UNIVERSAL TWO-PARAMETER $\mathcal{W}_\infty$-ALGEBRA AND VERTEX ALGEBRAS OF TYPE $\mathcal{W}(2,3,\ldots,N)$

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ABSTRACT. We prove the longstanding physics conjecture that there exists a unique two-parameter $\mathcal{W}_\infty$-algebra which is freely generated of type $\mathcal{W}(2,3,\ldots)$, and generated by the weights 2 and 3 fields. Subject to some mild constraints, all vertex algebras of type $\mathcal{W}(2,3,\ldots,N)$ for some N can be obtained as quotients of this universal algebra. As an application, we show that the structure constants for the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ are rational functions of k and n, and we classify all coincidences among the simple quotients $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ for $n \geq 3$. We also obtain many new coincidences between $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ and other vertex algebras of type $\mathcal{W}(2,3,\ldots,N)$ which arise as cosets of affine vertex algebras or non-principal $\mathcal{W}$-algebras.

1. INTRODUCTION

Associated to a simple, finite-dimensional Lie algebra $\mathfrak{g}$, a nilpotent element $f \in \mathfrak{g}$, and a complex parameter k, is a vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$ known as an *affine $\mathcal{W}$-algebra*. These are among the most important and best studied examples of vertex algebras in both the physics and mathematics literature. The first algebra of this kind other than the Virasoro algebra is the Zamolodchikov $\mathcal{W}_3$-algebra [Zam], and is associated to $\mathfrak{sl}_3$ with its principal nilpotent element f_{prin}. It is of type $\mathcal{W}(2,3)$, meaning that it has a minimal strong generating set consisting of a field in weights 2 and 3. Its structure is more complicated than that of affine vertex algebras since the OPE relation of the weight 3 field with itself contains nonlinear terms. Similarly, $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ was defined in [FL] and is of type $\mathcal{W}(2,3,\ldots,n)$. For an arbitrary $\mathfrak{g}$, the definition of $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{prin}})$ via quantum Drinfeld-Sokolov reduction was given by Feigin and Frenkel in [FFI]. This algebra is of type $\mathcal{W}(d_1,\ldots,d_m)$, where $d_1,\ldots,d_m$ are the degrees of the fundamental invariants of $\mathfrak{g}$. The definition of $\mathcal{W}^k(\mathfrak{g}, f)$ for an arbitrary nilpotent element f is due to Kac, Roan, and Wakimoto [KRW], and is a generalization of the quantum Drinfeld-Sokolov reduction. Although the generating fields of $\mathcal{W}(\mathfrak{g}, f)$ close nonlinearly under OPE, $\mathcal{W}(\mathfrak{g}, f)$ is *freely generated*, meaning that it has a PBW basis consisting of normally ordered monomials in the generators and their derivatives. Equivalently, it has the graded character of a differential polynomial ring, and its associated variety is an affine space [ArII].

We denote by $\mathcal{W}_k(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$ by its maximal proper graded ideal. In the case $f = f_{\mathrm{prin}}$, it was conjectured by Frenkel, Kac and Wakimoto [FKW] and proven by Arakawa [ArII, ArIII] that for an admissible level k, $\mathcal{W}_k(\mathfrak{g}, f_{\mathrm{prin}})$ is C_2-cofinite and rational. These are known as *minimal models* and are a generalization of the Virasoro minimal models [GKO]. There are other known C_2-cofinite, rational $\mathcal{W}$-algebras, not all of which are at admissible levels; see for example [ArI, AMI, Kaw, KWIII, CLIII].

The affine $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{prin}})$ are closely related to the classical $\mathcal{W}$-algebras which arose in the context of integrable hierarchies of soliton equations in the work of Adler, Gelfand, Dickey, Drinfeld, and Sokolov [Ad, GD, Di, DS]. The KdV hierarchy, which corresponds to the Virasoro algebra, was generalized by Drinfeld and Sokolov to an integrable hierarchy associated to any simple Lie algebra. The corresponding classical $\mathcal{W}$-algebras are Poisson vertex algebras and can be realized as the quasi-classical limits of affine $\mathcal{W}$-algebras [FBZ]. For a general nilpotent element $f \in \mathfrak{g}$, $\mathcal{W}^k(\mathfrak{g}, f)$ can also be regarded as a chiralization of the finite $\mathcal{W}$-algebra $\mathcal{W}^{\mathrm{fin}}(\mathfrak{g}, f)$ [DSKII]. These were defined by Premet [Pre], generalizing some examples that were originally studied by Kostant [Kos]. We can view $\mathcal{W}^{\mathrm{fin}}(\mathfrak{g}, f)$ as a quantization of the ring of functions on the Slodowy slice $S_f \subseteq \mathfrak{g} \cong \mathfrak{g}^*$ associated to f. Similarly, $\mathcal{W}^k(\mathfrak{g}, f)$ is a quantization of the ring of functions on the arc space of S_f, which is the inverse limit of the finite jet schemes [AMII].

Universal two-parameter $\mathcal{W}_\infty$-algebra. It is a longstanding conjecture in the physics literature that there exists a unique two-parameter $\mathcal{W}_\infty$-algebra of type $\mathcal{W}(2, 3, \dots)$ denoted by $\mathcal{W}_\infty[\mu]$, which interpolates between all the type A principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$, in the following sense.

(1) All structure constants appearing in the OPE algebra among the generators of $\mathcal{W}_\infty[\mu]$ are continuous functions of the central charge c and the parameter μ.
(2) If we set $\mu = n$, there is a truncation at weight $n + 1$ that allows all fields in weights $d \geq n + 1$ to be eliminated in the simple quotient of $\mathcal{W}_\infty[\mu]$, and this quotient is isomorphic to $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ as a one-parameter family of vertex algebras.

This conjecture appears in a number of papers including [YW, BK, B-H, BS, GGII, Pro]. Considerable evidence for the existence and uniqueness of $\mathcal{W}_\infty[\mu]$ was given in [YW], and later in [GGII], by solving Jacobi identities in low weights. It was observed that the structure constants in the OPEs of the first few generators depend on two free parameters, and it was conjectured that the full OPE algebra is determined recursively and consistently from this data. In the quasi-classical limit, the existence of a Poisson vertex algebra of type $\mathcal{W}(2, 3, \dots)$ which interpolates between the classical $\mathcal{W}$-algebras of $\mathfrak{sl}_n$ for all n, has been known for over twenty years; see [KZ, KM], and more recently [DSKV]. It can be defined using an affinization of Feigin's $\mathfrak{gl}_\lambda$-algebra of matrices of complex size λ, which is a certain quotient of $U(\mathfrak{sl}_2)$ that interpolates between the Lie algebras $\mathfrak{gl}_n$ for all n in an appropriate sense [F]. We mention that $\mathcal{W}_\infty[\mu]$ has recently become important in the conjectured duality between families of two-dimensional conformal field theories and higher spin gravity on three-dimensional Anti-de-Sitter space [GGI, GGII, GH]. A two-parameter family of conformal field theories corresponds to higher spin gravity, and the parameters relate to the 't Hooft coupling of the gravity. Strong generators of the symmetry algebra of the conformal field theory correspond to higher spin fields, where the conformal weight equals the spin. In [HII], an interpretation of the $\mathcal{W}$-algebra of $\mathfrak{sl}_{-n}$ was given by formally replacing n with $-n$ in the structure constants of $\mathcal{W}_\infty[\mu]$, and a coset realization of this algebra was proposed. It was also observed in [B-H] that other algebras of type $\mathcal{W}(2, 3, \dots, N)$ such as the parafermion algebra $N^k(\mathfrak{sl}_2)$, should also arise as quotients of this universal algebra.

Main result. In this paper we prove the existence and uniqueness of a vertex algebra $\mathcal{W}(c, \lambda)$ which is freely generated of type $\mathcal{W}(2, 3, \dots)$, depending on two parameters c and λ. Its quotients include *all* one-parameter vertex algebras of type $\mathcal{W}(2, 3, \dots, N)$ satisfying

some mild hypotheses, including $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ for $n \geq 3$. We use a different parameter λ which is related to μ by

$$\lambda = \frac{(\mu - 1)(\mu + 1)}{(\mu - 2)(3\mu^2 - \mu - 2 + c(\mu + 2))}.$$

This choice is not canonical but is more natural than μ because $\mathcal{W}(c, \lambda)$ is then defined over the polynomial ring $\mathbb{C}[c, \lambda]$. In other words, all structure constants appearing in the OPE algebra of the generators will lie in $\mathbb{C}[c, \lambda]$. The algebra is generated by a Virasoro field L of central charge c and a primary weight 3 field W^3 which is normalized so that $(W^3)_{(5)}W^3 = \frac{c}{3}1$. The remaining strong generators W^i of weight $i \geq 4$ are defined inductively by

$$W^i = (W^3)_{(1)}W^{i-1}, \qquad i \geq 4.$$

We show that by imposing all Jacobi identities among the generators, the structure constants in the OPEs of $L(z)W^i(w)$ and $W^i(z)W^j(w)$ are uniquely determined as polynomials in c and λ, for all i and j. The idea of determining the OPEs among the generators of a vertex algebra by imposing Jacobi identities has appeared in a number of papers in the physics literature including [KauWa, Bow, B-V, HI]. If all Jacobi identities can be solved, this procedure is enough to establish uniqueness, but to rigorously construct a vertex algebra in this way from generators and relations, it is necessary to invoke a deep result of [DSKI] which we call the *Kac-de Sole correspondence*. This is an equivalence between the categories of freely generated vertex algebras and nonlinear Lie conformal algebras. Roughly speaking, the OPE algebra of a set of free generators for a vertex algebra determines a nonlinear Lie conformal algebra. Conversely, associated to a nonlinear Lie conformal algebra A is its universal enveloping vertex algebra, which is a certain quotient of the tensor algebra of A, and is always freely generated. It will be important for us to relax the notion of nonlinear Lie conformal algebra by omitting a subset of Jacobi identities. The resulting structure is called a *degenerate* nonlinear conformal algebra in [DSKI]. Its universal enveloping vertex algebra is still defined but need not be freely generated.

Quotients of $\mathcal{W}(c, \lambda)$ and the classification of vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$. $\mathcal{W}(c, \lambda)$ has a conformal weight grading

$$\mathcal{W}(c, \lambda) = \bigoplus_{n \geq 0} \mathcal{W}(c, \lambda)[n],$$

where each $\mathcal{W}(c, \lambda)[n]$ is a free $\mathbb{C}[c, \lambda]$-module and $\mathcal{W}(c, \lambda)[0] \cong \mathbb{C}[c, \lambda]$. There is a symmetric bilinear form on $\mathcal{W}(c, \lambda)[n]$ given by

$$\langle \omega, \nu \rangle : \mathcal{W}(c, \lambda)[n] \otimes_{\mathbb{C}[c, \lambda]} \mathcal{W}(c, \lambda)[n] \to \mathbb{C}[c, \lambda], \qquad \langle \omega, \nu \rangle = \omega_{(2n-1)}\nu.$$

The level n Shapovalov determininant $\det_n \in \mathbb{C}[c, \lambda]$ is just the determinant of this form. It turns out that $\det_n$ is nonzero for all n; equivalently, $\mathcal{W}(c, \lambda)$ is a simple vertex algebra over $\mathbb{C}[c, \lambda]$.

Let p be an irreducible factor of $\det_{N+1}$ and let $I = (p) \subseteq \mathbb{C}[c, \lambda] \cong \mathcal{W}(c, \lambda)[0]$ be the corresponding ideal. Consider the quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/I \cdot \mathcal{W}(c, \lambda),$$

where $I \cdot \mathcal{W}(c, \lambda)$ is the vertex algebra ideal generated by I. This is a vertex algebra over the ring $\mathbb{C}[c, \lambda]/I$, which is no longer simple. It contains a singular vector ω in weight $N + 1$, which lies in the maximal proper ideal $\mathcal{I} \subseteq \mathcal{W}^I(c, \lambda)$ graded by conformal weight.

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If p does not divide $\det_m$ for any $m < N+1$, ω will have minimal weight among elements of $\mathcal{I}$. Often, there exists a localization R of $\mathbb{C}[c,\lambda]/I$ such that ω has the form

(1.1)
$$W^{N+1} - P(L, W^3, \ldots, W^{N-1}),$$

in the localization $\mathcal{W}_R^I(c,\lambda) = R \otimes_{\mathbb{C}[c,\lambda]/I} \mathcal{W}^I(c,\lambda)$. Here P is a normally ordered polynomial in the fields $L, W^3, \ldots, W^{N-1}$, and their derivatives, with coefficients in R. If this is the case, there will exist relations

$$W^m = P_m(L, W^3, \ldots, W^N)$$

for all $m \geq N+1$ expressing W^m in terms of $L, W^3, \ldots, W^N$ and their derivatives. The simple quotient $\mathcal{W}_R^I(c,\lambda)/\mathcal{I}$ will then be of type $\mathcal{W}(2,3,\ldots,N)$. Conversely, we will show that any simple one-parameter vertex algebra of type $\mathcal{W}(2,3,\ldots,N)$ satisfying some mild hypotheses, can be obtained as the simple quotient of $\mathcal{W}_R^I(c,\lambda)$ for some I and R. This reduces the classification of such vertex algebras to the classification of prime ideals $I = (p) \subseteq \mathbb{C}[c,\lambda]$ such that p divides $\det_{N+1}$ but does not divide $\det_m$ for $m < N+1$, and $\mathcal{W}^I(c,\lambda)$ contains a singular vector of the form (1.1), possibly after localizing.

