) \in \Sigma \mapsto \left\langle \omega_{x,y,t,\eth}, \omega_{x,y,t,\eth} \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \in \left(\ell^1(\mathbb{Z}^2), \|\cdot\|_{\ell^1(\mathbb{Z}^2)}\right)$$

is continuous as desired (to make notations clear: we pick a sequence $(\theta_n)_{n\in\mathbb{N}}$ converging to some θ, and we choose $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, and $t_n \in [0,1]^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, we have $(x_n, y_n, t_n, \theta_n - \frac{p}{q}) \in \Sigma$, and then we set, in the notations of Lemma (3.2.4), the functions $\xi_n = f_{t_n}(x_n, y_n, \eth_n, \cdot)$ and $\xi_\infty = f_{t_\infty}(x_\infty, y_\infty, \eth_\infty, \cdot)$).

We conclude our proof by observing that by Theorem (3.4.2):

$$\begin{aligned}
\mathsf{D}_\theta^{p,q,d}(\xi_{r(\eth)}) &= \sup\left\{ \frac{\left\| \alpha_{\eth,d}^{x,y,\frac{1}{2}xy} \xi_{r(\eth)} - \xi_{r(\eth)} \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2, \|(x,y)\| \leqslant 1 \right\} \\
&= \sup\left\{ \left\|\omega_{x,y,t,\eth}\right\|_{\mathscr{H}_\theta^{p,q,d}} : \|(x,y)\| = 1, t \in [0,1] \right\}
\end{aligned}$$

as stated. □

We now prove that D-norms on Heisenberg modules form continuous fields.

Proposition 3.6.2. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$. Let $\xi \in \mathcal{S}(\mathbb{C}^d)$. Let $r : \mathbb{R} \setminus \{\frac{p}{q}\} \to \mathbb{R} \setminus \{0\}$ be a continuous function. If $(\theta_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to θ_∞ and such that $\theta_k \neq \frac{p}{q}$ for all $k \in \overline{\mathbb{N}}$, then:*

$$\lim_{k\to\infty} \mathsf{D}_{\theta_k}^{p,q,d}(\xi_{r(\theta_k)}) = \mathsf{D}_{\theta_\infty}^{p,q,d}(\xi_{r(\theta_\infty)}),$$

where $\xi_{r(\theta)} : t \in \mathbb{R} \mapsto \xi(r(\theta)t)$ for all $\theta \neq \frac{p}{q}$.

Proof. The result is trivial if $\xi = 0$, which is equivalent to $\mathsf{D}_\vartheta^\rho(\xi) = 0$ for all $\vartheta \in \mathbb{R}$ with $\vartheta \neq \frac{p}{q}$.

Now, fix $\theta \in \mathbb{R} \setminus \{\frac{p}{q}\}$. We shall prove that $\vartheta \mapsto \mathsf{D}_\vartheta^\rho(\xi_{r(\vartheta)})$ is continuous at θ.

Let $\delta_1 > 0$ such that $I = [\theta - \delta_1, \theta + \delta_1] \subseteq \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$.

Let:

$$\mathrm{Y} = \left\{ (x,y,t) \in \mathbb{R}^3 : \|(x,y)\| = 1, t \in [0,1] \right\}.$$

Let $\xi \in \mathcal{S}(\mathbb{C}^d)$. We set:

$$t \in (0,\infty) \mapsto \omega_{x,y,t,\eth} = \frac{\exp(i\pi\eth t^2 xy)\sigma_{\eth,d}^{tx,ty}\xi_{r(\eth+p/q)} - \xi_{r(\eth+p/q)}}{2\pi|\eth|t}.$$

By Lemma (3.6.1):

$$\begin{aligned}\mathsf{D}_{\vartheta}^{p,q,d}(\xi_{r(\vartheta)})^2 &= \sup\left\{ \left\| \left\langle \omega_{x,y,t,\vartheta-\frac{p}{q}}, \omega_{x,y,t,\vartheta-\frac{p}{q}} \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right\|_{\mathcal{A}_\vartheta} : (x,y,t) \in \mathrm{Y} \right\} \\ &\leqslant \sup\left\{ \left\| \left\langle \omega_{x,y,t,\vartheta-\frac{p}{q}}, \omega_{x,y,t,\eth,\vartheta-\frac{p}{q}} \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right\|_{\ell^1(\mathbb{Z}^2)} : (x,y,t) \in \mathrm{Y} \right\}\end{aligned}$$

where Y is a compact subset of $\mathbb{R}^3$, independent of ϑ.

Now, since:

$$\nu : (x,y,t,\vartheta) \in \mathrm{Y} \times I \mapsto \left\langle \omega_{x,y,t,\vartheta-\frac{p}{q}}, \omega_{x,y,t,\vartheta-\frac{p}{q}} \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}}$$

is continuous in $\left(\ell^1(\mathbb{Z}^2), \|\cdot\|_{\ell^1(\mathbb{Z}^2)}\right)$ by Lemma (3.6.1), it is uniformly continuous on the compact $\mathrm{Y}_2 = \mathrm{Y} \times I$.

Let $\|(z,w,s,\hbar)\|_\infty = \max\{|z|,|w|,|s|,|\hbar|\}$ for all $(z,w,s,\hbar) \in \mathbb{R}^4$.

Let $\delta_2 > 0$ be chosen so that for all $(x,y,t,\eth),(z,w,r,s) \in \mathrm{Y}_2$ with $\|(x,y,t,\eth) - (z,w,s,\hbar)\|_\infty < \delta_2$ we have:

$$|\nu(x,y,t,\eth) - \nu(z,w,r,\hbar)| < \frac{\varepsilon}{4}.$$

Let $G \subseteq \mathrm{Y}_2$ be a δ_2-dense finite subset of Y_2 in the sense of the norm $\|\cdot\|_\infty$. Let:

$$F = \left\{ (z,w,r) \in \mathbb{R}^3 : \exists \hbar \in \mathbb{R} \quad (z,w,r,\hbar) \in G \right\}.$$

By construction, F is finite and δ_2-dense in Y for the restriction of $\|\cdot\|_\infty$ to $\mathbb{R}^3 \sim \mathbb{R}^3 \times \{0\}$.

Fix any $\eth \in [\theta - \frac{p}{q} - \delta_1, \theta - \frac{p}{q} + \delta_1]$ and set $\vartheta = \eth + \frac{p}{q}$. Now, let $(x,y,t) \in \mathrm{Y}$. There exists $(z,w,r) \in F$ with $\max\{|x-z|,|y-w|,|t-r|\} < \delta_2$. We then observe:

$$\frac{\|\alpha_{\eth,d}^{tx,ty,\frac{t^2xy}{2}}\xi_{r(\vartheta)} - \xi_{r(\vartheta)}\|^2_{\mathscr{H}_{\vartheta}^{p,q,d}}}{2\pi|\eth|t}$$

$$\leqslant \left| \frac{\left\| \alpha_{\eth,d}^{tx,ty,\frac{t^2xy}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|t} - \frac{\left\| \alpha_{\eth,d}^{rz,rw,\frac{r^2zw}{2}} \xi_{r(\vartheta)} - \xi_{\vartheta} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|r} \right| + \frac{\left\| \alpha_{\eth,d}^{rz,rw,\frac{r^2zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|r}$$

$$\leqslant \left| \left\| \left\langle \omega_{x,y,t,\eth}, \omega_{x,y,t,\eth} \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right\|_{\mathcal{A}_\vartheta} - \left\| \left\langle \omega_{z,w,r,\eth}, \omega_{z,w,r,\eth} \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right\|_{\mathcal{A}_\vartheta} \right| + \frac{\left\| \alpha_{\eth,d}^{rz,rw,\frac{r^2zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|r}$$

$$\leqslant \left\| \left\langle \omega_{x,y,t,\eth}, \omega_{x,y,t,\eth} \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} - \left\langle \omega_{z,w,r,\eth}, \omega_{z,w,r,\eth} \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right\|_{\mathcal{A}_\vartheta} + \frac{\left\| \alpha_{\eth,d}^{rz,rw,\frac{r^2zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|r}$$

$$\leqslant \left\| \left\langle \omega_{x,y,t,\eth}, \omega_{x,y,t,\eth} \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} - \left\langle \omega_{z,w,r,\eth}, \omega_{z,w,r,\eth} \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right\|_{\ell^1(\mathbb{Z}^2)} + \frac{\left\| \alpha_{\eth,d}^{rz,rw,\frac{r^2zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|r}$$

$$\leqslant \frac{\varepsilon}{4} + \frac{\left\| \alpha_{\eth,d}^{z,w,\frac{r^2zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)} \right\|^2_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi|\eth|r}.$$

Let $\mathsf{F}_\vartheta^{p,q,d}(\eta)$ be given by:

$$\mathsf{F}_\vartheta^{p,q,d}(\eta) = \max \left\{ \frac{\left\| \alpha_{\eth,d}^{rz,rw,\frac{r^2zw}{2}} \eta - \eta \right\|_{\mathscr{H}_\vartheta^{p,q,d}}}{2\pi\left(\vartheta + \frac{p}{q}\right) r} : (z,w,r) \in F \right\}$$

for all $\eta \in \mathcal{S}(\mathbb{C}^d)$.