In addition to $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$, there are many other one-parameter vertex algebras of type $\mathcal{W}(2,3,\ldots,N)$ for some N. Here is a short list of examples, which is by no means exhaustive.

(1) The parafermion algebra $N^k(\mathfrak{sl}_2) = \mathrm{Com}(\mathcal{H}, V^k(\mathfrak{sl}_2))$. Here $V^k(\mathfrak{sl}_2)$ denotes the universal affine vertex algebra of $\mathfrak{sl}_2$, $\mathcal{H}$ is the Heisenberg algebra corresponding to the Cartan subalgebra $\mathfrak{h}$, and the commutant means the subalgebra of $V^k(\mathfrak{sl}_2)$ which commutes with $\mathcal{H}$. This of type $\mathcal{W}(2,3,4,5)$ [DLY].

(2) The coset of $V^k(\mathfrak{gl}_n)$ inside $V^k(\mathfrak{sl}_{n+1})$. We call this the algebra of *generalized parafermions* since in the case $n = 1$ it is just $N^k(\mathfrak{sl}_2)$. We will show that it is of type $\mathcal{W}(2,3,\ldots,n^2+3n+1)$, which was conjectured in [B-H].

(3) The coset of the Heisenberg algebra $\mathcal{H}$ inside the Bershadsky-Polyakov algebra, which is the $\mathcal{W}$-algebra associated to $\mathfrak{sl}_3$ with its non-principal nilpotent element. This coset is of type $\mathcal{W}(2,3,4,5,6,7)$ [ACLI].

(4) The coset of the Heisenberg algebra $\mathcal{H}$ inside the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_4, f_{\mathrm{subreg}})$ associated to $\mathfrak{sl}_4$ with its subregular nilpotent element. This is of type $\mathcal{W}(2,3,4,5,6,7,8,9)$ [CLIII].

(5) The coset of $V^{k+1}(\mathfrak{gl}_{n-2})$ inside the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{min}})$ associated to $\mathfrak{sl}_n$ with its minimal nilpotent element f_{min}, for $n \geq 3$. This is of type $\mathcal{W}(2,3,\ldots,n^2-2)$ [ACKL].

(6) The coset of $V^k(\mathfrak{sl}_n)$ inside $V^{k+1}(\mathfrak{sl}_n) \otimes L_{-1}(\mathfrak{sl}_n)$. We will show that it is of type $\mathcal{W}(2,3,\ldots,n^2+2n)$, which was conjectured in [B-H].

Unlike $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$, the above algebras are not freely generated. In fact, it is a folklore conjecture that the $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ are the *only* freely generated vertex algebras of this kind. Note that Example (5) is a generalization of Example (3), which is just the case $n = 3$. Also, Examples (3) and (4) are part of the family of cosets of $\mathcal{H}$ inside the algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{subreg}})$ associated to $\mathfrak{sl}_n$ with its subregular nilpotent element f_{subreg}. Conjecturally, these are of type $\mathcal{W}(2,3,\ldots,2n+1)$, and are not freely generated.

All the vertex algebras in Examples (1)-(6), as well as $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ for $n \geq 3$, arise as quotients of $\mathcal{W}_R^I(c,\lambda)$ for some prime ideal $I = (p) \subseteq \mathbb{C}[c,\lambda]$ and some localization R of $\mathbb{C}[c,\lambda]/I$. We shall explicitly describe I for Examples (1)-(4), as well as $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$. For Examples (5) and (6), as well as $\mathrm{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{subreg}}))$, we shall give a conjectural

description of I. A remarkable feature of these ideals is that they are organized into infinite families that admit a uniform description, and the corresponding varieties $V(I) \subseteq \mathbb{C}^2$ are rational curves, possibly singular. We speculate that *all* prime ideals $I = (p)$ such that p divides $\det_n$ for some n, correspond to rational curves.

The vertex algebras $\mathcal{W}_R^I(c, \lambda)$ for $I = (p)$ are one-parameter families in the sense that the ring R has Krull dimension 1. It is also important to consider $\mathcal{W}^I(c, \lambda)$ when $I \subseteq \mathbb{C}[c, \lambda]$ is a *maximal* ideal, which has the form $I = (c - c_0, \lambda - \lambda_0)$ for some $c_0, \lambda_0 \in \mathbb{C}$. Then $\mathcal{W}^I(c, \lambda)$ and its quotients are ordinary vertex algebras over $\mathbb{C}$. Given two maximal ideals $I_0 = (c - c_0, \lambda - \lambda_0)$ and $I_1 = (c - c_1, \lambda - \lambda_1)$, let $\mathcal{W}_0$ and $\mathcal{W}_1$ be the simple quotients of $\mathcal{W}^{I_0}(c, \lambda)$ and $\mathcal{W}^{I_1}(c, \lambda)$. There is a very simple criterion for $\mathcal{W}_0$ and $\mathcal{W}_1$ to be isomorphic. We must have $c_0 = c_1$, and if this central charge is 0 or -2, there is no restriction on λ_0, λ_1. For all other values of the central charge, we must have $\lambda_0 = \lambda_1$.

This observation explains a number of recently discovered coincidences between simple vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$ at special parameter values, and simple principal $\mathcal{W}$-algebras of type A. For example, for all integers $k \geq 3$, the simple parafermion algebra $N_k(\mathfrak{sl}_2)$ is isomorphic to a principal $\mathcal{W}$-algebra of $\mathfrak{sl}_k$ which is C_2-cofinite and rational [ALY]. There are similar coincidences between the Heisenberg cosets in Examples (3) and (4) at values where they are C_2-cofinite and rational, and principal $\mathcal{W}$-algebras in type A [ACLI, CLIII]. The method of proving these coincidences relies on rationality, although other coincidences of this kind among more general $\mathcal{W}$-algebras and their cosets which are not necessarily rational, have been conjectured in [ACKL, CS, C, CG]. Our main result provides another way to establish coincidences among vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$, which need not be rational or C_2-cofinite. For example, we shall classify all coincidences among the simple algebras $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{prin}})$ for $n \geq 3$, which settles a conjecture of Gaberdiel and Gopakumar [GGII]. We also classify all coincidences between $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{prin}})$ and the simple quotients of the vertex algebras in Examples (1)-(4) above. Many of these coincidences are new.

Note that $\mathcal{W}(c, \lambda)$ has many rational quotients, since $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{prin}})$ is such a quotient for any admissible level k by [ArIII]. Due to the above coincidences, all known rational quotients of $\mathcal{W}(c, \lambda)$ are of the form $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{prin}})$. An interesting question that we do not address is whether $\mathcal{W}(c, \lambda)$ admits rational quotients that are *not* of this kind.

Organization. This paper is organized as follows. In Section 2, we review the basic definitions and examples of vertex algebras that we need. In Section 3 we review the Kac-de Sole correspondence between the categories of freely generated vertex algebras and nonlinear Lie conformal algebras. In Section 4, we discuss the notion of vertex algebras and nonlinear Lie conformal algebras over commutative rings, which is a straightforward generalization of the usual notions. In Section 5, we give the main result, which is the existence and uniqueness of the vertex algebra $\mathcal{W}(c, \lambda)$. In Section 6, we discuss localizations and quotients of $\mathcal{W}(c, \lambda)$. In Sections 7 and 8, we give the explicit generators for the ideals in $\mathbb{C}[c, \lambda]$ corresponding to $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ and $\text{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1}))$. In Section 9, we give conjectural generators for the ideals in $\mathbb{C}[c, \lambda]$ that correspond to two additional families, namely $\text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{min}}))$ and $\text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}}))$. In Section 10, we classify all coincidences among the simple algebras $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{prin}})$, as well as the coincidences between $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{prin}})$ and several other families of vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$. Finally, in Section 11 we discuss a one-parameter deformation of the $\mathcal{W}_{1+\infty}$-algebra with central charge c.

2. Vertex algebras

In this section, we define vertex algebras, which have been discussed from various different points of view in the literature (see for example [Bor, FLM, FHL, K, FBZ]). We will follow the formalism developed in [LZ] and partly in [LiI]. Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$, and let z, w be formal variables, and let $\mathrm{QO}(V)$ denote the space of linear maps

$$V \to V((z)) = \{\sum_{n \in \mathbb{Z}} v(n) z^{-n-1} | v(n) \in V, \ v(n) = 0 \text{ for } n \gg 0\}.$$

Each element $a \in \mathrm{QO}(V)$ can be represented as a power series

$$a = a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]].$$

We assume that $a = a_0 + a_1$ where $a_i : V_j \to V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$.

For each $n \in \mathbb{Z}$, we have a nonassociative bilinear operation on $\mathrm{QO}(V)$, defined on homogeneous elements a and b by

$$a(w)_{(n)}b(w) = \mathrm{Res}_z a(z)b(w) \, \iota_{|z|>|w|}(z-w)^n - (-1)^{|a||b|}\mathrm{Res}_z b(w)a(z) \, \iota_{|w|>|z|}(z-w)^n.$$

Here $\iota_{|z|>|w|}f(z,w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$. For $a, b \in \mathrm{QO}(V)$, we have the following identity of power series known as the *operator product expansion* (OPE) formula.

(2.1) $$a(z)b(w) = \sum_{n \geq 0} a(w)_{(n)}b(w) \, (z-w)^{-n-1} + : a(z)b(w) : .$$

Here $: a(z)b(w) : \, = a(z)_- b(w) + (-1)^{|a||b|}b(w)a(z)_+$, where $a(z)_- = \sum_{n<0} a(n)z^{-n-1}$ and $a(z)_+ = \sum_{n \geq 0} a(n)z^{-n-1}$. Often, (2.1) is written as

$$a(z)b(w) \sim \sum_{n \geq 0} a(w)_{(n)}b(w) \, (z-w)^{-n-1},$$

where $\sim$ means equal modulo the term $: a(z)b(w) :$, which is regular at $z = w$.

Note that $: a(w)b(w) :$ is a well-defined element of $\mathrm{QO}(V)$. It is called the *Wick product* or *normally ordered product* of a and b, and it coincides with $a_{(-1)}b$. For $n \geq 1$ we have

$$n! \, a(z)_{(-n-1)}b(z) = : (\partial^n a(z))b(z) :, \qquad \partial = \frac{d}{dz}.$$

For $a_1(z), \ldots, a_k(z) \in \mathrm{QO}(V)$, the k-fold iterated Wick product is defined inductively by

(2.2) $$: a_1(z)a_2(z) \cdots a_k(z) : \, = : a_1(z)b(z) :, \qquad b(z) = : a_2(z) \cdots a_k(z) : .$$

We often omit the formal variable z when no confusion can arise.

A subspace $\mathcal{A} \subseteq \mathrm{QO}(V)$ containing 1 which is closed under all the above products will be called a *quantum operator algebra* (QOA). We say that $a, b \in \mathrm{QO}(V)$ are *local* if

$$(z-w)^N[a(z), b(w)] = 0$$

for some $N \geq 0$. Here $[,]$ denotes the super bracket. This condition implies that $a_{(n)}b = 0$ for $n \geq N$, so (2.1) becomes a finite sum. Finally, a *vertex algebra* will be a QOA whose elements are pairwise local. This notion is well known to be equivalent to the notion of a vertex algebra in the sense of [FLM].

A vertex algebra $\mathcal{A}$ is said to be *generated* by a subset $S = \{\alpha^i \mid i \in I\}$ if $\mathcal{A}$ is spanned by words in the letters α^i, and all products, for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* $\mathcal{A}$ if $\mathcal{A}$ is spanned by words in the letters α^i, and all products for $n < 0$. Equivalently, $\mathcal{A}$ is spanned by

$$\{: \partial^{k_1}\alpha^{i_1} \cdots \partial^{k_m}\alpha^{i_m} : \mid i_1, \ldots, i_m \in I, \ k_1, \ldots, k_m \geq 0\}.$$

Suppose that S is an ordered strong generating set $\{\alpha^1, \alpha^2, \ldots\}$ for $\mathcal{A}$ which is at most countable. We say that S *freely generates* $\mathcal{A}$, if $\mathcal{A}$ has a PBW basis consisting of all normally ordered monomials

(2.3)
$$: \partial^{k_1^1}\alpha^{i_1} \cdots \partial^{k_{r_1}^1}\alpha^{i_1}\partial^{k_1^2}\alpha^{i_2} \cdots \partial^{k_{r_2}^2}\alpha^{i_2} \cdots \partial^{k_1^n}\alpha^{i_n} \cdots \partial^{k_{r_n}^n}\alpha^{i_n} :, \qquad 1 \leq i_1 < \cdots < i_n,$$
$$k_1^1 \geq k_2^1 \geq \cdots \geq k_{r_1}^1, \quad k_1^2 \geq k_2^2 \geq \cdots \geq k_{r_2}^2, \quad \cdots, \quad k_1^n \geq k_2^n \geq \cdots \geq k_{r_n}^n.$$

Conformal structure. A conformal structure with central charge c on a vertex algebra $\mathcal{A}$ is a Virasoro vector $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \in \mathcal{A}$ satisfying

(2.4)
$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1},$$

such that in addition, $L_{-1}\alpha = \partial\alpha$ for all $\alpha \in \mathcal{A}$, and L_0 acts diagonalizably on $\mathcal{A}$. We say that a has conformal weight d if $L_0(\alpha) = d\alpha$, and we denote the conformal weight d subspace by $\mathcal{A}[d]$. In all our examples, the conformal weight grading will be by $\mathbb{Z}_{\geq 0}$, that is,

$$\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}[d].$$

As a matter of notation, we say that a vertex algebra $\mathcal{A}$ is of type

$$\mathcal{W}(d_1, d_2, \ldots)$$

if it has a minimal strong generating set consisting of one field in each conformal weight $d_1, d_2, \ldots$. If $\mathcal{A}$ is freely generated of type $\mathcal{W}(d_1, d_2, \ldots)$, it has graded character

(2.5)
$$\chi(\mathcal{A}, q) = \sum_{n \geq 0} \dim(\mathcal{A})[n])q^n = \prod_{i \geq 1}\prod_{k \geq 0} \frac{1}{1 - q^{d_i + k}}.$$

Important identities. We recall some important identities that hold in any vertex algebra $\mathcal{A}$. For any fields $a, b, c \in \mathcal{A}$, we have

(2.6)
$$(\partial a)_{(n)}b = -na_{n-1} \qquad \forall n \in \mathbb{Z},$$

(2.7)
$$: (: ab :)c : - : abc : = \sum_{n \geq 0} \frac{1}{(n+1)!}\left(: (\partial^{n+1}a)(b_{(n)}c) : + (-1)^{|a||b|}(\partial^{n+1}b)(a_{(n)}c) : \right).$$

(2.8)
$$a_{(n)}(: bc :) - : (a_{(n)}b)c : - (-1)^{|a||b|} : b(a_{(n)}c) : = \sum_{i=1}^{n} \binom{n}{i}(a_{(n-i)}b)_{(i-1)}c, \qquad \forall n \geq 0,$$

(2.9)
$$a_{(n)}b = \sum_{p \in \mathbb{Z}} (-1)^{p+1}(b_{(p)}a)_{(n-p-1)}1, \qquad \forall n \in \mathbb{Z}.$$

Given fields a, b, c and integers $m, n \geq 0$, the following identities are known as Jacobi relations of type (a, b, c).

$$(2.10) \qquad a_{(r)}(b_{(s)}c) = (-1)^{|a||b|} b_{(s)}(a_{(r)}c) + \sum_{i=0}^{r} \binom{r}{i} (a_{(i)}b)_{(r+s-i)}c.$$

For each triple of fields a, b, c, these identities are nontrivial only for finitely many choices of r, s.