We thus have proven:

$$\mathsf{F}_\vartheta^{p,q,d}(\xi_{r(\vartheta)})^2 \leqslant \mathsf{D}_\vartheta^{p,q,d}(\xi_{r(\vartheta)})^2 \leqslant \frac{\varepsilon}{4} + \mathsf{F}_\vartheta^{p,q,d}(\xi_{r(\vartheta)})^2.$$

Therefore:

$$\left| \mathsf{D}_\vartheta^{p,q,d}(\xi_{r(\vartheta)})^2 - \mathsf{D}_\theta^{p,q,d}(\xi_{r(\theta)})^2 \right| \leqslant \frac{\varepsilon}{2} + |\mathsf{F}_\vartheta^{p,q,d}(\xi_{r(\vartheta)})^2 - \mathsf{F}_\theta^{p,q,d}(\xi_{r(\theta)})^2|. \tag{3.6.3}$$

Note that for any $\eta \in \mathcal{S}(\mathbb{C}^d)$, the quantity $\mathsf{F}_\vartheta^{p,q,d}(\eta)$ is finite as the maximum of finitely many values. Also note that the set F does not change with $\vartheta \in I$ — the only dependence of $\mathsf{F}_\vartheta^{p,q,d}$ on ϑ is via the choice of the quantum torus norm $\|\cdot\|_{\mathscr{H}_\vartheta^{p,q,d}}$.

Now the key observation is that $\vartheta \in I \mapsto \mathsf{F}_\vartheta^{p,q,d}(\xi)$ is continuous. Fix $(z,w,r) \in F$. By Proposition (3.2.5), the function:

$$\vartheta \in I \mapsto \left\| \frac{\alpha_{\eth,d}^{z,w,\frac{zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)}}{2\pi|\eth|r} \right\|_{\mathscr{H}_\vartheta^{p,q,d}}$$

is continuous (we note that $\vartheta \in I \mapsto \alpha_{\eth,d}^{z,w,\frac{zw}{2}} \xi_{r(\vartheta)} - \xi_{r(\vartheta)}$ satisfies the necessary hypothesis, owing to ξ being a Schwarz function and r being continuous. The details follow similar lines to our proof of Lemma (3.6.1) and we shall omit them this time around).

Thus $\vartheta \in I \mapsto \mathsf{F}_{\vartheta}^{p,q,d}(\xi_{r(\vartheta)})$ is the maximum of finitely many continuous functions, and is therefore continuous as well.

Thus there exists $\delta_3 > 0$ such that for all $\vartheta \in [\theta - \delta_3, \theta + \delta_3]$ we have:

$$|\mathsf{F}_{\vartheta}^{p,q,d}(\xi_{r(\vartheta)})^2 - \mathsf{F}_{\theta}^{p,q,d}(\xi_{r(\theta)})^2| < \frac{\varepsilon}{2}.$$

Thus if $\delta = \min\{\delta_1, \delta_3\} > 0$ then for all $\vartheta \in [\theta - \delta, \theta + \delta]$ we have:

$$\left| \mathsf{D}_{\vartheta}^{p,q,d}(\xi_{r(\vartheta)})^2 - \mathsf{D}_{\theta}^{p,q,d}(\xi_{r(\theta)})^2 \right| < \varepsilon.$$

Since $\mathsf{D}_{\vartheta}(\xi_{r(\vartheta)}) \geqslant 0$ for all $\vartheta \in [\theta - \delta, \theta + \delta]$ and $\sqrt[2]{\cdot}$ is a continuous function on $[0, \infty)$, we have shown that:

$$\vartheta \mapsto \mathsf{D}_{\vartheta}^{p,q,d}(\xi_{r(\vartheta)})$$

is continuous. □

Corollary 3.6.3. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$. If $\xi \in \mathcal{S}(\mathbb{C}^d)$, then:*

$$\theta \in \mathbb{R} \setminus \left\{ \frac{p}{q} \right\} \mapsto \mathsf{D}_{\theta}^{p,q,d}(\xi)$$

is continuous.

Proof. This follows from Proposition (3.6.2) using $r : x \in \mathbb{R} \mapsto 1$. □

3.7. **Convergence.** We now present our main convergence result for the modular propinquity. Our first step consists in finding an appropriate choice of anchors. We establish two lemmas to this end. The first lemma extends Lemma (3.5.10) by proving that while the range of the operators involved in Lemma (3.5.10) depends on the parameters used to define the Heisenberg modules, its dimension does not. The second lemma then uses the particular basis of Hermite functions obtained in the first lemma to construct our anchors.

Lemma 3.7.1. *For all $j \in \mathbb{N}$ and $\eth > 0$, let:*

$$\psi_{\eth}^{j} : x \in [0, \infty) \mapsto \eth \exp\left(-\frac{\pi \eth r^2}{2} \right) \mathrm{L}_j \left(\pi \eth r^2 \right)$$

where $\mathrm{L}_j : t \in \mathbb{R} \mapsto \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{j}{j-k} t^k$ is the j^{th} Laguerre polynomial.

Note that $\psi_{\eth}^{j}(t) = \eth \psi_1^j(\sqrt{\eth} t)$ for all $t \geqslant 0$, with ψ_1^j the j-Laguerre function.

Let f be compactly supported continuous. For all $j \in \mathbb{N}$ and $\eth \neq 0$, we set:

$$\mathrm{C}_{\eth}^{j}(f) = \frac{1}{j} \sum_{k=0}^{j} \frac{j+1-k}{j+1} \left\langle f \psi_{\eth}^{j}, \psi_{\eth}^{j} \right\rangle_{L^2(\mathbb{R}, r\,dr)} \psi_{\eth}^{j}.$$

For all $\varepsilon > 0$ and $\eth_0 \neq 0$, there exists $N \in \mathbb{N}$ and $\delta \in (0, |\eth_0|)$ such that, for all $\eth \in [\eth_0 - \delta, \eth_0 + \delta]$, we have:

$$\left\| f - \sum_{j=0}^{N} \mathrm{C}^j_{\eth}(f) \right\|_{L^1(\mathbb{R}_+, rdr)} \leqslant \varepsilon.$$

Proof. We fix $\eth_0 \neq 0$. By [50, Theorem 6.2.1], as in the proof of Lemma (3.5.10), there exists $N > 0$ such that:

$$\left\| f - \sum_{j=0}^{N} \mathrm{C}^j_{\eth_0}(f) \right\|_{L^1(\mathbb{R}_+, rdr)} \leqslant \frac{\varepsilon}{2}.$$

Let $Q = \sum_{j=0}^{N} \frac{N+1-j}{N+1}$.

Let $K_1 > 0$ be chosen so that $f(x) = 0$ whenever $x \geqslant K_1$ (as f is compactly supported by assumption). Let $M_1 = \int_0^\infty |f(r)|\, rdr$.

Let $M_2 = \max\{\|\psi^j_{\eth_0}\|_{L^1(\mathbb{R}_+, rdr)} : j \in \{0, \ldots, N\}\}$. Now, there exists $C > 0$ and $K_2 > 0$ such that for all $x \geqslant K_2$ and for all $j \in \{0, \ldots, N\}$:

$$|\psi^j_1(r)| \leqslant C \exp\left(-\frac{r^2}{4}\right).$$

Indeed, one checks trivially that $\lim_{r\to\infty} \exp\left(\frac{1}{4}r^2\right)\psi^j_1(r) = 0$ and once again, we work with finitely many functions.

Let $K_3 \geqslant K_2 > 0$ be chosen so that $\int_{K_3}^\infty \exp\left(-\frac{r^2}{4}\right) rdr \leqslant \frac{\varepsilon}{16 C M_3 Q N}$.