Affine vertex algebras. Given a simple, finite-dimensional Lie algebra $\mathfrak{g}$, the *universal affine vertex algebra* $V^k(\mathfrak{g})$ is freely generated by fields X^ξ which are linear in $\xi \in \mathfrak{g}$ and satisfy

$$(2.11) \qquad X^\xi(z)X^\eta(w) \sim k(\xi, \eta)(z - w)^{-2} + X^{[\xi, \eta]}(w)(z - w)^{-1},$$

where $(\cdot, \cdot)$ denotes the normalized Killing form $\frac{1}{2h^\vee}\langle \cdot, \cdot \rangle$. For all $k \neq -h^\vee$, $V^k(\mathfrak{g})$ has a conformal vector

$$(2.12) \qquad L^\mathfrak{g} = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{n} : X^{\xi_i} X^{\xi_i'} :$$

of central charge

$$c = \frac{k \dim(\mathfrak{g})}{k + h^\vee}.$$

Here ξ_i runs over a basis of $\mathfrak{g}$, and ξ_i' is the dual basis with respect to $(\cdot, \cdot)$.

As a module over the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}$, $V^k(\mathfrak{g})$ is isomorphic to the vacuum $\hat{\mathfrak{g}}$-module, and is freely generated $\{X^{\xi_i}\}$ as ξ_i runs over a basis of $\mathfrak{g}$. For generic values of k, the vacuum module is irreducible and $V^k(\mathfrak{g})$ is a simple vertex algebra. At certain special values of k, $V^k(\mathfrak{g})$ is not simple. We denote by $L_k(\mathfrak{g})$ the quotient by its maximal proper ideal graded by conformal weight.

Affine $\mathcal{W}$-algebras. Given a simple Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$, there is a vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$ known as an affine $\mathcal{W}$-algebra. The standard construction is via the quantum Drinfeld-Sokolov reduction and its generalizations, and the reader is referred to [FFI, KRW] for details. We briefly recall some basic features of the $\mathcal{W}$-algebras of $\mathfrak{sl}_n$ associated to the principal, subregular, and minimal nilpotent elements.

(1) For $n \geq 3$, the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is freely generated of type

$$\mathcal{W}(2, 3, \ldots, n).$$

It is known to be generated by the weights 2 and 3 fields [ALY], and the Virasoro element L has central charge

$$c = -\frac{(n-1)(n^2 + nk - n - 1)(n^2 + k + nk)}{n + k}.$$

(2) For $n \geq 3$, the subregular $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ is freely generated of type

$$\mathcal{W}(1, 2, 3, \ldots, n - 1, n/2, n/2).$$

The weight 1 field generates a Heisenberg subalgebra $\mathcal{H}$, and the Virasoro field has central charge

$$c = -\frac{((k+n)(n-1) - n)((k+n)(n-2)n - n^2 + 1)}{k + n}.$$

(3) For $n \geq 4$, the minimal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n, f_{\min})$ is freely generated of type

$$\mathcal{W}\big(1^{(n-2)^2}, 2, (3/2)^{2(n-2)}\big).$$

The $(n-2)^2$ fields in weight 1 generate an affine vertex algebra

$$V^{k+1}(\mathfrak{gl}_{n-2}) = \mathcal{H} \otimes V^{k+1}(\mathfrak{sl}_{n-2}),$$

and the fields in weight $3/2$ are primary and transform under $\mathfrak{gl}_{n-2}$ as the sum of the standard representation $\mathbb{C}^{n-2}$ and its dual. The Virasoro element has central charge

$$c = -\frac{5k + 6k^2 + 4n + 5kn - n^2 - kn^2}{k + n}.$$

The complete OPE algebra of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ is not known in general, although for $n \leq 5$ it has been worked out explicitly in the physics literature [Zh]. On the other hand, the OPE algebra of all minimal $\mathcal{W}$-algebras, including $\mathcal{W}^k(\mathfrak{sl}_n, f_{\min})$, is simpler and appears in [KWII]. By a theorem of Genra [Gen], $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{subreg}})$ coincides with the Feigin-Semikhatov algebra $\mathcal{W}_n^{(2)}$ at level k [FS], and part of the OPE algebra appears in [FS].

Coset construction. Given a vertex algebra $\mathcal{V}$ and a subalgebra $\mathcal{A} \subseteq \mathcal{V}$, the *coset* or *commutant* of $\mathcal{A}$ in $\mathcal{V}$, denoted by $\mathrm{Com}(\mathcal{A}, \mathcal{V})$, is the subalgebra of elements $v \in \mathcal{V}$ such that

$$[a(z), v(w)] = 0, \qquad \forall a \in \mathcal{A}.$$

This was introduced by Frenkel and Zhu in [FZ], generalizing earlier constructions in [GKO, KP]. Equivalently, $v \in \mathrm{Com}(\mathcal{A}, \mathcal{V})$ if and only if $a_{(n)}v = 0$ for all $a \in \mathcal{A}$ and $n \geq 0$. Note that if $\mathcal{V}$ and $\mathcal{A}$ have Virasoro elements $L^{\mathcal{V}}$ and $L^{\mathcal{A}}$, $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ has Virasoro element $L = L^{\mathcal{V}} - L^{\mathcal{A}}$ as long as $L^{\mathcal{V}} \neq L^{\mathcal{A}}$.

Principal $\mathcal{W}$-algebras as cosets. When $\mathfrak{g}$ is of type A, D, or E, there is a coset realization of $\mathcal{W}^k(\mathfrak{g}, f_{\mathrm{prin}})$ that was recently established in [ACLII]. Define

$$(2.13) \qquad\qquad \mathcal{C}^k(\mathfrak{g}) = \mathrm{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

By Theorem 6.10 and Example 7.13 of [CLII], $L_1(\mathfrak{g})^G$ is the orbifold limit of $\mathcal{C}^k(\mathfrak{g})$ in the sense that

$$\lim_{k \to \infty} \mathcal{C}^k(\mathfrak{g}) \cong L_1(\mathfrak{g})^G.$$

Here G is a connected Lie group with Lie algebra $\mathfrak{g}$, whose action on $L_1(\mathfrak{g})$ is infinitesimally generated by the zero modes of the generating fields. It is well known that $L_1(\mathfrak{g})^G$ is isomorphic to $\mathcal{W}^{1-h^{\vee}}(\mathfrak{g}, f_{\mathrm{prin}})$.

Theorem 2.1. [ACLII] *For generic values of k,*

$$\mathcal{C}^k(\mathfrak{g}) \cong \mathcal{W}^{k'}(\mathfrak{g}, f_{prin}), \qquad k' = -h^{\vee} + \frac{k + h^{\vee}}{k + h^{\vee} + 1}, \quad -h^{\vee} + \frac{k + h^{\vee} + 1}{k + h^{\vee}}.$$

In particular, this isomorphism holds for all real numbers $k > -h^{\vee} - 1$.

In the case $\mathfrak{g} = \mathfrak{sl}_n$, the coset realization of $\mathcal{W}^{k'}(\mathfrak{sl}_n, f_{\mathrm{prin}})$ is very useful for our purposes because the weight 3 field of $\mathcal{C}^k(\mathfrak{sl}_n)$ appears explicitly in [BBSSII].

Coincidences. For later use, we mention some recently established coincidences between certain cosets of affine or $\mathcal{W}$-algebras, and principal, rational $\mathcal{W}$-algebras of type A. First, recall the universal parafermion algebra $N^k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, V^k(\mathfrak{sl}_2))$, as well as its simple quotient $N_k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, L_k(\mathfrak{sl}_2)$ [DLY].

Theorem 2.2. [ALY] *For all positive integers $k \geq 3$,*

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_{k'}(\mathfrak{sl}_k, f_{prin}), \qquad k' = -k + \frac{2+k}{1+k}, \ -k + \frac{1+k}{2+k},$$

which has central charge $c = \frac{2(k-1)}{k+2}$, and is rational and C_2-cofinite.

Next, recall the *Bershadsky-Polyakov algebra*, which is just the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_{\min})$. As in [ACLI], it is convenient to introduce a shift of level; we define

$$\mathcal{W}^\ell = \mathcal{W}^{\ell-3/2}(\mathfrak{sl}_3, f_{\min}),$$

and we denote its simple quotient by $\mathcal{W}_\ell$. The weight 1 field generates a Heisenberg subalgebra $\mathcal{H}$. We define $\mathcal{C}^\ell = \text{Com}(\mathcal{H}, \mathcal{W}^\ell)$ and $\mathcal{C}_\ell = \text{Com}(\mathcal{H}, \mathcal{W}_\ell)$, which is the simple quotient of $\mathcal{C}^\ell$.

Theorem 2.3. [ACLI] *For all $\ell \in \mathbb{Z}_+$,*

$$\mathcal{C}_\ell \cong \mathcal{W}_{\ell'}(\mathfrak{sl}_{2\ell}, f_{prin}), \qquad \ell' = -2l + \frac{3+2l}{1+2l}, \ -2l + \frac{1+2l}{3+2l},$$

which has central charge $c = -\frac{3(2\ell-1)^2}{2\ell+3}$ and is C_2-cofinite and rational.

Finally, recall the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$, and its simple quotient $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$. Let $\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}}))$ and $\mathcal{C}_k = \text{Com}(\mathcal{H}, \mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}}))$ which is the simple quotient of $\mathcal{C}^k$.

Theorem 2.4. [CLIII] *Let $n \geq 3$ and $k = \frac{1}{3}(n-8)$, and assume that $n-1$ is co-prime to at least one of $n+1$ and $n+4$. Then*

$$\mathcal{C}_k \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{4+n}{1+n}, \ -n + \frac{1+n}{4+n},$$

and is C_2-cofinite and rational.

These theorems were proven by exhibiting $V_k(\mathfrak{sl}_2)$, $\mathcal{W}_\ell$, and $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$, respectively, as simple current extensions of $V_L \otimes \mathcal{W}$. Here V_L is a certain rank one lattice vertex algebra that extends $\mathcal{H}$, and $\mathcal{W}$ is the appropriate principal $\mathcal{W}$-algebra of type A. This approach does not easily generalize if the vertex algebras involved are not rational, or if the larger vertex algebra is not a simple current extension of a tensor product of two nice subalgebras. One of the goals of this paper will be to establish many more coincidences of this kind. For example, we shall classify all cases where the above cosets $N_k(\mathfrak{sl}_2)$, $\mathcal{C}_\ell$, and $\mathcal{C}_k$, are isomorphic to a principal $\mathcal{W}$-algebras of type A. There are several families of coincidences which do not involve rational or C_2-cofinite vertex algebras.

3. Nonlinear Lie conformal algebras and the Kac-de Sole correspondence

We recall the main results of an important paper of Kac and de Sole [DSKI], which gives an equivalence between the categories of nonlinear Lie conformal algebras and freely generated vertex algebras. Kac and de Sole use the language of lambda-brackets rather than the equivalent language of OPEs, which we briefly explain. If a vertex algebra $\mathcal{V}$ has a strong generating set $\{\alpha^1, \alpha^2, \dots\}$, the OPE relation

$$\alpha^i(z)\alpha^j(w) \sim \sum_{n \geq 0} (\alpha^i_{(n)}\alpha^j)(w)(z-w)^{-n-1},$$

is written in the form

$$[\alpha^i_\lambda \alpha^j] = \sum_{n \geq 0} \frac{\lambda^n}{n!} \alpha^i_{(n)}\alpha^j.$$

Here λ is a formal variable and the coefficient of λ^n corresponds to $1/n!$ times the pole of order $n+1$ appearing in the OPE formula. The operator $\partial = \frac{d}{dz}$ corresponds to the translation operator T, and a number of standard vertex algebra identities can be written elegantly in this formalism. For example, the following identities correspond to (2.6)-(2.8):

(3.1) $$[Ta_\lambda b] = -\lambda[a_\lambda b], \qquad [a_\lambda Tb] = (\lambda + T)(a_\lambda b).$$

(3.2) $$: (: ab :)c : - : a(: bc :) : = : \left(\int_0^T d\lambda\, a \right)[b_\lambda c] : + (-1)^{|a||b|} : \left(\int_0^T d\lambda\, b \right)[a_\lambda c] : .$$

(3.3) $$[a_\lambda : bc :] = : [a_\lambda b]c : + (-1)^{|a||b|} : b[a_\lambda c] : + \int_0^\lambda d\mu\, [[a_\lambda b]_\mu c].$$

Let A denote the $\mathbb{C}$-span of $\{\alpha^i\}$ and their T-translates, which has the structure of a free $\mathbb{C}[T]$-module. We can regard the lambda-bracket as a $\mathbb{C}$-bilinear product

$$[-_\lambda-] : A \otimes A \to \mathbb{C}[\lambda] \otimes \mathcal{T}(A),$$

where $\mathcal{T}(A)$ denotes the tensor algebra. This structure captures the same information as the OPE algebra of the generators of $\mathcal{V}$. Accordingly, one defines a *nonlinear conformal algebra* to be a $\mathbb{C}[T]$-module A possessing a grading

$$A = \bigoplus_{\Delta \in \Gamma \setminus \{0\}} A[\Delta]$$

for some semigroup Γ, and a $\mathbb{C}$-bilinear operation

$$[-_\lambda-] : A \otimes A \to \mathbb{C}[\lambda] \otimes \mathcal{T}(A)$$

satisfying (3.1) together with the following grading condition:

(3.4) $$\Delta([a_\lambda b]) < \Delta(a) + \Delta(b).$$

Here the grading on A is extended to a grading

$$\mathcal{T}(A) = \bigoplus_{\Delta \in \Gamma \setminus \{0\}} \mathcal{T}(A)[\Delta]$$

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by additivity, so that $\Delta(1) = 0$ and $\Delta(a \otimes b) = \Delta(a) + \Delta(b)$. We have the induced filtration $\mathcal{T}_\Delta(A) = \bigoplus_{\Delta' \leq \Delta} \mathcal{T}(A)[\Delta']$. It is shown that under these assumptions, the normally ordered product

$$N : \mathcal{T}(A) \otimes \mathcal{T}(A) \to \mathcal{T}(A),$$

and the lambda-bracket

$$L_\lambda : \mathcal{T}(A) \otimes \mathcal{T}(A) \to \mathbb{C}[\lambda] \otimes \mathcal{T}(A),$$

can be defined on $\mathcal{T}(A)$ such that identities analogous to (3.1)-(3.3) hold for all $a, b, c \in \mathcal{T}(A)$. See Lemma 3.2 of [DSKI] for the precise statement.

Let $\mathcal{M}_\Delta(A) \subseteq \mathcal{T}_\Delta(A)$ denote the span of elements of the form

$$a \otimes \left((b \otimes c - (-1)^{|b||c|} c \otimes b) \otimes d - N \left(\int_{-T}^0 d\lambda \, L_\lambda(b, c), d \right) \right),$$

for all $b, c \in A$ and $a, d \in \mathcal{T}(A)$ such that $a \otimes b \otimes c \otimes d \in \mathcal{T}_\Delta(A)$. Let $\mathcal{M}(A) = \bigcup_{\Delta \in \Gamma} \mathcal{M}_\Delta(A)$.

A nonlinear conformal algebra A is called a *nonlinear Lie conformal algebra* if it satisfies two additional axioms:

(3.5)
$$[a_\lambda b] = -(-1)^{|a||b|}[b_{-\lambda-T}a], \qquad \forall a, b \in A,$$

which is known as *skew-symmetry* and is analogous to (2.9), and

(3.6)
$$L_\lambda(a, L_\mu(b, c)) - (-1)^{|a||b|} L_\mu(b, L_\lambda(a, c)) - L_{\lambda+\mu}(L_\lambda(a, b), c) \in \mathbb{C}[\lambda, \mu] \otimes M_{\Delta'}(A),$$

for all $a, b, c \in A$ and some $\Delta' < \Delta(a) + \Delta(b) + \Delta(c)$. This is known as the *Jacobi identity* and is analogous to (2.10). The main result (Theorem 3.9) of [DSKI] is that for any nonlinear Lie conformal algebra A, the quotient

(3.7)
$$U(A) = \mathcal{T}(A)/\mathcal{M}(A)$$

has a canonical vertex algebra structure, and is known as the *universal enveloping vertex algebra* of A. A key ingredient is showing that $\mathbb{C}[\lambda] \otimes \mathcal{M}(A)$ is a two-sided ideal of $\mathbb{C}[\lambda] \otimes \mathcal{T}(A)$ under the operations N and L_λ. If $\{\alpha^1, \alpha^2, \dots\}$ is an ordered generating set for A as a free $\mathbb{C}[T]$-module, $U(A)$ is then freely generated by the corresponding fields, which we also denote by $\{\alpha^1, \alpha^2, \dots\}$. In particular, the monomials (2.3) form a PBW basis of $U(A)$.

In this paper, we shall always take the semigroup Γ to be $\mathbb{Z}_{\geq 0}$. In this case, $U(A)$ will be graded by conformal weight if we declare $T^k(a)$ to have weight $\Delta(a) + k$. If $\Delta(\alpha^i) = d_i > 0$, $U(A)$ will be freely generated of type $\mathcal{W}(d_1, d_2, \dots)$ and will have graded character (2.5). In this paper, we will only consider examples where $U(A)$ has a Virasoro vector L whose L_0-eigenspace decomposition coincides with the above conformal weight grading.

Conversely, suppose that $\mathcal{V}$ is a conformal vertex algebra freely generated by a set $\{\alpha^1, \alpha^2, \dots\}$ and has conformal weight grading $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n$, where α^i has conformal weight $d_i > 0$. Then the $\mathbb{C}$-span of $\{\alpha^1, \alpha^2, \dots\}$ and their T-translates forms a nonlinear Lie conformal algebra with $\Delta(\alpha^i) = d_i$.

Given a nonlinear Lie conformal algebra A with ordered basis $\{\alpha^1, \alpha^2, \dots\}$ as a $\mathbb{C}[T]$-module, we can define a *category* $C(A)$ whose objects are vertex algebras with strong generators $\{\alpha^1, \alpha^2, \dots\}$ and OPE relations determined by the lambda-bracket in A. The morphisms in $C(A)$ are vertex algebra homomorphisms. By Proposition 3.11 of [DSKI], the freely generated vertex algebra $U(A)$ is characterized by a universal property: any other object in $C(A)$ is a homomorphic image of $U(A)$.

If $U(A)$ is graded by conformal weight as above, $C(A)$ has another universal object $S(A)$ which is the quotient of $U(A)$ by its maximal proper ideal graded by conformal weight. It is the unique simple graded object in $C(A)$ and is a homomorphic image of any object in $C(A)$.

As pointed out on the last page of [DSKI], it is interesting to relax the Jacobi identity (3.6). A nonlinear conformal algebra A satisfying (3.5) but not all of the identites (3.6) is called *degenerate*. The universal enveloping vertex algebra $U(A)$ can still be defined as a quotient of the tensor algebra $\mathcal{T}(A)$. For simplicity, we restrict to case $\Gamma = \mathbb{Z}_{\geq 0}$. For $a, b, c \in \mathcal{T}(A)$ with $\Delta(a) + \Delta(b) + \Delta(c) - 1 = \Delta$, note that

$$L_\lambda(a, L_\mu(b, c)) - (-1)^{|a||b|} L_\mu(b, L_\lambda(a, c)) - L_{\lambda+\mu}(L_\lambda(a, b), c) \in \mathbb{C}[\lambda, \mu] \otimes \mathcal{T}(A)_\Delta.$$

Let $\mathcal{J}_\Delta(A)$ be the span of the coefficients of $\lambda^i \mu^j$ in all such expressions, for all $i, j \geq 0$. Then we define $\mathcal{M}'_\Delta(A) \subseteq \mathcal{T}_\Delta(A)$ to be the span of $\mathcal{M}_\Delta(A)$ together with the following elements:

(1) $N(a, b)$ and $N(b, a)$ for all $a \in \mathcal{T}_{\Delta_1}(A)$ and $b \in \mathcal{J}_{\Delta_2}(A)$ with $\Delta_1 + \Delta_2 = \Delta$.
(2) The coefficient of λ^i for all $i \geq 0$ in $L_\lambda(a, b)$ and $L_\lambda(b, a)$ for all $a \in \mathcal{T}_{\Delta_1}(A)$ and $b \in \mathcal{J}_{\Delta_2}(A)$, with $\Delta_1 + \Delta_2 - 1 = \Delta$.

Setting

$$(3.8) \qquad\qquad \mathcal{M}'(A) = \bigcup_{\Delta \in \Gamma} \mathcal{M}'_\Delta(A),$$

it follows that $\mathbb{C}[\lambda] \otimes \mathcal{M}'(A)$ is a two-sided ideal of $\mathbb{C}[\lambda] \otimes \mathcal{T}(A)$ under the operations N and L_λ. Setting

$$(3.9) \qquad\qquad U(A) = \mathcal{T}(A)/\mathcal{M}'(A),$$

by a similar argument as in the case of nonlinear Lie conformal algebras, $U(A)$ has a vertex algebra structure. Note that if A is a nonlinear Lie conformal algebra, $\mathcal{M}(A) = \mathcal{M}'(A)$, so this definition of $U(A)$ agrees with (3.7)

Suppose that A has basis $\{\alpha^1, \alpha^2, \dots\}$ as a $\mathbb{C}[T]$-module, where $\Delta(\alpha^i) = d_i > 0$. Then $U(A)$ is strongly generated by the corresponding fields $\{\alpha^1, \alpha^2, \dots\}$, but is not freely generated unless A is a nonlinear Lie conformal algebra. If we declare that $T^k(\alpha^i)$ has conformal weight $d_i + k$, $U(A)$ will be graded by conformal weight. We can still consider the category $C(A)$ consisting of vertex algebras graded by conformal weight with strong generators $\{\alpha^1, \alpha^2, \dots\}$, and OPE relations determined by A. Then $U(A)$ is a universal object in $C(A)$ which admits a homomorphism to any other object in the category, and $C(A)$ contains another universal object $S(A)$ which is the simple quotient of $U(A)$ by its maximal proper graded ideal.

Unlike the category of commutative rings, not every vertex algebra is the quotient of a freely generated vertex algebra with the same strong generating set. This is because Jacobi relations that fail to hold will give rise to *null fields*, which are elements of $\mathcal{M}'(A)$ that correspond to nontrivial normally ordered polynomial relation among the generators and their derivatives. A typical example (to be discussed in more detail later) is the parafermion algebra $N^k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, V^k(\mathfrak{sl}_2))$, which is of type $\mathcal{W}(2, 3, 4, 5)$. These fields close nonlinearly and the OPE algebra can be found in the paper [DLY]. Starting in weight 8, there are Jacobi relations that do not hold identically, which give rise to nontrivial relations among the generators and their derivatives.

4. Vertex algebras over commutative rings

Let R be a finitely generated, unital, commutative $\mathbb{C}$-algebra. A *vertex algebra over R* will be an R-module $\mathcal{A}$ with a vertex algebra structure which we define as in Section 2. A more comprehensive theory of vertex algebras over commutative rings has recently been developed by Mason [Ma], but the main difficulties are not present when R is a $\mathbb{C}$-algebra. First, given an R-module M, we define $\mathrm{QO}_R(M)$ to be the set of R-module homomorphisms $a : M \to M((z))$, which can be represented power series

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \mathrm{End}_R(M)[[z, z^{-1}]].$$

Here $a(n) \in \mathrm{End}_R(M)$ is an R-module endomorphism, and for each $v \in M$, $a(n)v = 0$ for $n \gg 0$. Clearly $\mathrm{QO}_R(M)$ is an R-module, and we define the products $a_{(n)}b$ as before, which are R-module homomorphisms from $\mathrm{QO}_R(M) \otimes_R \mathrm{QO}_R(M) \to \mathrm{QO}_R(M)$. A QOA will be an R-module $\mathcal{A} \subseteq \mathrm{QO}_R(M)$ containing 1 and closed under all products. Locality is defined as before, and a vertex algebra over R is a QOA $\mathcal{A} \subseteq \mathrm{QO}_R(M)$ whose elements are pairwise local. The OPE formula (2.1) still holds, as well as the fact that there is a faithful representation $\mathcal{A} \to \mathrm{QO}_R(\mathcal{A})$, which gives us the state field correspondence.

In particular, we say that a set $S = \{\alpha^i | \ i \in I\}$ generates $\mathcal{A}$ if $\mathcal{A}$ is spanned as an R-module by all words in α^i and the above products. We say that S *strongly generates* $\mathcal{A}$ if $\mathcal{A}$ is spanned as an R-module by all iterated Wick products of these generators and their derivatives. Note that we do *not* assume that $\mathcal{A}$ is a free R-module. If $S = \{\alpha^1, \alpha^2, \dots\}$ is an ordered strong generating set for $\mathcal{A}$ which is at most countable, we say that S *freely generates* $\mathcal{A}$, if $\mathcal{A}$ has an R-basis consisting of all normally ordered monomials of the form (2.3). In particular, this implies that $\mathcal{A}$ is a free R-module.