Last, let $M_3 = \max\left\{ \left| \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \right| : j \in \{0, \ldots, N\} \right\}$.

For all $j \in \{0, \ldots, N\}$, the function ψ^j_1 is continuous on $\mathbb{R}_+$, so $(\psi^j_1)^2$ is continuous on $[0, K_3]$, and thus the family $\{\psi^j_1, (\psi^j_1)^2 : j \in \{0, \ldots, N\}\}$ is uniformly equicontinuous on this compact interval. Thus there exists $\delta_1 > 0$ such that if $|x - y| \leqslant \delta_1$ then $|(\psi^j_1)^2(x) - (\psi^j_1)^2(y)| \leqslant \frac{\varepsilon}{8Q\eth_0^2 M_1 M_2 N}$ and $|\psi^j_1(x) - \psi^j_1(y)| \leqslant \frac{\varepsilon}{16 Q \eth_0 N M_3}$ for all $j \in \{0, \ldots, N\}$ (note that of course, it is important here that we work with finitely many functions, so we trivially have a uniformly equicontinuous family).

Using the continuity of the square root function and the square function, there exists $\delta_2 \in (0, \delta_1)$ such that if $|\eth - \eth_0| \leqslant \delta_2$ then $|\sqrt{\eth} - \sqrt{\eth_0}| \leqslant \frac{\delta_1}{K_1}$ and $|\eth^2 - \eth_0^2| \leqslant \frac{\varepsilon}{8QM_2^2 N}$. Therefore, $|\sqrt{\eth}x - \sqrt{\eth_0}x| \leqslant \delta_1$ for all $x \in \mathbb{R}_+$ with $|x| \leqslant K_1$, so that for all $j \in \{0, \ldots, N\}$:

$$\begin{aligned}
\left| \int_0^\infty f(r) \left((\psi^j_{\eth})^2(r) - (\psi^j_{\eth_0}(r))^2 \right) rdr \right| &\leqslant M_1 \sup_{|r| \leqslant K_1} |\eth^2 (\psi^j_1)^2(\sqrt{\eth} r) - \eth_0^2 (\psi^j_1)^2(\sqrt{\eth_0} r)| \\
&\leqslant M_1 \sup_{|r| \leqslant K_1} \eth_0^2 |(\psi^j_1)^2(\sqrt{\eth} r) - (\psi^j_1)^2(\sqrt{\eth_0} r)| \\
&\quad + 2M_2 |\eth^2 - \eth_0^2| \\
&\leqslant \frac{\varepsilon}{4QM_2 N}.
\end{aligned}$$

We thus conclude that for all $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2]$ and $j \in \{0, \ldots, N\}$:

$$\left| \left\langle f\psi^j_\eth, \psi^j_\eth \right\rangle_{L^2(\mathbb{R}_+, rdr)} - \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \right| \leqslant \frac{\varepsilon}{4QM_2N}$$

Moreover, for $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2]$:

$$\begin{aligned}
&\left| \int_0^\infty |\psi^j_\eth(r)| - |\psi^j_{\eth_0}(r)| \, rdr \right| \\
&\quad \leqslant \int_0^{K_3} |\eth\psi^j_1(\sqrt{\eth}r) - \eth_0\psi^j_1(\sqrt{\eth_0}r)| \, rdr + \int_{K_3}^\infty |\eth\psi^j_1(\sqrt{\eth}r) - \eth_0\psi^j_1(\sqrt{\eth_0}r)| \, rdr \\
&\quad \leqslant 2\eth_0 \left(\frac{\varepsilon}{16\eth_0 QM_3N} + \int_{\complement K} 2C \exp(-Dr^2) \, rdr \right) \\
&\quad \leqslant \frac{\varepsilon}{4QM_3N}.
\end{aligned}$$

Therefore, for all $j \in \{0, \ldots, N\}$ and for all $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2]$:

$$\begin{aligned}
&\left\| \left\langle f\psi^j_\eth, \psi^j_\eth \right\rangle_{L^2(\mathbb{R}_+, rdr)} \psi^j_\eth - \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \psi^j_{\eth_0} \right\|_{L^1(\mathbb{R}_+, rdr)} \\
&\qquad \leqslant \left\| \left\langle f\psi^j_\eth, \psi^j_\eth \right\rangle_{L^2(\mathbb{R}_+, rdr)} \psi^j_\eth - \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \psi^j_\eth \right\|_{L^1(\mathbb{R}_+, rdr)} \\
&\qquad\quad + \left| \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \right| \left\| \psi^j_\eth - \psi^j_{\eth_0} \right\|_{L^1(\mathbb{R}_+, rdr)} \\
&\qquad \leqslant \|\psi^j_\eth\|_{L^1(\mathbb{R}_+, rdr)} \left| \left\langle f\psi^j_\eth, \psi^j_\eth \right\rangle_{L^2(\mathbb{R}_+, rdr)} - \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \right| \\
&\qquad\quad + M_3 \|\psi^j_\eth - \psi^j_{\eth_0}\|_{L^1(\mathbb{R}_+, rdr)} \\
&\qquad \leqslant \frac{\varepsilon}{4QNM_2} M_2 + M_3 \frac{\varepsilon}{4QNM_3} \\
&\qquad \leqslant \frac{\varepsilon}{2QN}.
\end{aligned}$$

Consequently, for all $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2]$ and $j \in \{0, \ldots, N\}$:

$$\begin{aligned}
&\left\| \mathrm{C}^j_\eth(f) - \mathrm{C}^j_{\eth_0}(f) \right\|_{L^1(\mathbb{R}_+, rdr)} \\
&\quad \leqslant \sum_{k=1}^j \frac{j+1-k}{j+1} \left\| \left\langle f\psi^j_\eth, \psi^j_\eth \right\rangle_{L^2(\mathbb{R}_+, rdr)} \psi^j_\eth - \left\langle f\psi^j_{\eth_0}, \psi^j_{\eth_0} \right\rangle_{L^2(\mathbb{R}_+, rdr)} \psi^j_{\eth_0} \right\|_{L^1(\mathbb{R}_+, rdr)} \\
&\quad \leqslant \frac{\varepsilon}{2N}.
\end{aligned}$$

Thus for all $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2]$:

$$\left\| f - \sum_{j=1}^N C^j_\eth(f) \right\|_{L^1(\mathbb{R}_+, rdr)}$$

$$\begin{aligned}
&\leqslant \left\| f - \sum_{j=1}^{N} C^j_{\eth_0}(f) \right\|_{L^1(\mathbb{R}_+, rdr)} + \left\| \sum_{j=0}^{N} (C^j_{\eth} - C^j_{\eth_0}) \right\|_{L^1(\mathbb{R}_+, rdr)} \\
&\leqslant \frac{\varepsilon}{2} + N \max\{ \| C^j_{\eth}(f) - C^j_{\eth_0}(f) \|_{L^1(\mathbb{R}_+, rdr)} : j \in \{0, \ldots, N\}\} \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This concludes our lemma. □

Lemma 3.7.2. *Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$. Let $\eth_0 \neq 0$. For all $\varepsilon > 0$, there exist $\delta \in \left(0, \frac{|\eth_0|}{2}\right)$ and a finite subset $\mathfrak{F}$ of $\mathscr{D}_1\left(\mathsf{D}^{p,q,d}_{\eth + \frac{p}{q}}\right) \setminus \{0\}$ for $\|\cdot\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth_0}}$ such that, for all $\eth \in [\eth_0 - \delta, \eth_0 + \delta]$, the set:*

$$\left\{ \frac{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth_0}(\omega)}{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\omega)} \omega : \omega \in \mathfrak{F} \right\}$$

is a ε-dense subset of $\mathscr{D}_1\left(\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}\right)$ for $\|\cdot\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}}$.