Conformal structure and graded character. Let $\mathcal{V}$ be a vertex algebra over R and let $c \in R$. Suppose that $\mathcal{V}$ contains a field L satisfying (2.4), such that L_0 acts on $\mathcal{V}$ by ∂ and L_1 acts diagonalizably, and we have an R-module decomposition

$$\mathcal{V} = \bigoplus_{d \in R} \mathcal{V}[d],$$

where $\mathcal{V}[d]$ is the L_0-eigenspace with eigenvalue d. If each $\mathcal{V}[d]$ is in addition a free R-module of finite rank, we have a well defined graded character

$$\chi(\mathcal{V}, q) = \sum_{d \in R} \mathrm{rank}_R(\mathcal{V}[n]) q^d.$$

In all our examples, the grading will be by $\mathbb{Z}_{\geq 0}$ regarded as a subsemigroup of R, and $\mathcal{V}[0] \cong R$. A typical example is $V^k(\mathfrak{g})$ where k is regarded as a formal variable, so $V^k(\mathfrak{g})$ is a vertex algebra over the polynomial ring $\mathbb{C}[k]$. As such, it has no conformal vector, but if we define R to be the localization $D^{-1}\mathbb{C}[k]$ where D is the multiplicatively closed set generated by $(k - h^\vee)$, then $V^k(\mathfrak{g})$ has Virasoro vector $L^{\mathfrak{g}}$ given by (2.12).

In [CLI, CLII], a more general class of examples called *deformable families* were considered. These are vertex algebras over certain rings of rational functions of degree at most zero in a parameter κ. This notion was used to study orbifolds and cosets of affine vertex algebras and $\mathcal{W}$-algebras by considering the limit as $\kappa \to \infty$, which has a simpler structure.

Here we allow R to be the ring of functions on a variety $X \subseteq \mathbb{C}^n$, so that $X = \mathrm{Specm}(R)$. We can think of the vertex algebra as being defined on X in the sense that for each point $p \in X$, the evaluation at p yields an ordinary vertex algebra over $\mathbb{C}$. There is also the notion of specialization along a subvariety. If $I \subseteq R$ is an ideal corresponding to a subvariety $Y \subseteq X$, the reduction mod I is the quotient

$$\mathcal{V}^I = \mathcal{V}/I \cdot \mathcal{V},$$

where $I \cdot \mathcal{V}$ is the ideal generated by I. This is a vertex algebra over the ring R/I.

Simplicity. Let $\mathcal{V}$ be a vertex algebra over a ring R with weight grading

$$\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}[n], \qquad \mathcal{V}[0] \cong R.$$

A vertex algebra ideal $\mathcal{I} \subseteq \mathcal{V}$ is called graded if

$$\mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I}[n], \qquad \mathcal{I}[n] = \mathcal{I} \cap \mathcal{V}[n].$$

We say that $\mathcal{V}$ is *simple* if there are no graded ideals $\mathcal{I}$ such that $\mathcal{I}[0] \neq \{0\}$.

Shapovalov determinant and spectrum. Given a simple vertex algebra $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}[n]$ over R, $\mathcal{V}[n]$ has a symmetric bilinear form

(4.1) $$\langle, \rangle_n : \mathcal{V}[n] \otimes_R \mathcal{V}[n] \to R, \qquad \langle \omega, \nu \rangle_n = \omega_{(2n-1)}\nu.$$

Define the level n Shapovalov determinant $\det_n \in R$ to be the determinant of the matrix of $\langle, \rangle_n$. If R is a unique factorization ring, each irreducible factor a of $\det_n$ give rise to a prime ideal $(a) \subseteq R$. Note that if $a | \det_n$, then $a | \det_m$ for all $m > n$. We define the *Shapovalov spectrum* $\mathrm{Shap}(\mathcal{V})$ to be the set of distinct prime ideals of the form $(a) \subseteq R$ such that a is a divisor of $\det_n$ for some n. We say that $(a) \in \mathrm{Shap}(\mathcal{V})$ has level n if a divides $\det_n$ but does not divide $\det_m$ for $m < n$.

Note that the varieties $V(a)$ for $a \in \mathrm{Shap}(\mathcal{V})$ are precisely the subvarieties where reduction along the subvariety gives rise to a vertex algebra that has a nontrivial maximal graded ideal. For example, if $\mathcal{V} = V^k(\mathfrak{g})$ and $R = D^{-1}\mathbb{C}[k]$ where D is the multiplicative set generated by $k - h^\vee$ as above, $\mathrm{Shap}(V^k(\mathfrak{g}))$ is the set of all ideals $(k - k_0)$ where the vacuum module $V^{k_0}(\mathfrak{g})$ is reducible. This was described explicitly by Kac and Gorelik in [KG].

Kac-de Sole correspondence over rings. In a similar way to [DSKI], one can define the notions of nonlinear conformal algebra and nonlinear Lie conformal algebra over a ring R. A nonlinear conformal algebra over R will be a free $R[T]$-module A with a grading

$$A = \bigoplus_{\Delta \in \Gamma \setminus \{0\}} A[\Delta]$$

for some semigroup Γ, and an R-module homomorphism

$$[-_\lambda-] : A \otimes_R A \to R[\lambda] \otimes_R \mathcal{T}(A)$$

satisfying (3.1) and (3.4). Here $\mathcal{T}(A)$ is the tensor algebra of A in the category of R-modules, which inherits the Γ-grading as before. Then the normally ordered product

$$N : \mathcal{T}(A) \otimes_R \mathcal{T}(A) \to \mathcal{T}(A),$$

and the lambda-bracket

$$L_\lambda : \mathcal{T}(A) \otimes_R \mathcal{T}(A) \to R[\lambda] \otimes_R \mathcal{T}(A),$$

can be defined on $\mathcal{T}(A)$ as in Section 3, and the identities analogous to (3.1)-(3.3) hold for all $a, b, c \in \mathcal{T}(A)$.

Here we only need the case $\Gamma = \mathbb{Z}_{\geq 0}$. The subspaces $\mathcal{M}_\Delta(A)$ and $\mathcal{M}(A)$ are defined as in Section 3. We call A a *nonlinear Lie conformal algebra* over R if it also satisfies (3.5) and (3.6). As in Section 3, we can define the universal enveloping vertex algebra $U(A) = \mathcal{T}(A)/\mathcal{M}(A)$, which is a vertex algebra over R. It is graded by conformal weight if we declare $T^k(a)$ to have weight $\Delta(a) + k$. We only consider examples where $U(A)$ contains a Virasoro element L whose L_0-eigenspace grading coincides with the weight grading. If $\{\alpha^1, \alpha^2, \dots\}$ is an ordered basis of A as a free $R[T]$-module, $U(A)$ is freely generated by the corresponding fields as a vertex algebra over R. In particular, each weighted component $U(A)[n]$ is a free R-module. If $\Delta(\alpha^i) = d_i > 0$, the graded character of $U(A)$ is given by (2.5) with $\dim(U(A)[n])$ replaced by $\mathrm{rank}_R(U(A)[n])$.

Finally, suppose that A is a degenerate nonlinear conformal algebra over R, meaning that it satisfies (3.5) but not (3.6). Then the universal enveloping vertex algebra $U(A) = \mathcal{T}(A)/\mathcal{M}'(A)$ is still defined, where $\mathcal{M}'(A)$ is defined as in Section 3. If $\{\alpha^1, \alpha^2, \dots\}$ is an ordered basis for A as a $R[T]$-module, $U(A)$ is strongly generated by the corresponding fields, but is not freely generated. It is still the direct sum of its weight graded pieces $U(A)[n]$, but since nontrivial normally ordered polynomial relations among the monomials hold, $U(A)[n]$ need not be a free R-module.

5. MAIN CONSTRUCTION

Throughout this section we shall work over the polynomial ring $\mathbb{C}[c, \lambda]$ where c, λ are formal variables. We wish to construct a nonlinear Lie conformal algebra $\mathcal{L}(c, \lambda)$ over $\mathbb{C}[c, \lambda]$ with generators $\{L, W^i | \, i \geq 3\}$ and grading $\Delta(L) = 2$, $\Delta(W^i) = i$. Equivalently, the universal enveloping vertex algebra

$$\mathcal{W}(c, \lambda) = U(\mathcal{L}(c, \lambda))$$

is freely generated by the corresponding fields, which we also denote by $\{L, W^i | \, i \geq 3\}$.

We will first construct a nonlinear conformal algebra $\mathcal{L}(c, \lambda)$ which satisfies (3.5) but only a subset of the Jacobi identities (3.6). A priori, it is not clear that it satisfies all such identities and is a nonlinear Lie conformal algebra. The vertex algebra $\mathcal{W}(c, \lambda)$ is therefore well-defined but might not be freely generated. However, by considering a family of quotients of $\mathcal{W}(c, \lambda)$ whose graded characters are known, we will show that $\mathcal{W}(c, \lambda)$ must in fact be freely generated.

Notationally, we prefer to work with the OPE rather than lambda-bracket formalism, so the identities (3.1)-(3.3) and (3.5)-(3.6) are replaced with (2.6)-(2.10). Even though the existence of $\mathcal{L}(c, \lambda)$ must be established before we know that $\mathcal{W}(c, \lambda)$ exists, we shall work with the fields $L, W^i \in \mathcal{W}(c, \lambda)$, with the caveat that knowing the OPEs $L(z)W^i(w)$ and $W^i(z)W^j(w)$ is equivalent to knowing the lambda-bracket of the corresponding elements of $\mathcal{L}(c, \lambda)$ up to null fields, that is, elements of $\mathcal{M}'(\mathcal{L}(c, \lambda))$.

We postulate that $\mathcal{W}(c, \lambda)$ has the following features.

(1) $\mathcal{W}(c, \lambda)$ possesses a Virasoro field L of central charge c and a weight 3 primary field W^3, so that

(5.1)
$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1},$$
$$L(z)W^3(w) \sim 3W^3(w)(z-w)^{-2} + \partial W^3(w)(z-w)^{-1}.$$

(2) The fields W^i are defined inductively by
$$W^i = (W^3)_{(1)}W^{i-1}, \qquad i \geq 4.$$

We have
$$L(z)W^i(w) \sim \cdots + iW^i(w)(z-w)^{-2} + \partial W^i(w)(z-w)^{-1},$$

and the fields $\{L, W^i \mid i \geq 3\}$ close under OPE. In particular, $\mathcal{W}(c, \lambda)$ is generated as a vertex algebra by $\{L, W^3\}$, and is strongly generated by $\{L, W^i \mid i \geq 3\}$.

(3) W^3 is nondegenerate in the sense that $W^3_{(5)}W^3 \neq 0$. Without loss of generality we may normalize W^3 so that
$$(W^3)_{(5)}W^3 = \frac{c}{3}1.$$

(4) There is a $\mathbb{Z}_2$-action on $\mathcal{W}(c, \lambda)$ defined by
$$L \mapsto L, \qquad W^3 \mapsto -W^3.$$

This forces $W^i \mapsto (-1)^i W^i$ for all $i \geq 3$, and each term appearing in $W^i_{(k)}W^j$ has weight $i + j - k - 1$ and eigenvalue $(-1)^{i+j}$ under this symmetry.

For notational convenience, we sometimes denote L by W^2. The grading and symmetry assumptions have the following consequence. For $k = 0, 1$, and $2 \leq i \leq j$, $(W^i)_{(k)}W^j$ depends only on $L, W^3, \ldots, W^{i+j-2}$ and their derivatives. This is because all terms must have eigenvalue is $(-1)^{i+j}$ under $\mathbb{Z}_2$. Then ∂W^{i+j-3} cannot appear in $(W^i)_{(1)}W^j$, and neither W^{i+j-1}, $\partial^2 W^{i+j-3}$ nor $: LW^{i+j-3} :$, can appear in $(W^i)_{(0)}W^j$. Similarly, for $k \geq 2$, $(W^i)_{(k)}W^j$ only depends on $L, W^3, \ldots, W^{i+j-4}$ and their derivatives.

In particular, for all i, j such that $2 \leq i \leq j$, we can write

(5.2)
$$W^i_{(1)}W^j = a_{i,j}W^{i+j-2} + C_{i,j},$$

where $a_{i,j}$ denotes the coefficient of W^{i+j-2} and $C_{i,j}$ is a normally ordered polynomial in $L, W^3, \ldots, W^{i+j-4}$ and their derivatives. By assumption, $a_{2,j} = j$, $a_{3,j} = 1$, $C_{2,j} = 0$, and $C_{3,j} = 0$ for all $j \geq 2$. Similarly,

(5.3)
$$W^i_{(0)}W^j = b_{i,j}\partial W^{i+j-2} + D_{i,j},$$

where $b_{i,j}$ denotes the coefficient of ∂W^{i+j-2} and $D_{i,j}$ is a normally ordered polynomial in $L, W^3, \ldots, W^{i+j-4}$ and their derivatives. By assumption, $b_{2,j} = 1$ and $D_{2,j} = 0$ for all $j \geq 2$.

We let $S_{n,k}$ denote the set of products
$$\{W^i_{(k)}W^j \mid i + j = n, \ k \geq 0\},$$

which vanish for $k > n - 1$. Our strategy is as follows. First, by imposing the identities (2.6)-(2.9), and all Jacobi relations of type (W^i, W^j, W^k) for $i + j + k \leq 9$, this determines $S_{n,k}$ for all $n \leq 7$ and $k \geq 0$ uniquely. If we then assume $S_{m,k}$ to be known for $m \leq n$ and $k \geq 0$, we show that by imposing a subset of Jacobi relations of type (W^i, W^j, W^k) with $i + j + k = n + 3$, this uniquely determines $S_{n+1,k}$.