Proof. Let $\varepsilon > 0$ be given. By Lemmas (3.5.8), (3.5.6) and (3.5.9), there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, if $g : (x,y) \in \mathbb{R}^2 \mapsto f\left(\sqrt{x^2+y^2}\right)$ then:

(1) $\int_{\mathbb{R}_+} f(r)\, rdr = \frac{1}{2\pi}$,
(2) $\iint_{\mathbb{R}^2} g(x,y) \|(x,y)\| \, dxdy \leqslant \frac{\varepsilon}{48\pi\eth_0} = \frac{\frac{\varepsilon}{16}}{2\pi \frac{3\eth_0}{2}}$,
(3) $\mathsf{D}^{p,q,d}_{\vartheta}(\sigma^g_{\eth,d}\xi) \leqslant (1 + \frac{\varepsilon}{16})\mathsf{D}^{p,q,d}_{\vartheta}(\xi)$ for all $\vartheta \neq \frac{p}{q}$ and all $\xi \in \mathcal{S}(\mathbb{C}^d)$.

as in Lemma (3.5.11), for instance given by Lemma (3.5.8). By Lemma (3.5.6), we have that for all $\omega \in \mathcal{S}(\mathbb{C}^d)$, we have:

$$\left\| \omega - \sigma^g_{\eth,d}\omega \right\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} \leqslant \frac{\varepsilon}{16} \mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\omega),$$

for all $\eth \in \left[\frac{\eth_0}{2}, \frac{3\eth_0}{2}\right]$.

We apply Lemma (3.7.1) to obtain some $N \in \mathbb{N}$ and $\delta_0 \in \left(0, \frac{|\eth_0|}{2}\right)$ such that:

$$\left\| f - \sum_{j=0}^{N} \mathsf{C}^j_{\eth}(f) \right\|_{L^1(\mathbb{R}_+, rdr)} \leqslant \frac{\varepsilon}{8}.$$

We now note that thanks to a change of variable in the definition of the operator $\sigma^g_{\eth,d}$, it is sufficient to prove our result for $\eth > 0$. We shall henceforth assume $\eth > 0$.

Let $h_\eth : (x,y) \in \mathbb{R}^2 \mapsto \sum_{j=0}^{N} \mathsf{C}^j_{\eth}(f)(\sqrt{x^2+y^2})$ for all $\eth \neq 0$.

For each $\eth \in [\eth_0 - \delta_0, \eth_0 + \delta_0]$, we then have, in a manner similar to the proof of Lemma (3.5.10):

$$\left|\left|\left| \sigma^g_{\eth,d} - \sigma^{h_\eth}_{\eth,d} \right|\right|\right|_{\mathscr{H}^{p,q,d}_{\theta}} \leqslant \| f - \sum_{j=0}^{N} \mathsf{C}^j_{\eth}(f) \|_{L^1(\mathbb{R}_+, rdr)} \leqslant \frac{\varepsilon}{8}.$$

Consequently, for all $\eth \in [\eth_0 - \delta_0, \eth_0 + \delta_0]$ and for all $\omega \in \mathscr{D}_1\left(\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}\right)$, we have:

$$\left\|\omega - \sigma^{h_\eth}_{\eth,d}\omega\right\|_{\mathscr{H}^{p,q,d}} \leqslant \frac{3\varepsilon}{16},$$

therefore:

$$\left\|\omega - \frac{1}{1+\frac{\varepsilon}{16}}\sigma^{h_\eth}_{\eth,d}\omega\right\|_{\mathscr{H}^{p,q,d}_{\eth+\frac{p}{q}}} \leqslant \frac{\frac{\varepsilon}{16}}{1+\frac{\varepsilon}{16}}\|\omega\|_{\mathscr{H}^{p,q,d}} + \frac{3\varepsilon}{16} \leqslant \frac{\varepsilon}{4}$$

while $\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}\left(\frac{1}{1+\frac{\varepsilon}{16}}\sigma^{h_\eth}_{\eth,d}\omega\right) \leqslant 1$.

Our efforts thus far show that the range of $\sigma^{h_\eth}_{\eth,d}$ is of dimension $N+1$, spanned by:

$$\left\{\mathcal{H}^j_\eth = \eth^{\frac{1}{4}}\mathcal{H}^j_1\left(\sqrt{\eth}\cdot\right) : j \in \{0,\ldots,N\}\right\}$$

where:

$$\mathcal{H}^j_1 : t \in \mathbb{R} \mapsto \frac{(2)^{\frac{1}{4}}}{\sqrt{j!2^j}}\exp\left(-\frac{t^2\sqrt{2\pi}}{2}\right)\mathrm{H}_j\left(t\sqrt{2\pi}\right)$$

and H_j is the j^{th} Hermite polynomial, as seen in Lemma (3.5.10).

Let $V = \mathbb{C}^N$. For any $\eth > 0$ we define $\eta_\eth : (c_j)_{j\in\{1,\ldots,N\}} \in V \mapsto \sum_{j=1}^N c_j\mathcal{H}^j_\eth$. The map $\eta_\eth$ is a linear injection from V to $\mathcal{S}\otimes\mathbb{C}^d$. For each $\eth > 0$ and $c \in V$, we set $\|c\|_\eth = \mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_\eth(c))$; of course $\|\cdot\|_\eth$ is a norm on V.

We now set $\|c\|_V = \sup_{\eth\in[\eth_0-\delta_0,\eth_0+\delta_0]}\|c\|_\eth$. By construction, it is sufficient to check that $\|\cdot\|_V$ is valued in $\mathbb{R}_+$ (i.e. is never infinite) to conclude that $\|\cdot\|_V$ is a norm on V.

Let $c = (c_j)_{j\in\{0,\ldots,N\}} \in V$. Note that for all $t \in \mathbb{R}$ and $\eth > 0$:

$$\begin{aligned}\eta_\eth(c)(t) &= \sum_{j=0}^N c_j\mathcal{H}^j_\eth(t)\\ &= (\eth)^{\frac{1}{4}}\sum_{j=0}^N \mathcal{H}^j_1(\sqrt{\eth}t)\\ &= (\eth)^{\frac{1}{4}}\,\eta_1(c)(\sqrt{\eth}t).\end{aligned} \tag{3.7.1}$$

and of course, $\eta_1 \in \mathcal{S}(\mathbb{C}^d)$. Thus by Proposition (3.6.2), we conclude that $\eth \in [\eth_0-\delta_0,\eth_0+\delta_0] \mapsto \mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_\eth(c))$ is continuous as well as the product of the two continuous functions $\eth \in [\eth_0-\delta_0,\eth_0+\delta_0] \mapsto \mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_1(c)(\sqrt{\eth}\cdot))$ (by Proposition (3.6.2)), and $\eth \in [\eth_0-\delta_0,\eth_0+\delta_0] \mapsto \sqrt[4]{\eth}$.

Therefore, it reaches its maximum on the compact $[\eth_0-\delta_0,\eth_0+\delta_0]$, which is by definition the number $\|c\|_V$.

Thus $\|\cdot\|_V$ is a norm on V.

We now make another observation. We have, for all $\eth \in [\eth_0 - \delta_0, \eth_0 + \delta_0]$:

$$\begin{aligned}\left|\|c\|_{\eth} - \|d\|_{\eth_0}\right| &\leqslant \left|\|c\|_{\eth} - \|c\|_{\eth_0}\right| + \left|\|c\|_{\eth_0} - \|d\|_{\eth_0}\right| \\ &\leqslant \left|\|c\|_{\eth} - \|c\|_{\eth_0}\right| + |\|c-d\|_{\eth_0} \\ &\leqslant \left|\|c\|_{\eth} - \|c\|_{\eth_0}\right| + |\|c-d\|_{V} .\end{aligned}$$

Thus the function:

$$\mathsf{n} : (\eth, c) \in [\eth_0 - \delta_0, \eth_0 + \delta_0] \times V \mapsto \|c\|_{\eth}$$

is continuous. It is in particular continuous on the compact $[\eth_0 - \delta_0, \eth_0 + \delta_0] \times B$ where B is the closed unit ball for $\|\cdot\|_V$.

Therefore there exists $k > 0$ such that for all $c \in V$, $\eth \in [\eth_0 - \delta_0, \eth_0 + \delta_0]$, we have:

$$k\|c\|_V \leqslant \|c\|_{\eth} \leqslant \|c\|_V .$$

Let now $E = \{c \in V : \|c\|_V \leqslant \frac{1}{k}\}$. Since V is finite dimensional, E is compact. Therefore, the function n is uniformly continuous on the compact $[\eth_0 - \delta_0, \eth_0 + \delta_0] \times E$. Let $\delta_1 \in (0, \delta_0)$ be chosen so that, for all $\eth \in [\eth_0 - \delta_1, \eth_0 + \delta_1]$, and for all $c, d \in E$ with $\|c-d\|_V \leqslant \delta_1$, we have:

$$|\mathsf{n}(\eth, c) - \mathsf{n}(\eth_0, d)| \leqslant \frac{k\varepsilon}{8}.$$

In particular, for all $\eth \in [\eth_0 - \delta_1, \eth_0 + \delta_1]$ and all $c \in E$, we have:

$$\left|\|c\|_{\eth} - \|c\|_{\eth_0}\right| \leqslant \frac{k\varepsilon}{8}.$$

Let $\mathfrak{E} = \frac{1}{1+\frac{\varepsilon}{\ }}\sigma^{g}_{\eth_0,d}\left(\mathscr{D}_1\left(\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth_0}\right)\right)$. By definition, $\mathfrak{E}$ is a bounded subset of V which is finite dimensional. Thus $\mathfrak{E}$ is totally bounded for $\|\cdot\|_V$. Let $\mathfrak{F}$ be a finite $\frac{\varepsilon}{8}$-dense subset of $\mathfrak{E}$ for $\|\cdot\|_V$. We assume $0 \notin \mathfrak{F}$ (we can simply pick a $\frac{\varepsilon}{16}$-dense subset of $\mathfrak{E}$ and then remove 0 from it if needed).

For all $c \in \mathfrak{F}$, the function:

$$\mathsf{l}_c : \eta \in [\eth_0 - \delta_1, \eth_0 + \delta_1] \mapsto \frac{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth_0}(\eta_{\eth_0}(c)) - \mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_{\eth_0}(c))}{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_{\eth_0}(c))}$$

is continuous on a compact, and it is null at $\eth_0$; hence there exists $\delta_2 \in (0, \delta_1)$ such that for all $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2]$ and $c \in \mathfrak{F}$, we have:

$$\mathsf{l}_c(\eth) \leqslant \frac{k\varepsilon}{4}.$$

We emphasize that in the definition of l_c, for any $c \in \mathfrak{F}$, only involves the element $\eta_{\eth_0}(c)$, and the only dependence on the variable is through the choice of D-norm.

Last, for all $d \in \mathfrak{F}$, the function $\eth \in [\eth_0 - \delta_2, \eth_0 + \delta_2] \mapsto \|\eta_{\eth}(d) - \eta_{\eth_0}(d)\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}}$ is continuous by Proposition (3.2.5) and Expression (3.7.1). Therefore, since $\mathfrak{F}$ is finite, there exists $\delta_3 \in (0, \delta_2)$ such that for all $\eth \in [\eth_0 - \delta_3, \eth_0 + \delta_3]$ and for all $d \in \mathfrak{F}$:

$$\left\|\eta_{\eth}(d) - \eta_{\eth_0}(d)\right\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} \leqslant \frac{\varepsilon}{4}.$$

Fix now $\eth \in [\eth_0 - \delta_3, \eth_0 + \delta_3]$. Let now $\eta \in \mathscr{D}_1\left(\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}\right)$. Let $c \in V$ so that $\frac{1}{1+\frac{\varepsilon}{16}}\sigma^{h_\eth}_{\eth,d}(\eta) = \eta_\eth(c)$ and note that by construction, $\|\eta - \eta_\eth(c)\|_{\mathscr{H}^{p,q,d}_{\eth+\frac{p}{q}}} \leqslant \frac{\varepsilon}{4}$ while $\mathsf{D}_{\frac{p}{q}+\eth}(\eta_\eth(c)) \leqslant 1$.

Since $\mathsf{D}^{p,q,d}_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}}(\eta_\eth(c)) \leqslant 1$ so $\|c\|_V \leqslant \frac{1}{k}$. Thus, $\|\eta_{\eth_0}(c)\| \leqslant 1 + \frac{k\varepsilon}{8}$.

Thus, $\frac{1}{1+\frac{k}{8}\varepsilon}\eta_{\eth_0}(c) \in \mathfrak{E}$, and thus there exists $d \in \mathfrak{F}$ such that $\|\frac{1}{1+\frac{k}{8}\varepsilon}c - d\|_V \leqslant \frac{\varepsilon}{8}$. Thus:

$$\|c-d\|_V \leqslant \frac{\frac{k}{8}\varepsilon}{1+\frac{k}{8}\varepsilon}\|c\|_V + \frac{\varepsilon}{8} \leqslant \frac{\varepsilon}{4}.$$

We conclude by observing that:

$$\begin{aligned}
&\left\| \frac{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth_0}(\eta_{\eth_0}(d))}{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_{\eth_0}(d))}\eta_{\eth_0}(d) - \eta \right\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} \\
&\qquad \leqslant \left\| \frac{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth_0}(\eta_{\eth_0}(d))}{\mathsf{D}^{p,q,d}_{\frac{p}{q}+\eth}(\eta_{\eth_0}(d))}\eta_{\eth_0}(d) - \eta_\eth(c) \right\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} + \frac{\varepsilon}{4} \\
&\qquad \leqslant \mathsf{l}_d(\eth)\|d\|_V + \|\eta_{\eth_0}(d) - \eta_\eth(c)\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} + \frac{\varepsilon}{4} \\
&\qquad \leqslant \frac{\varepsilon}{4} + \|\eta_\eth(d) - \eta_{\eth_0}(d)\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} + \|\eta_\eth(d) - \eta_\eth(c)\|_{\mathscr{H}^{p,q,d}_{\frac{p}{q}+\eth}} + \frac{\varepsilon}{4} \\
&\qquad \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \|c-d\|_V + \frac{\varepsilon}{4} \leqslant \varepsilon.
\end{aligned}$$

This concludes our lemma. ☐

We now summarize, in the following lemma, all the elements of the proof of convergence for quantum tori, as worked in [25], which we will employ in the current paper.

Notation 3.7.3. Let μ be the probability Haar measure on the 2-torus $\mathbb{T}^2$.

For any $f \in L^1(\mathbb{T}^2, \mu)$ and $\theta \in \mathbb{R}$, we denote by β^f_θ the operator on $\mathcal{A}_\theta$ defined for all $a \in \mathcal{A}_\theta$ by:

$$\beta^f_\theta(a) = \int_{\mathbb{T}^2} f(z)\beta^z_\theta(a)\,d\mu(z)$$

which is continuous with $\left|\left|\left|\beta^f_\theta\right|\right|\right|_{\mathcal{A}_\theta} \leqslant \|f\|_{L^1(\mathbb{T}^2,\mu)}$.

Lemma 3.7.4. *Let ℓ be a continuous length function on $\mathbb{T}^2$. Let $\theta \in \mathbb{R}$ and $\varepsilon > 0$. There exists $\delta_\varepsilon > 0$, a trace-class operator T on $\ell^2(\mathbb{Z}^2)$ with nonempty 1-level set and operator norm equals to 1, a finite dimensional subspace $V \subseteq \ell^1(\mathbb{Z}^2)$ and a nonnegative continuous function $\mathsf{Fe} : \mathbb{T}^2 \mapsto [0,\infty)$ such that, for all $\vartheta \in [\theta - \delta_\varepsilon, \theta + \delta_\varepsilon]$:*

(1) if $a \in V$ then $a^* \in V$,

(2) *$\beta_\theta^{\mathsf{Fe}}$ is a finite rank operator and $\beta_\theta^{\mathsf{Fe}}(\mathfrak{sa}(\mathcal{A}_\theta)) = V$,*
(3) *$1_{\mathcal{A}_\vartheta} \in V$,*
(4) *the function:*
$$(\vartheta, a) \in \left(\mathbb{R} \setminus \left\{\frac{p}{q}\right\}\right) \times V \mapsto \mathsf{L}_\vartheta(a)$$
is continuous,
(5) *if $\tau : f \in \ell^1(\mathbb{Z}^2) \mapsto f(0)$, i.e. the restriction of the unique β_ϑ-invariant tracial state of $\mathcal{A}_\vartheta$ to $\ell^1(\mathbb{Z}^2)$ (noting τ does not depend on ϑ), and if $E = V \cap \ker(\tau)$ while Σ is the unit sphere in E for $\|\cdot\|_{\ell^1(\mathbb{Z}^2)}$, then for any $a \in \Sigma$, if $s(a,\vartheta) = \frac{\mathsf{L}_\theta(a)}{\mathsf{L}_\vartheta(a)} > 0$ then:*
$$|||\pi_\theta(a)T - T\pi_\vartheta(s(a)a)|||_{\ell^2(\mathbb{Z}^2)} \leqslant \mathsf{L}(a)\varepsilon,$$
while:
$$|1 - s(a,\vartheta)| < \varepsilon;$$
(6) *the length of the bridge $\left(\mathfrak{B}(\ell^2(\mathbb{Z}^2)), T, \pi_\theta, \pi_\vartheta\right)$ is no more than ε, where $\mathfrak{B}(\ell^2(\mathbb{Z}^2))$ is the C*-algebra of all bounded linear operators on $\ell^2(\mathbb{Z}^2)$.*

Proof. The construction of the bridges in this lemma is the matter of [25] — including the construction of T. We will only need its existence and the properties listed here, which involve all the work in [25] to be established.