Note that we are not checking *all* Jacobi relations of type (W^i, W^j, W^k) for $i + j + k = n + 3$ at this stage; we leave open the possibility that some of them might not vanish but instead give rise to nontrivial null fields. This has the effect that as we proceed through the induction, the OPEs $W^i(z)W^j(w)$ are uniquely determined modulo null fields that only depend on $L, W^3, \ldots, W^{i+j-2}$ and their derivatives. For convenience, we suppress from our notation the fact that OPEs are determined up to null fields. At the end of the induction, we obtain the existence of a (possibly degenerate) nonlinear conformal algebra $\mathcal{L}(c, \lambda)$ over $\mathbb{C}[c, \lambda]$ with generators $\{L, W^i \mid i \geq 3\}$, satisfying (3.5). We then invoke the Kac-de Sole correspondence to conclude that the universal enveloping vertex algebra $\mathcal{W}(c, \lambda)$ indeed exists. Since the lambda-brackets in $\mathcal{L}(c, \lambda)$ are unique up to null fields, the OPE algebra of $\mathcal{W}(c, \lambda)$ is unique.

Step 1: $S_{n,k}$ for $n \leq 7$. Since $W^4 = (W^3)_{(1)}W^3$ by definition, the most general OPE of W^3 with itself that is compatible with the $\mathbb{Z}_2$-symmetry is

$$W^3(z)W^3(w) \sim \frac{c}{3}(z-w)^{-6} + a_0 L(w)(z-w)^{-4} + a_1 \partial L(w)(z-w)^{-3}$$
$$+ W^4(w)(z-w)^{-2} + \left(a_2 \partial W^4 + a_3 \partial^3 L\right)(w)(z-w)^{-1}.$$

It is not difficult to check that imposing the Jacobi relations of type (L, W^3, W^3) forces

(5.4)
$$W^3(z)W^3(w) \sim \frac{c}{3}(z-w)^{-6} + 2L(w)(z-w)^{-4} + \partial L(w)(z-w)^{-3}$$
$$+ W^4(w)(z-w)^{-2} + \left(\frac{1}{2}\partial W^4 - \frac{1}{12}\partial^3 L\right)(w)(z-w)^{-1}.$$

The most general OPE of L and W^4 compatible with the $\mathbb{Z}_2$-symmetry is

$$L(z)W^4(w) \sim a_0(z-w)^{-6} + a_1 L(w)(z-w)^{-4} + a_2 \partial L(w)(z-w)^{-3}$$
$$+ 4W^4(w)(z-w)^{-2} + \partial W^4(w)(z-w)^{-1}.$$

By imposing the Jacobi relations of type (L, L, W^4) and (L, W^3, W^3) this forces

(5.5)
$$L(z)W^4(w) \sim 3c(z-w)^{-6} + 10L(w)(z-w)^{-4} + 3\partial L(w)(z-w)^{-3}$$
$$+ 4W^4(w)(z-w)^{-2} + \partial W^4(w)(z-w)^{-1}.$$

Next, the most general OPE of L and W^5 compatible with the $\mathbb{Z}_2$-symmetry is

$$L(z)W^5(w) \sim b_0 W^3(z)(z-w)^{-4} + b_1 \partial W^3(z)(z-w)^{-3}$$
$$+ 5W^5(w)(z-w)^{-2} + \partial W^5(w)(z-w)^{-1}.$$

Similarly, the most general OPE of W^3 and W^4 compatible with the $\mathbb{Z}_2$-symmetry is

$$W^3(z)W^4(w) \sim a_0 W^3(w)(z-w)^{-4} + a_1 \partial W^3(w)(z-w)^{-3} + W^5(w)(z-w)^{-2}$$
$$+ \left(a_2 : L\partial W^3 : +a_3 : (\partial L)W^3 : +a_4 \partial W^5 + a_5 \partial^3 W^3\right)(w)(z-w)^{-1},$$

It turns out that unlike (5.4) and (5.5), if we impose all Jacobi identities of type (L, W^3, W^4), (L, L, W^5), and (W^3, W^3, W^3) there is an additional free parameter λ in the above OPEs. They have the form

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(5.6)
$$L(z)W^5(w) \sim \big(185 - 80\lambda(2+c)\big)W^3(z)(z-w)^{-4} + \big(55 - 16\lambda(2+c)\big)\partial W^3(z)(z-w)^{-3}$$
$$+ 5W^5(w)(z-w)^{-2} + \partial W^5(w)(z-w)^{-1}.$$

(5.7)
$$W^3(z)W^4(w) \sim \Big(31 - 16\lambda(2+c)\Big)W^3(w)(z-w)^{-4} + \frac{8}{3}\Big(5 - 2\lambda(2+c)\Big)\partial W^3(w)(z-w)^{-3}$$
$$+ W^5(w)(z-w)^{-2} + \Big(\frac{2}{5}\partial W^5 + \frac{32}{5}\lambda : L\partial W^3 : -\frac{48}{5}\lambda : (\partial L)W^3 :$$
$$+ \frac{2}{15}\Big(-5 + 2\lambda(-1+c)\Big)\partial^3 W^3\Big)(w)(z-w)^{-1}.$$

Letting $L = W^2$, by imposing Jacobi identities of type (W^i, W^j, W^k) for $i + j + k \leq 9$, we have shown that all terms in the OPE of $W^i(z)W^j(w)$ for $2 \leq i \leq j$ and $i + j \leq 7$ are uniquely determined as $\mathbb{C}[c, \lambda]$-linear combinations of normally ordered monomials in the generators W^i for $i \leq 5$. By imposing the identities (2.6) and (2.9), this uniquely determine the OPEs

$$\partial^a W^i(z)\partial^b W^j(w), \qquad a, b \geq 0, \qquad i > j \geq 2, \qquad i + j \leq 7.$$

Step 2: Induction. We make the following inductive hypothesis.

(1) For $2 \leq i \leq j$ and $i + j \leq n$, all terms in the OPE of $W^i(z)W^j(w)$ have been expressed as $\mathbb{C}[c, \lambda]$-linear combinations of normally monomials in $L, W^i, \ldots, W^{n-2}$ and their derivatives. The OPEs are weight homogeneous and compatible with $\mathbb{Z}_2$-symmetry, i.e., all terms appearing in $W^i_{(k)}W^j$ have weight $i + j - k - 1$ and eigenvalue $(-1)^{i+j}$.

(2) We impose (2.6) and (2.9), which then determines all OPEs of the form

$$\partial^a W^i(z)\partial^b W^j(w), \qquad a, b \geq 0, \qquad i + j \leq n.$$

(3) $a_{i,j}$ and $b_{i,j}$ are independent of c, λ for $i + j \leq n$. Here $a_{i,j}$ and $b_{i,j}$ are given by (5.2) and (5.3), respectively.

By *inductive data*, we mean the collection of all OPE relations

(5.8)
$$\partial^a W^i(z)\partial^b W^j(w), \qquad a, b \geq 0, \qquad i + j \leq n.$$

If we impose (2.7) and (2.8), it is a consequence of our inductive hypothesis that if α is a normally ordered polynomial in the generators $L, W^3, \ldots, W^{n-i}$ and their derivatives of weight $w \leq n + 1 - i$, the OPE $W^i(z)\alpha(w)$ is uniquely determined by the inductive data.

Lemma 5.1. *If we impose the Jacobi identity*

(5.9)
$$L_{(2)}(W^3_{(0)}W^{n-2}) = (L_{(2)}W^3)_{(0)}W^{n-2} + W^3_{(0)}(L_{(2)}W^{n-2})$$
$$+ 2(L_{(1)}W^3)_{(1)}W^{n-2} + (L_{(0)}W^3)_{(2)}W^{n-2},$$

we must have

$$b_{3,n-2} = \frac{2}{n-1}.$$

In particular, $b_{3,n-2}$ is independent of c and λ.

Proof. Recall that
$$W^3_{(0)}W^{n-2} = b_{3,n-2}\partial W^{n-1} + D_{3,n-2}$$
where $D_{3,n-2}$ is a normally ordered polynomial in $L, W^3, \ldots, W^{n-3}$ and their derivatives. Then
$$L_{(2)}(W^3_{(0)}W^{n-2}) = b_{3,n-2}L_{(2)}\partial W^{n-1} + L_{(2)}D_{3,n-2}.$$
Also, $L_{(2)}D_{3,n-2}$ has no terms depending on W^{n-1} by inductive assumption, so it does not contribute to the coefficient of ∂W^{n-1}. We have
$$L_{(2)}\partial W^{n-1} = -(\partial L)_{(2)}W^{n-1} + \partial(L_{(2)}W^{n-1}).$$
Note that $L_{(2)}W^{n-1} \in S_{n+1,2}$ and is therefore not yet known, but by weight and parity considerations it only depends on $L, W^3, \ldots, W^{n-3}$ and their derivatives. Then $\partial(L_{(2)}W^{n-1})$ only depends on $L, W^3, \ldots, W^{n-3}$. Modulo terms which only depend on $L, W^3, \ldots, W^{n-3}$ and their derivatives, we have
$$L_{(2)}\partial W^{n-1} \equiv -(\partial L)_{(2)}W^{n-1} = 2L_{(1)}W^{n-1} = 2(n-1)W^{n-1}.$$
So the left hand side of (5.9) is $2(n-1)b_{3,n-2}W^{n-1}$ up to terms which do not depend on W^{n-1}.

Next, the term $(L_{(2)}W^3)_{(0)}W^{n-2}$ from (5.9) vanishes because W^3 is assumed primary. The term
$$W^3_{(0)}(L_{(2)}W^{n-2})$$
from (5.9) has no contribution to the coefficient of W^{n-1}, since $L_{(2)}W^{n-2}$ only depends on $L, W^3, \ldots, W^{n-4}$ and their derivatives. The term
$$2(L_{(1)}W^3)_{(1)}W^{n-2}$$
from (5.9) contributes $6W^3_{(1)}W^{n-2} = 6W^{n-1}$. The term
$$(L_{(0)}W^3)_{(2)}W^{n-2} = \partial W^3_{(2)}W^{n-2} = -2W^3_{(1)}W^{n-2} = -2W^{n-1}.$$
We conclude that
$$2(n-1)b_{3,n-2} = 4,$$
hence the lemma follows. $\square$

Lemma 5.2. *All coefficients $b_{i,n+1-i}$ for $3 \le i \le \frac{n}{2}$ are independent of c, λ and are determined uniquely by imposing Jacobi relations of type (W^i, W^j, W^k) for $i + j + k = n + 3$.*

Proof. We first impose the Jacobi relation
$$(5.10) \qquad W^3_{(1)}(W^3_{(0)}W^{n-3}) = (W^3_{(1)}W^3)_{(0)}W^{n-3} + W^3_{(0)}(W^3_{(1)}W^{n-3}) + (W^3_{(0)}W^3)_{(1)}W^{n-3}.$$
Since $W^3_{(0)}W^{n-3} = \frac{2}{n-2}\partial W^{n-2} + D_{3,n-3}$, the left hand side of (5.10) is

$$(5.11)$$
$$W^3_{(1)}\left(\frac{2}{n-2}\partial W^{n-2} + D_{3,n-3}\right) = -\frac{2}{n-2}(\partial W^3)_{(1)}W^{n-2} + \frac{2}{n-2}\partial\left((W^3)_{(1)}W^{n-2}\right) + W^3_{(1)}D_{3,n-3}$$
$$= \frac{2}{n-2}\left(\frac{2}{n-1}\partial W^{n-1} + D_{3,n-2}\right) + \frac{2}{n-2}\partial W^{n-1} + W^3_{(1)}D_{3,n-3}$$
$$= \frac{2(1+n)}{(-2+n)(-1+n)}\partial W^{n-1} + \frac{2}{n-2}D_{3,n-2} + W^3_{(1)}D_{3,n-3}.$$

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Next,

$$(W^3_{(1)}W^3)_{(0)}W^{n-3} = W^4_{(0)}W^{n-3} = b_{4,n-3}\partial W^{n-1} + D_{4,n-3},$$

$$W^3_{(0)}(W^3_{(1)}W^{n-3}) = W^3_{(0)}W^{n-2} = \frac{2}{n-1}\partial W^{n-1} + D_{3,n-2},$$

(5.12)
$$(W^3_{(0)}W^3)_{(1)}W^{n-3} = \left(\frac{1}{2}\partial W^4 - \frac{1}{12}\partial^3 L\right)_{(1)}W^{n-3}$$

$$= -\frac{1}{2}W^4_{(0)}W^{n-3} = -\frac{1}{2}\left(b_{4,n-3}\partial W^{n-1} + D_{4,n-3}\right).$$

Collecting terms, we conclude that

$$b_{4,n-3} = \frac{12}{(n-1)(n-2)}.$$

Inductively, we impose the Jacobi relation
(5.13)
$$W^3_{(1)}(W^{i-1}_{(0)}W^{n+1-i}) = (W^3_{(1)}W^{i-1})_{(0)}W^{n+1-i} + W^{i-1}_{(0)}(W^3_{(1)}W^{n+1-i}) + (W^3_{(0)}W^{i-1})_{(1)}W^{n+1-i}.$$

The left side of (5.13) is

(5.14)
$$W^3_{(1)}\left(b_{i-1,n+1-i}\partial W^{n-2} + D_{i-1,n+1-i}\right) = b_{i-1,n+1-i}W^3_{(1)}\partial W^{n-2} + W^3_{(1)}D_{i-1,n+1-i}$$

$$= b_{i-1,n+1-i}\left(\frac{2}{n-1}\partial W^{n-1} + D_{3,n-2} + \partial W^{n-1}\right) + W^3_{(1)}D_{i-1,n+1-i}.$$

The right side of (5.13) is

(5.15) $b_{i,n+1-i}\partial W^{n-1} + D_{i,n+1-i} + b_{i-1,n+2-i}\partial W^{n-1} + D_{i-1,n+2-i} - \frac{2}{i}\left(\partial W^{n-1} + D_{i,n+1-i}\right).$

Recall that $b_{i-1,n+1-i}$ and $D_{i-1,n+1-i}$ are part of our inductive data, and we are assuming inductively that $b_{i-1,n+1-i}$ is independent of c, λ. Since $D_{i-1,n+1-i}$ is a normally ordered polynomial of weight $n-2$ in $L, W^3, \dots, W^{n-4}$ and their derivatives, it follows that $W^3_{(1)}D_{n-1,n+1-i}$ is uniquely determined by inductive data. This shows that $b_{i,n+1-i}$ is uniquely determined from inductive data together with $b_{i-1,n+2-i}$ for $4 \le i < \frac{n}{2}$. Finally, since $b_{3,n-2}$ is independent of c, λ, it is clear that each $b_{i,n+1-i}$ is independent of c, λ. $\square$

Similarly, we will show that $a_{4,n-3} = \frac{4}{n-2}$ and that $a_{i,n+1-i}$ is determined from the Jacobi identities for all i; see Lemma 5.4 below. Combining these observations, we obtain

Lemma 5.3. *The products* $(W^i)_{(0)}W^{n+1-i}$ *for* $3 \le i \le \frac{n}{2}$, *are uniquely determined from inductive data, Jacobi relations of type* (W^i, W^j, W^k) *for* $i + j + k = n + 3$, *and elements of* $S_{n+1,1}$.

Proof. Since $b_{i,n+1-i}$ is determined from this data, it suffices to show that $D_{i,n+1-i}$ is also determined for $3 \le i \le \frac{n}{2}$. It follows from (5.12) that modulo terms which are determined from inductive data,

(5.16)
$$D_{4,n-3} \equiv -\frac{2(n-4)}{n-2}D_{3,n-2}.$$

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Using (5.14) and (5.15) in the case $i = 5$, we get

(5.17)
$$\frac{12}{(n-2)(n-3)} D_{3,n-3} \equiv D_{5,n-4} + D_{4,n-3} - \frac{2}{5} D_{5,n-4},$$

modulo inductive data. Similarly, (5.14) and (5.15) show that there are nontrivial relations $D_{i,n+1-i} \equiv p_i(n) D_{3,n-2}$ for rational functions $p_i(n)$ for all i, modulo inductive data. So it is enough to find three linearly independent relation between $D_{3,n-2}$, $D_{4,n-3}$, and $D_{5,n-4}$. From the Jacobi relation

$$W^3_{(0)}(W^4_{(1)} W^{n-4}) = (W^3_{(0)} W^4)_{(1)} W^{n-4} + W^4_{(1)}(W^3_{(0)} W^{n-4}),$$

we get

$$\frac{4}{n-3} D_{3,n-2} \equiv -\frac{2}{5} D_{5,n-4} + \frac{2}{n-3} D_{4,n-3} + \frac{2}{n-3} \partial C_{4,n-3},$$

modulo inductive data. Since $C_{4,n-3}$ is determined by $S_{n+1,1}$, we get the relation

(5.18)
$$\frac{4}{n-3} D_{3,n-2} \equiv -\frac{2}{5} D_{5,n-4} + \frac{2}{n-3} D_{4,n-3}$$

modulo inductive data together with data determined by $S_{n+1,1}$. Then (5.16)-(5.18) are the desired linearly independent relations. $\qquad\square$

Lemma 5.4. *The set $S_{n+1,1}$ of products $W^i_{(1)} W^{n+1-i}$ is uniquely determined from inductive data, Jacobi relations of type (W^i, W^j, W^k) for $i + j + k = n + 3$, and the sets $S_{n+1,\ell}$ for $\ell \geq 2$.*

Proof. By assumption $L_{(1)} W^{n-1} = (n-1) W^{n-1}$ and $W^3_{(1)} W^{n-2} = W^{n-1}$, so we begin with the case $i \geq 4$. We impose the Jacobi relation

(5.19)
$$W^{n-3}_{(1)}(W^3_{(1)} W^3) = W^3_{(1)}(W^{n-3}_{(1)} W^3) + (W^{n-3}_{(1)} W^3)_{(1)} W^3 + (W^{n-3}_{(0)} W^3)_{(2)} W^3.$$

By (2.9), the left hand side of (5.19) is

$$W^{n-3}_{(1)} W^4 = \sum_{i \geq 0} (-1)^i \frac{1}{i!} \partial^i (W^4_{(i+1)} W^{n-3}).$$

Therefore modulo derivatives of elements of $S_{n+1,\ell}$ for $\ell \geq 2$,

$$W^{n-3}_{(1)} W^4 \equiv W^4_{(1)} W^{n-3} = a_{4,n-3} W^{n-1} + C_{4,n-3}.$$

As for the right hand side, modulo terms which are either derivatives of elements of $S_{n+1,\ell}$ for $\ell \geq 2$, or are determined by inductive data, we have

$$W^3_{(1)}(W^{n-3}_{(1)} W^3) \equiv W^3_{(1)}(W^3_{(1)} W^{n-3}) = W^3_{(1)} W^{n-2} = W^{n-1},$$

$$(W^{n-3}_{(1)} W^3)_{(1)} W^3 \equiv (W^3_{(1)} W^{n-3})_{(1)} W^3 \equiv W^{n-2}_{(1)} W^3 \equiv W^{n-1},$$

$$(W^{n-3}_{(0)} W^3)_{(2)} W^3 = \left(\sum_{i \geq 0} (-1)^{i+1} \frac{1}{i!} \partial^i (W^3_{(i)} W^{n-3}) \right)_{(2)} W^3$$

$$= \left(-\frac{2}{n-2} \partial W^{n-2} + \partial W^{n-2} \right)_{(2)} W^3 = \left(\frac{4}{n-2} - 2 \right) W^{n-2}_{(1)} W^3$$

$$\equiv \left(\frac{4}{n-2} - 2 \right) W^3_{(1)} W^{n-2} = \left(\frac{4}{n-2} - 2 \right) W^{n-1}.$$

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Collecting terms, we see that
$$a_{4,n-3} = \frac{4}{n-2},$$
and that $C_{4,n-3}$ is uniquely determined modulo inductive data and derivatives of elements of $S_{n+1,\ell}$ for $\ell \geq 2$.

Next, we impose the Jacobi relation

(5.20) $\qquad W^3_{(1)}(W^4_{(1)} W^{n-4}) = W^4_{(1)}(W^3_{(1)} W^{n-4}) + (W^3_{(1)} W^4)_{(1)} W^{n-4} + (W^3_{(0)} W^4)_{(2)} W^{n-4}.$

Since $W^4_{(1)} W^{n-4} = \frac{4}{n-3} W^{n-2} + C_{4,n-4}$ where $C_{4,n-4}$ depends only on $L, W^3, \ldots, W^{n-4}$ and their derivatives and is determined by inductive data, the left hand side of (5.20) is
$$W^3_{(1)}\left(\frac{4}{n-3} W^{n-2} + C_{4,n-4}\right) \equiv \frac{2}{n-1}\frac{4}{n-3} W^{n-1}.$$

For the right hand side, we have
$$W^4_{(1)}(W^3_{(1)} W^{n-4}) = W^4_{(1)} W^{n-3} = \frac{4}{n-2} W^{n-1} + C_{4,n-3},$$
$$(W^3_{(1)} W^4)_{(1)} W^{n-4} = W^5_{(1)} W^{n-4} = a_{5,n-4} W^{n-1} + C_{5,n-4},$$
$$(W^3_{(0)} W^4)_{(2)} W^{n-4} = \left(\frac{2}{5}\partial W^5 + \frac{32}{5}\lambda : L\partial W^3 : -\frac{48}{5}\lambda : (\partial L)W^3 :\right.$$
$$\left. +\frac{2}{15}\left(-5 + 2\lambda(-1+c)\right)\partial^3 W^3\right)_{(2)} W^{n-4}$$
$$\equiv -\frac{4}{5}\left(a_{5,n-4} W^{n-1} + C_{5,n-4}\right).$$

It is immediate that
$$a_{5,n-4} = \frac{20}{(n-2)(n-3)},$$
and that $C_{5,n-4}$ is determined uniquely modulo inductive data and derivatives of elements of $S_{n+1,\ell}$ for $\ell \geq 2$. Similarly, for $i > 4$, by imposing the Jacobi relation

(5.21) $\qquad W^3_{(1)}(W^i_{(1)} W^{n-i}) = W^i_{(1)}(W^3_{(1)} W^{n-i}) + (W^3_{(1)} W^i)_{(1)} W^{n-i} + (W^3_{(0)} W^i)_{(2)} W^{n-i},$

the same argument shows that both $a_{i+1,n-i}$ and $C_{i+1,n-i}$ are uniquely determined modulo inductive data and derivatives of elements of $S_{n+1,\ell}$ for $\ell \geq 2$. This shows that $S_{n+1,1}$ is uniquely determined modulo this data. $\qquad \square$

Lemma 5.5. *The set $S_{n+1,2}$ of products $W^i_{(2)} W^{n+1-i}$ is uniquely determined from inductive data, Jacobi relations of type (W^i, W^j, W^k) for $i + j + k = n + 3$, and the sets $S_{n+1,\ell}$ for $\ell \geq 3$.*

Proof. First, since $L_{(2)} W^{n-1} = L_{(2)}(W^3_{(1)} W^{n-2})$, we need to impose the Jacobi identity

(5.22)
$$L_{(2)}(W^3_{(1)} W^{n-2}) = W^3_{(1)}(L_{(2)} W^{n-2}) + (L_{(2)} W^3)_{(1)} W^{n-2}$$
$$+ 2(L_{(1)} W^3)_{(2)} W^{n-2} + (L_{(0)} W^3)_{(3)} W^{n-2}.$$

By inductive assumption, $L_{(2)} W^{n-2}$ is known and is expressible in terms of L, W^i for $i \leq n-4$. Then $W^3_{(1)}(L_{(2)} W^{n-2})$ is also determined by inductive data. The term $(L_{(2)} W^3)_{(1)} W^{n-2}$ vanishes because W^3 is primary, and the remaining terms are expressible in terms of

$W^3_{(2)}W^{n-2}$ together with inductive data. So determining $L_{(2)}W^{n-1}$ is equivalent to determining $W^3_{(2)}W^{n-2}$.

For the purpose of determining $W^3_{(2)}W^{n-2}$, we impose the Jacobi relation

(5.23) $\qquad W^{n-3}_{(1)}(W^3_{(2)}W^3) = W^3_{(2)}(W^{n-3}_{(1)}W^3) + (W^{n-3}_{(1)}W^3)_{(2)}W^3 + (W^{n-3}_{(0)}W^3)_{(3)}W^3.$

The left hand side of (5.23) is $W^{n-3}_{(1)}\partial L$, which is known by inductive data. Next, by (2.9), we have

$$W^3_{(2)}(W^{n-3}_{(1)}W^3) + (W^{n-3}_{(1)}W^3)_{(2)}W^3 = \sum_{i \geq 1}(-1)^{i+1}\frac{1}{i!}\partial^i\left((W^{n-3}_{(1)}W^3)_{(i+2)}W^3\right),$$

Since $(W^{n-3}_{(1)}W^3) = W^{n-2}$ modulo terms which depend only on $L, W^3, \ldots, W^{n-4}$, and is known by inductive data, the sum $W^3_{(2)}(W^{n-3}_{(1)}W^3) + (W^{n-3}_{(1)}W^3)_{(2)}W^3$ in (5.23) is determined by inductive data together with derivatives of elements of $S_{n+1,\ell}$ for $\ell \geq 3$. Finally, the remaining term in (5.23) is

$$(W^{n-3}_{(0)}W^3)_{(3)}W^3 \equiv -(W^3_{(0)}W^{n-3})_{(3)}W^3 = -\frac{2}{n-2}\partial W^{n-2}_{(3)}W^3 = \frac{6}{n-2}W^{n-2}_{(2)}W^3$$

$$= \frac{6}{n-2}\sum_{i \geq 0}(-1)^{i+1}\frac{1}{i!}\partial^i(W^3_{(i+2)}W^{n-2}),$$

modulo inductive data. Therefore $W^3_{(2)}W^{n-2}$ is expressible in terms of inductive data together with derivatives of elements of $S_{n+1,\ell}$ for $\ell > 2$. Since $L_{(2)}W^{n-1}$ can be expressed in terms of $W^3_{(2)}W^{n-2}$, the same holds for $L_{(2)}W^{n-1}$.