We note that Assertions (1), (2), (3) and (4) were established in [49]; a summary is presented in [25, Theorem 3.19] (all these assertions are extended to fuzzy tori in [21]).

Assertion (5) is established as part of [25, Claim 5.15]. Assertion (6) is [25, Claim 5.15]. Of course, the computation of the length of the bridges defined in Assertion (6) provides the upper bound on the quantum propinquity between quantum tori in [25]. □

Corollary 3.7.5. *Let $\varepsilon > 0$; let $\delta > 0$ be given by Lemma (3.7.4). If for all $a \in E \setminus \{0\}$ and $\vartheta \in [\theta - \delta, \theta + \delta]$, we set $s(a,\vartheta) = \frac{\mathsf{L}_\theta(a)}{\mathsf{L}_\vartheta(a)} > 0$, then:*
$$|||\pi_\theta(a)T - T\pi_\vartheta(s(a)a)|||_{\ell^2(\mathbb{Z}^2)} \leqslant \mathsf{L}_\vartheta(a)\varepsilon.$$

Proof. Fix $\vartheta \in [\theta - \delta, \theta + \delta]$. If $\mathsf{L}_\vartheta(a) = 0$ then $a \in \mathbb{R}1_{\mathfrak{A}_\vartheta}$; as $a \in E$ we conclude that $a = 0$. Thus $s(a,\vartheta)$ is well-defined.

We then note that $s(a) = s(ra)$ for any $r > 0$ by definition. Moreover, if $a \in E$ and $a \neq 0$, then $\frac{1}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a \in \Sigma$ and thus by Lemma (3.7.4) :
$$\begin{aligned}
&|||\pi_\vartheta(a)T - T\pi_\vartheta(s(a)a)|||_{\ell^2(\mathbb{Z}^2)} \\
&\quad = \|a\|_{\ell^1(\mathbb{Z}^2)} \left|\left|\left|\pi_\theta\left(\frac{1}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a\right) T - T\pi_\vartheta\left(\frac{s(\|a\|_{\ell^1(\mathbb{Z}^2)}^{-1} a)}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)} \\
&\quad \leqslant \|a\|_{\ell^1(\mathbb{Z}^2)} \mathsf{L}_\vartheta\left(\frac{1}{\|a\|_{\ell^1(\mathbb{Z}^2)}} a\right)\varepsilon = \mathsf{L}_\vartheta(a)\varepsilon.
\end{aligned}$$
This concludes our corollary. □

We now conclude our paper with the main result of its second part, which demonstrates that the modular propinquity endows the moduli space of Heisenberg modules over quantum 2-tori with a nontrival geometry.

Theorem 3.7.6. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. For all $\theta \in \mathbb{R}$, we equip the quantum torus $\mathcal{A}_\theta$ with the L-seminorm:*

$$\mathsf{L}_\theta : a \in \mathfrak{sa}(\mathfrak{A}) \mapsto \sup\left\{ \frac{\left\|\beta_\theta^{\exp(ix),\exp(iy)} a - a\right\|_{\mathcal{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

For all $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $d \in q\mathbb{N} \setminus \{0\}$ and for all $\theta \in \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$, we endow the Heisenberg module $\mathscr{H}_\theta^{p,q,d}$ with the D-norm:

$$\mathsf{D}_\theta^{p,q,d} : \xi \in \mathscr{H}_\theta^{p,q,d} \mapsto$$
$$\sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\|\exp\left(i\pi\left(\theta - \frac{p}{q}\right)xy\right)\sigma_{\mathfrak{d},d}^{x,y}\xi - \xi\right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi\left(\theta - \frac{p}{q}\right)\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

Let $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$. Let $d \in q\mathbb{N} \setminus \{0\}$. For any $\theta \in \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$, we have:

$$\lim_{\vartheta \to \theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathcal{A}_\vartheta, \mathsf{L}_\vartheta\right),\right.$$
$$\left.\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathcal{A}_\theta, \mathsf{L}_\theta\right)\right) = 0.$$

Proof. Let $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$. Let $d \in q\mathbb{N} \setminus \{0\}$. Let $X = \mathbb{R} \setminus \left\{\frac{p}{q}\right\}$.

Let $\theta \in X$ and let $\varepsilon > 0$.

We shall apply Lemma (3.7.4) and use its notations for $\frac{\varepsilon}{16} > 0$ (rather than ε).

To begin with, for all $\vartheta \in \mathbb{R}$, we note that if $a \in \ell^1(\mathbb{Z}^2)$, then $\beta_\vartheta^z(a) = \beta^z(a)$ does not depend on $\vartheta \in \mathbb{R}$ for any $z \in \mathbb{T}^2$. Thus, the restriction of $\beta_\vartheta^{\mathsf{Fe}}$ to $\ell^1(\mathbb{Z}^2)$ is independent of ϑ, valued in V, and will be denoted by β^{Fe}.

By Lemma (3.7.2), there exists a finite subset $F = \{\omega_j : j \in \{1,\ldots,N\}\}$ of $\mathscr{D}_1\left(\mathscr{H}_\theta^{p,q,d}\right) \setminus \{0\}$ for some $N \in \mathbb{N}$ and $\delta_0 > 0$ such that, for all $\vartheta \in [\theta - \delta_0, \theta + \delta_0]$, the set:

$$\left\{ \frac{\mathsf{D}_\theta^{p,q,d}(\omega_j)}{\mathsf{D}_\vartheta^{p,q,d}(\omega_j)}\omega_j : j \in \{1,\ldots,N\} \right\}$$

is $\frac{\varepsilon}{16}$-dense in $\mathscr{D}_1\left(\mathsf{D}_\vartheta^{p,q,d}\right)$.

We thus record:

Summary 3.7.7. Any modular bridge from $\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathcal{A}_\vartheta, \mathsf{L}_\vartheta\right)$ to $\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathcal{A}_\vartheta, \mathsf{L}_\vartheta\right)$ whose anchors are $(\omega_j)_{j\in\{1,\ldots,N\}}$ and co-anchors are $\left(\frac{\mathsf{D}_\theta^{p,q,d}(\omega_j)}{\mathsf{D}_\vartheta^{p,q,d}(\omega_j)}\omega_j\right)_{j\in\{1,\ldots,N\}}$, has imprint at most $\frac{\varepsilon}{16}$.

For each $\omega \in F$, the map $\vartheta \in X \mapsto \mathsf{D}_{\vartheta}^{p,q,d}(\omega)$ is continuous by Proposition (3.6.2). The function $\vartheta \mapsto \mathsf{L}_{\vartheta}(\langle \omega, \eta \rangle_{\mathscr{H}_{\vartheta}^{p,q,d}})$ is also continuous for all $\omega, \eta \in F$ (see Lemma (3.7.4)). Last, for any $\omega \in F$, we note that the continuous function $\vartheta \in X \mapsto \mathsf{D}_{\vartheta}^{p,q,d}(\omega)$, reaches its minimum on the compact $[\theta - \delta_{\frac{\varepsilon}{16}}, \theta + \delta_{\frac{\varepsilon}{16}}]$, and thus in particular, since $\omega \neq 0$ and $\mathsf{D}_{\vartheta}^{p,q,d}$ is a norm, this minimum is not zero (note that $\delta_{\frac{\varepsilon}{16}} > 0$ is given by Lemma (3.7.4)).

Thus the functions:

$$\mathsf{y}_{j,k}^{\Re} : \vartheta \in X \mapsto \frac{\mathsf{L}_{\theta}\left(\Re\beta^{\mathsf{Fe}}\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}}\right)}{\mathsf{L}_{\vartheta}\left(\Re\beta^{\mathsf{Fe}}\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}}\right)} - \frac{\mathsf{D}_{\theta}^{p,q,d}(\omega_j)\mathsf{D}_{\theta}^{p,q,d}(\omega_k)}{\mathsf{D}_{\vartheta}^{p,q,d}(\omega_j)\mathsf{D}_{\vartheta}^{p,q,d}(\omega_k)}$$

and

$$\mathsf{y}_{j,k}^{\Im} : \vartheta \in X \mapsto \frac{\mathsf{L}_{\theta}\left(\Im\beta^{\mathsf{Fe}}\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}}\right)}{\mathsf{L}_{\vartheta}\left(\Im\beta^{\mathsf{Fe}}\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}}\right)} - \frac{\mathsf{D}_{\theta}^{p,q,d}(\omega_j)\mathsf{D}_{\theta}^{p,q,d}(\omega_k)}{\mathsf{D}_{\vartheta}^{p,q,d}(\omega_j)\mathsf{D}_{\vartheta}^{p,q,d}(\omega_k)}$$

are continuous as well for all $j, k \in \{1, \ldots, N\}$. Consequently, the function:

$$\mathsf{y} = \max_{j,k \in \{1,\ldots,N\}} \left\{ \left|\mathsf{y}_{j,k}^{\Re}\right|, \left|\mathsf{y}_{j,k}^{\Im}\right| \right\}$$

is continuous as the maximum of finitely many continuous functions. We also note that $\mathsf{y}(\theta) = 0$.

Thus there exists $\delta_2 > 0$ such that:

$$|\mathsf{y}| < \frac{\varepsilon}{16} \text{ on } [\theta - \delta_2, \theta + \delta_2].$$

For each $j \in \{1, \ldots, N\}$, let:

$$\eta_j = \frac{\mathsf{D}_{\theta}(\omega_j)}{\mathsf{D}_{\vartheta}(\omega_j)}\omega_j.$$

By construction, we have:

$$\mathsf{D}_{\vartheta}(\eta_j) = \mathsf{D}_{\theta}(\omega_j).$$

Last, by Lemma (3.2.4), there exists $\delta_3 > 0$ such that for all $\vartheta \in [\theta - \delta_3, \theta + \delta_3]$ we have, for all $j, k \in \{1, \ldots, N\}$:

$$\left\| \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} - \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right\|_{\ell_1(\mathbb{Z}^2)} \leqslant \frac{\varepsilon}{16}.$$

Let $\delta_4 = \min\{\delta_{\frac{\varepsilon}{16}}, \delta_2, \delta_3\}$ and $\vartheta \in [\theta - \delta_4, \theta + \delta_4]$.

We now begin a string of inequalities for two given $j, k \in \{1, \ldots, N\}$. To begin with, we apply Lemma (3.7.4) to obtain for all $a \in \mathcal{A}_{\vartheta}$:

$$\|a - \beta^{\mathsf{Fe}}a\|_{\mathcal{A}_{\vartheta}} \leqslant \|\Re a - \beta^{\mathsf{Fe}}\Re a\|_{\mathcal{A}_{\vartheta}} + \|\Im a - \beta^{\mathsf{Fe}}\Im a\|_{\mathcal{A}_{\vartheta}} \leqslant \frac{\varepsilon}{16}\left(\mathsf{L}_{\vartheta}(\Re a) + \mathsf{L}_{\vartheta}(\Im a)\right) \leqslant \frac{\varepsilon}{8}\mathsf{L}_{\vartheta}(a).$$

Therefore, using the inner quasi-Leibniz inequality:

$$\left|\left|\left|\pi_{\theta}\left(\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}}\right) T - T\pi_{\vartheta}\left(\left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)} \tag{3.7.2}$$

$$
\begin{aligned}
&\leqslant \left\| \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} - \beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\theta}} \\
&\quad + \left\| \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} - \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} \\
&\quad + \left|\left|\left| \pi_{\theta} \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) T - T \pi_{\vartheta} \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right) \right|\right|\right|_{\ell^2(\mathbb{Z}^2)} \\
&\leqslant \frac{\varepsilon}{8} \left(\mathsf{L}_{\theta} \left(\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) + \mathsf{L}_{\vartheta} \left(\left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right) \right) \\
&\quad + \left|\left|\left| \pi_{\theta} \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) T - T \pi_{\vartheta} \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right) \right|\right|\right|_{\ell^2(\mathbb{Z}^2)} \\
&\leqslant \frac{\varepsilon}{8} (\mathsf{D}_{\theta}^{p,q,d}(\omega_j) \|\omega_k\|_{\mathscr{H}_{\theta}^{p,q,d}} + \mathsf{D}_{\theta}^{p,q,d}(\omega_k) \|\omega_j\|_{\mathscr{H}_{\theta}^{p,q,d}} \\
&\quad + \mathsf{D}_{\vartheta}^{p,q,d}(\eta_j) \|\eta_k\|_{\mathscr{H}_{\vartheta}^{p,q,d}} + \mathsf{D}_{\vartheta}^{p,q,d}(\eta_k) \|\eta_j\|_{\mathscr{H}_{\vartheta}^{p,q,d}}) \\
&\quad + \left|\left|\left| \pi_{\theta} \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) T - T \pi_{\vartheta} \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right) \right|\right|\right|_{\ell^2(\mathbb{Z}^2)} \\
&\leqslant \frac{\varepsilon}{2} + \left|\left|\left| \pi_{\theta} \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) T - T \pi_{\vartheta} \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right) \right|\right|\right|_{\ell^2(\mathbb{Z}^2)}.
\end{aligned}
$$

Our next step is to replace $\beta^{\mathsf{Fe}}\left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}}$ with $\beta^{\mathsf{Fe}}\left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}}$, because our work in [25] follows a single element in V from $\mathcal{A}_{\theta}$ to $\mathcal{A}_{\vartheta}$.

Now, noting that β^{Fe} has norm 1:

$$
\begin{aligned}
&\left\| \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} \\
&\qquad \leqslant \left\| \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} + \left\| \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} - \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} \\
&\qquad \leqslant \left\| \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} + \left\| \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} - \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} \\
&\qquad \leqslant \left\| \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} + \left\| \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} - \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\ell^1(\mathbb{Z}^2)} \\
&\qquad \leqslant \left\| \beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right\|_{\mathcal{A}_{\vartheta}} + \frac{\varepsilon}{16}.
\end{aligned}
$$

Thus we conclude, as $|||T|||_{\ell^2(\mathbb{Z}^2)} = 1$:

$$
\begin{aligned}
&\left|\left|\left| \pi_{\theta} \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) T - T \pi_{\vartheta} \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\vartheta}^{p,q,d}} \right) \right|\right|\right|_{\ell^2(\mathbb{Z}^2)} \\
&\leqslant \left|\left|\left| \pi_{\theta} \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) T - T \pi_{\vartheta} \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_{\theta}^{p,q,d}} \right) \right|\right|\right|_{\ell^2(\mathbb{Z}^2)} + \frac{\varepsilon}{16}.
\end{aligned} \tag{3.7.3}
$$

Inserting Inequality (3.7.3) in Inequality (3.7.2) we thus have:

$$\begin{aligned}(3.7.4)\quad &\left|\!\left|\!\left|\pi_\theta\left(\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(\left\langle\eta_j,\eta_k\right\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant \frac{9\varepsilon}{16} + \left|\!\left|\!\left|\pi_\theta\left(\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(\beta^{\mathsf{Fe}}\left\langle\eta_j,\eta_k\right\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}.\end{aligned}$$

We now insert the factor $s(\cdot)$ of Lemma (3.7.4) into our inequality. To make notations somewhat easier to read, we set:

$$s_{j,k}^{\Re} = s\left(\Re\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}},\vartheta\right) = \frac{\mathsf{L}_\theta\left(\Re\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)}{\mathsf{L}_\vartheta\left(\Re\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)}$$

and:

$$s_{j,k}^{\Im} = s\left(\Im\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}},\vartheta\right) = \frac{\mathsf{L}_\theta\left(\Im\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)}{\mathsf{L}_\vartheta\left(\Im\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)}.$$

We thus compute:

$$\begin{aligned}(3.7.5)\quad &\left|\!\left|\!\left|\Re\left(\pi_\theta\left(\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(\beta^{\mathsf{Fe}}\left\langle\eta_j,\eta_k\right\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right)\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant \left|\!\left|\!\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(\frac{\mathsf{D}_\theta^{p,q,d}(\omega_j)\mathsf{D}_\theta^{p,q,d(\omega_k)}}{\mathsf{D}_\vartheta^{p,q,d}(\omega_j)\mathsf{D}_\vartheta^{p,q,d}(\omega_j)}\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant \left|\!\left|\!\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(s_{j,k}^{\Re}\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\quad + \left|\!\left|\!\left|\pi_\vartheta\left(\left(s_{j,k}^{\Re} - \frac{\mathsf{D}_\theta^{p,q,d}(\omega_j)\mathsf{D}_\theta^{p,q,d}(\omega_k)}{\mathsf{D}_\vartheta^{p,q,d}(\omega_j)\mathsf{D}_\vartheta^{p,q,d}(\omega_k)}\right)\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant \left|\!\left|\!\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(s_{j,k}^{\Re}\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\quad + \left\|\left(s_{j,k}^{\Re} - \frac{\mathsf{D}_\theta^{p,q,d}(\omega_j)\mathsf{D}_\theta^{p,q,d}(\omega_k)}{\mathsf{D}_\vartheta^{p,q,d}(\omega_j)\mathsf{D}_\vartheta^{p,q,d}(\omega_k)}\right)\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right\|_{qt\vartheta}.\end{aligned}$$

By assumption on ϑ, we have:

$$(3.7.6)\qquad \left\|\left(s_{j,k}^{\Re} - \frac{\mathsf{D}_\theta^{p,q,d}(\omega_j)\mathsf{D}_\theta^{p,q,d}(\omega_k)}{\mathsf{D}_\vartheta^{p,q,d}(\omega_j)\mathsf{D}_\vartheta^{p,q,d}(\omega_k)}\right)\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right\|_{\mathcal{A}_\vartheta} \leqslant \mathsf{y}(\vartheta) < \frac{\varepsilon}{16},$$

and thus, plugging Inequality (3.7.6) in Inequality (3.7.5), we obtain:

$$\begin{aligned}(3.7.7)\quad &\left|\!\left|\!\left|\Re\left(\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(\Re\beta^{\mathsf{Fe}}\left\langle\eta_j,\eta_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant \left|\!\left|\!\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T - T\pi_\vartheta\left(s_{j,k}\Re\beta^{\mathsf{Fe}}\left\langle\omega_j,\omega_k\right\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)} + \frac{\varepsilon}{16}.\end{aligned}$$

The elements $\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\vartheta^{p,q,d}}$, for all $\vartheta\in X$, lie in V. We now wish them to lie in $E=\ker\tau\cap V$ with $\tau : f\in\ell^1(\mathbb{Z}^2)\mapsto f(0)$ to use Lemma (3.7.4). Again to ease notations, let:

$$\tau_\vartheta^{j,k}=\tau\left(\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right).$$

Of course, $\tau_\vartheta^{j,k}=\tau\left(\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right)=\tau\left(\Im\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\vartheta^{p,q,d}}\right)$.

We thus evaluate:

$$\begin{aligned}&\left|\left|\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T-T\pi_\vartheta\left(s_{j,k}\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant\left|\left|\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}-\tau_\theta^{j,k}\right)T-T\pi_\vartheta\left(s_{j,k}\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}-s_{j,k}\tau_\theta^{j,k}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\\ &\quad+\left|\tau_\theta^{j,k}-s_{j,k}\tau_\theta^{j,k}\right|.\end{aligned}$$

Now $\left|\tau_\theta-s_{j,k}\tau_\theta\right|\leqslant|1-s_{j,k}|\tau_{j,k}\leqslant|1-s_{j,k}|<\frac{\varepsilon}{16}$ since $\tau_{j,k}\leqslant\|\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\|_{\mathcal{A}_\theta}\leqslant 1$. We thus have:

$$\begin{aligned}&\left|\left|\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T-T\pi_\vartheta\left(s_{j,k}\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\\ &\leqslant\left|\left|\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}-\tau_\theta^{j,k}\right)T-T\pi_\vartheta\left(s_{j,k}\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}-s_{j,k}\tau_\theta^{j,k}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\\ &\qquad+\frac{\varepsilon}{16}.\end{aligned}\tag{3.7.8}$$

We are now in the position to apply Lemma (3.7.4) and conclude:

$$\begin{aligned}&\left|\left|\left|\pi_\theta\left(\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}-\tau_\theta^{j,k}\right)T-T\pi_\vartheta\left(s_{j,k}\Re\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}-s_{j,k}\tau_\theta^{j,k}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\\ &\qquad\leqslant\frac{\varepsilon}{16}.\end{aligned}\tag{3.7.9}$$

We now insert Inequality (3.7.9) into Inequality (3.7.8) and the result in Inequality (3.7.7) to conclude:

$$\left|\left|\left|\Re\left(\pi_\theta\left(\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T-T\pi_\vartheta\left(\beta^{\mathsf{Fe}}\langle\eta_j,\eta_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\leqslant\frac{3\varepsilon}{16}.\tag{3.7.10}$$

We get the same inequality as Inequality (3.7.10) for $\Im$ in place of $\Re$ by the same reasoning, so we get:

$$\left|\left|\left|\pi_\theta\left(\beta^{\mathsf{Fe}}\langle\omega_j,\omega_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)T-T\pi_\vartheta\left(\beta^{\mathsf{Fe}}\langle\eta_j,\eta_k\rangle_{\mathscr{H}_\theta^{p,q,d}}\right)\right|\right|\right|_{\ell^2(\mathbb{Z}^2)}\tag{3.7.11}$$

$$
\begin{aligned}
&\leqslant \left|\!\left|\!\left| \Re \left(\pi_\theta \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \right) T - T\pi_\vartheta \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \right) \right) \right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)} \\
&\quad + \left|\!\left|\!\left| \Im \left(\pi_\theta \left(\beta^{\mathsf{Fe}} \left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \right) T - T\pi_\vartheta \left(\beta^{\mathsf{Fe}} \left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \right) \right) \right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)} \\
&\leqslant \frac{3\varepsilon}{8}.
\end{aligned}
$$

Thus inserting Inequality (3.7.11) in Inequality (3.7.4), we conclude:

(3.7.12)

$$
\left|\!\left|\!\left| \pi_\theta \left(\left\langle \omega_j, \omega_k \right\rangle_{\mathscr{H}_\theta^{p,q,d}} \right) T - T\pi_\vartheta \left(\left\langle \eta_j, \eta_k \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}} \right) \right|\!\right|\!\right|_{\ell^2(\mathbb{Z}^2)} \leqslant \frac{9\varepsilon}{16} + \frac{3\varepsilon}{8} = \frac{15\varepsilon}{16}.
$$

By construction, the following is a modular bridge (note that $|||T|||\underset{=}{\overset{\ell^2(\mathbb{Z}^2)}{}}1$):

$$
\gamma_\vartheta = \left(\mathfrak{B}(\ell^2(\mathbb{Z})), T, \pi_\theta, \pi_\vartheta, (\omega_j)_{j\in\{1,\ldots,n\}}, (\eta_j)_{j\in\{1,\ldots,n\}} \right).
$$

By Lemma (3.7.4), the length of the basic bridge $\gamma_\flat$ is no more than $\frac{\varepsilon}{16}$, so the basic reach and the height of γ are bounded by $\frac{\varepsilon}{16}$. Now, Expression (3.7.12) states that the modular reach of γ is bounded above by $\frac{15\varepsilon}{16}$. Thus by Definition (2.3.16), the reach of γ is no more than $\frac{\varepsilon}{16} + \frac{15\varepsilon}{16} = \varepsilon$.

By Summary (3.7.7), the imprint of γ is no more than $\frac{\varepsilon}{16}$.

Thus by Definition (2.3.18), the length of γ is no more than $\varepsilon = \max\left\{\varepsilon, \frac{\varepsilon}{16}\right\}$. If we identify γ with the modular trek (γ), we conclude by Definition (2.4.6) that:

$$
\begin{aligned}
\Lambda^{\mathsf{mod}} \Bigg(& \left(\mathscr{H}_\vartheta^{p,q,d}, \left\langle \cdot, \cdot \right\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathcal{A}_\vartheta, \mathsf{L}_\vartheta \right), \\
& \left(\mathscr{H}_\theta^{p,q,d}, \left\langle \cdot, \cdot \right\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathcal{A}_\theta, \mathsf{L}_\theta \right) \Bigg) \leqslant \varepsilon.
\end{aligned}
$$

This concludes our proof. □

We conclude with an interesting observation. The proof of Theorem (3.7.6) reveals that the class of Heisenberg modules over all quantum 2-tori is actually iso-pivotal. Trivially, we can include in this class all the free modules over quantum 2-tori and keep the class iso-pivotal. Hence we conclude that the direct sum of metrized quantum vector bundles on the class of all free modules of finite rank and all Heisenberg modules over all quantum 2-tori is in fact jointly continuous with respect to the modular propinquity by Theorem (2.7.2). We thus get some additional convergence results from our work for our new metric.

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