Next, for $i \geq 3$ we impose the Jacobi relation

(5.24) $\qquad \begin{aligned} W^3_{(2)}(W^i_{(1)}W^{n-i}) &= W^i_{(1)}(W^3_{(2)}W^{n-i}) + (W^3_{(2)}W^i)_{(1)}W^{n-i} + 2(W^3_{(1)}W^i)_{(2)}W^{n-i} \\ &\quad + (W^3_{(0)}W^i)_{(3)}W^{n-i}. \end{aligned}$

This allows us to express $W^i_{(2)}W^{n+1-i}$ for all i in terms of inductive data together with derivatives of $S_{n+1,\ell}$ for $\ell \geq 3$. $\qquad\square$

Lemma 5.6. *For all $k > 2$, the set $S_{n+1,k}$ of products $W^i_{(k)}W^{n+1-i}$ is uniquely determined from inductive data, Jacobi relations of type (W^i, W^j, W^k) for $i + j + k = n + 3$, and the sets $S_{n+1,\ell}$ for $\ell > k$.*

Proof. The argument is the same as the proof of Lemma 5.5. Imposing the Jacobi relation

(5.25) $\qquad L_{(k)}(W^3_{(1)}W^{n-2}) = W^3_{(1)}(L_{(k)}W^{n-2}) + \sum_{i \geq 0}\binom{k}{i}(L_{(i)}W^3)_{(k+1-i)}W^{n-2},$

shows that determining $L_{(k)}W^{n-1}$ is equivalent to determining $W^3_{(k)}W^{n-2}$. Imposing the Jacobi relation

(5.26) $\qquad W^{n-3}_{(1)}(W^3_{(k)}W^3) = W^3_{(k)}(W^{n-3}_{(1)}W^3) + (W^{n-3}_{(1)}W^3)_{(k)}W^3 + (W^{n-3}_{(0)}W^3)_{(k+1)}W^3,$

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shows that $W^3_{(k)}W^{n-2}$, and hence $L_{(k)}W^{n-1}$, are determined from inductive data together with $S_{n+1,\ell}$ for $\ell > k$. Finally, imposing the Jacobi relation

$$(5.27) \qquad W^3_{(k)}(W^i_{(1)}W^{n-i}) = W^i_{(1)}(W^3_{(k)}W^{n-i}) + \sum_{r\geq 0}\binom{k}{r}(W^3_{(r)}W^i)_{(k+1-r)}W^{n-i},$$

shows that $W^i_{(k)}W^{n+1-i}$ can be expressed in terms of inductive data together with $S_{n+1,\ell}$ for $\ell > k$. $\qquad\square$

This process terminates after finitely many steps since all elements of $S_{n+1,k}$ vanish for $k > n$. Therefore we have proven the following

Theorem 5.7. *There exists a nonlinear conformal algebra $\mathcal{L}(c,\lambda)$ over the ring $\mathbb{C}[c,\lambda]$ satisfying (3.5), whose universal enveloping vertex algebra $\mathcal{W}(c,\lambda)$ has the following properties.*

(1) $\mathcal{W}(c,\lambda)$ has conformal weight grading

$$\mathcal{W}(c,\lambda) = \bigoplus_{n\geq 0}\mathcal{W}(c,\lambda)[n],$$

and $\mathcal{W}(c,\lambda)[0] \cong \mathbb{C}[c,\lambda]$.
(2) $\mathcal{W}(c,\lambda)$ is strongly generated by $\{L, W^i|\ i \geq 3\}$, and satisfies the OPE relations (5.4)-(5.7), together with the Jacobi identities (5.9)-(5.10), (5.13), (5.19)-(5.27), which appear in the above lemmas.
(3) $\mathcal{W}(c,\lambda)$ is the unique initial object in the category of vertex algebras with the above properties.

It is not yet apparent that all Jacobi identities of the form (W^i, W^j, W^k) hold, or equivalently, that $\mathcal{L}(c,\lambda)$ is a nonlinear Lie conformal algebra and $\mathcal{W}(c,\lambda)$ is freely generated.

Step 3: Free generation of $\mathcal{W}(c,\lambda)$. In order to prove that $\mathcal{W}(c,\lambda)$ is freely generated, we need to consider certain simple quotients of $\mathcal{W}(c,\lambda)$. It is useful to think of such quotients as being obtained by a two-step procedure. First, let

$$I \subseteq \mathbb{C}[c,\lambda] \cong \mathcal{W}(c,\lambda)[0]$$

be an ideal, and let $I \cdot \mathcal{W}(c,\lambda)$ denote the vertex algebra ideal generated by I, which is the set of all I-linear combinations of elements of $\mathcal{W}(c,\lambda)$. We define the quotient

$$(5.28) \qquad \mathcal{W}^I(c,\lambda) = \mathcal{W}(c,\lambda)/I \cdot \mathcal{W}(c,\lambda),$$

which has strong generators $\{L, W^i|\ i \geq 3\}$ satisfying the same OPE algebra as the corresponding generators of $\mathcal{W}(c,\lambda)$ where all structure constants in $\mathbb{C}[c,\lambda]$ are replaced by their images in $\mathbb{C}[c,\lambda]/I$. Even though $\mathcal{W}(c,\lambda)$ will turn out to be simple as a vertex algebra over $\mathbb{C}[c,\lambda]$, $\mathcal{W}^I(c,\lambda)$ need not be simple as a vertex algebra over $\mathbb{C}[c,\lambda]/I$.

We now consider localizations of $\mathcal{W}^I(c,\lambda)$. Let $D \subseteq \mathbb{C}[c,\lambda]/I$ be a multiplicatively closed subset, and let $R = D^{-1}\mathbb{C}[c,\lambda]/I$ denote the localization of $\mathbb{C}[c,\lambda]/I$ along D. Then we have the localization of $\mathbb{C}[c,\lambda]/I$-modules

$$\mathcal{W}^I_R(c,\lambda) = R \otimes_{\mathbb{C}[c,\lambda]/I} \mathcal{W}^I(c,\lambda),$$

which is a vertex algebra over R.

Theorem 5.8. *Let I, D, and R be as above, and let $\mathcal{W}$ be a simple vertex algebra over R with the following properties.*

(1) $\mathcal{W}$ is generated by a Virasoro field $\tilde{L}$ of central charge c and a weight 3 primary field $\tilde{W}^3$.

(2) *Setting* $\tilde{W}^2 = \tilde{L}$ *and* $\tilde{W}^i = \tilde{W}^3_{(1)}\tilde{W}^{i-1}$ *for* $i \geq 4$, *the OPE relations* (5.4)-(5.7) *are satisfied if the structure constants are replaced with their images in* R.

Then $\mathcal{W}$ *is the simple quotient of* $\mathcal{W}^I_R(c, \lambda)$ *by its maximal graded ideal* $\mathcal{I}$.

Proof. The assumption that $\{\tilde{L}, \tilde{W}^i | i \geq 3\}$ satisfy (5.4)-(5.7) is equivalent to the statement that the Jacobi relations of type $(\tilde{W}^i, \tilde{W}^j, \tilde{W}^k)$ for $i + j + k \leq 9$ hold in the corresponding (possibly degenerate) nonlinear conformal algebra. Then all OPE relations among the generators of $\mathcal{W}^I_R(c, \lambda)$ must hold among the fields $\{\tilde{L}, \tilde{W}^i | i \geq 3\}$, since they are formal consequences of (5.4)-(5.7) together with the Jacobi identities, which hold in $\mathcal{W}$. It follows that $\{\tilde{L}, \tilde{W}^i | i \geq 3\}$ close under OPE and strongly generate a vertex subalgebra $\mathcal{W}' \subseteq \mathcal{W}$, which must coincide with $\mathcal{W}$ since $\mathcal{W}$ is assumed to be generated by $\{\tilde{L}, \tilde{W}^3\}$ as a vertex algebra. So $\mathcal{W}$ has the same strong generating set and OPE algebra as $\mathcal{W}^I_R(c, \lambda)$. Since $\mathcal{W}$ is simple and the category of vertex algebras over R with this strong generating set and OPE algebra has a unique simple graded object, $\mathcal{W}$ must be the simple quotient of $\mathcal{W}^I_R(c, \lambda)$ by its maximal graded ideal. $\square$

Theorem 5.9. *For all* $n \geq 3$, *the algebra* $\mathcal{W}^k(\mathfrak{sl}_n, f_{prin})$ *which has central charge*

$$c(k) = -\frac{(n-1)(n^2 + nk - n - 1)(n^2 + k + nk)}{n+k},$$

is the simple quotient of $\mathcal{W}^I_R(c, \lambda)$ *for some prime ideal* $I \subseteq \mathbb{C}[c, \lambda]$ *and some localization* R *of* $\mathbb{C}[c, \lambda]/I$. *Moreover, the maximal proper graded ideal* $\mathcal{I} \subseteq \mathcal{W}^I_R(c, \lambda)$ *is generated by a singular vector of weight* $n+1$ *of the form*

(5.29) $$W^{n+1} - P(L, W^3, \ldots, W^{n-1}),$$

where P *is a normally ordered polynomial in* $L, W^3, \ldots, W^{n-1}$, *and their derivatives.*

Proof. Recall that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is freely generated by the Virasoro field L, a weight 3 primary field W^3 satisfying $(W^3)_{(5)}W^3 = \frac{c}{3}\mathbf{1}$, and fields $W^i = W^3_{(1)}W^{i-1}$ for $4 \leq i \leq n$. Since all Jacobi identities of type (W^i, W^j, W^k) hold for $i + j + k \leq 9$, and all structure constants are rational functions of k, it follows that there exists some rational function

$$\lambda(k) = \frac{f(k)}{g(k)}$$

such that (5.4)-(5.7) are satisfied if c and λ are replaced by $c(k)$ and $\lambda(k)$, respectively.

Let D be the multiplicative set generated by $(n + k)$ and all roots $g(k)$. By standard methods of elimination theory, letting $J \subseteq \mathbb{C}[c, \lambda, k]$ be the ideal generated by

$$c(n+k) + (n-1)(n^2 + nk - n - 1)(n^2 + k + nk), \qquad g(k)\lambda - f(k),$$

we can eliminate k to obtain an ideal $I \subseteq \mathbb{C}[c, \lambda]$ such that some localization R of $\mathbb{C}[c, \lambda]/I$ is isomorphic to $D^{-1}\mathbb{C}[k]$. Note that the variety $V(I)$ is a rational curve with rational parametrization $\lambda = \lambda(k)$ and $c = c(k)$, so that I is a prime ideal.

Since $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is of type $\mathcal{W}(2, 3, \ldots, n)$, there must be a singular vector in $\mathcal{W}^I_R(c, \lambda)$ of weight $n+1$ such that the coefficient of W^{n+1} is nonzero. If this coefficient is not invertible in R, we may localize R further (without changing notation) so it becomes invertible, and the singular vector has the form (5.29). Since $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is simple for generic values of k, it is simple as a vertex algebra over R, and hence must be the simple quotient

$\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$, by Theorem 5.8. Here $\mathcal{I}$ is the maximal proper graded ideal of $\mathcal{W}_R^I(c, \lambda)$. Finally, we need to show that (5.29) generates $\mathcal{I}$.

Let $\mathcal{I}' \subseteq \mathcal{I}$ be the ideal in $\mathcal{W}_R^I(c, \lambda)$ generated by (5.29). Since $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}}) \cong \mathcal{W}_R^I(c, \lambda)/\mathcal{I}$, $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is also a quotient of $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}'$. By Theorem 6.1, $\mathcal{W}_R^I(c, \lambda)(c, \lambda)/\mathcal{I}'$ is of type $\mathcal{W}(2, 3, \dots, n)$. Since $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is freely generated, there can be no more relations in $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$ than in $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}'$, so $\mathcal{I}' = \mathcal{I}$. $\qquad\square$

Remark 5.10. *It is not obvious that the generator of the ideal I is a polynomial in both n and k. Later, we will show this by giving an explicit formula.*

Corollary 5.11. *All Jacobi identities of type (W^i, W^j, W^k) hold identically in $\mathcal{L}(c, \lambda)$, so $\mathcal{L}(c, \lambda)$ is a nonlinear Lie conformal algebra with generators $\{L, W^i | i \geq 3\}$. Equivalently, $\mathcal{W}(c, \lambda)$ is freely generated by $\{L, W^i | i \geq 3\}$ and has graded character*

$$(5.30) \qquad \chi(\mathcal{W}(c, \lambda), q) = \sum_{n \geq 0} \text{rank}_{\mathbb{C}[c, \lambda]}(\mathcal{W}(c, \lambda)[n])q^n = \prod_{n \geq 2} \frac{1}{(1 - q^n)^{n-1}}.$$

Moreover, for any prime ideal $I \subseteq \mathbb{C}[c, \lambda]$, $\mathcal{W}^I(c, \lambda)$ is freely generated by $\{L, W^i | i \geq 3\}$ as a vertex algebra over $\mathbb{C}[c, \lambda]/I$, and

$$(5.31) \qquad \chi(\mathcal{W}^I(c, \lambda), q) = \sum_{n \geq 0} \text{rank}_{\mathbb{C}[c, \lambda]/I}(\mathcal{W}^I(c, \lambda)[n])q^n = \prod_{n \geq 2} \frac{1}{(1 - q^n)^{n-1}},$$

and for any localization $R = D^{-1}\mathbb{C}[c, \lambda]/I$ along a multiplicatively closed set $D \subseteq \mathbb{C}[c, \lambda]/I$, $\mathcal{W}_R^I(c, \lambda)$ is freely generated by $\{L, W^i | i \geq 3\}$ and

$$(5.32) \qquad \chi(\mathcal{W}_R^I(c, \lambda), q) = \sum_{n \geq 0} \text{rank}_R(\mathcal{W}_R^I(c, \lambda)[n])q^n = \prod_{n \geq 2} \frac{1}{(1 - q^n)^{n-1}}.$$

Proof. If some Jacobi identity of type (W^i, W^j, W^k) does not hold, there would be a null vector of weight N in $\mathcal{W}(c, \lambda)$ for some N. Then the rank of $\mathcal{W}(c, \lambda)[N]$ would be strictly smaller than that given by (5.30), and the same would hold in any quotient of $\mathcal{W}(c, \lambda)[N]$, as well as any localization of such a quotient. But since $\mathcal{W}^k(\mathfrak{sl}_N, f_{\text{prin}})$ is a localization of such a quotient and is freely generated of type $\mathcal{W}(2, 3, \dots, N)$, this is impossible. $\qquad\square$

Corollary 5.12. *$\mathcal{W}(c, \lambda)$ is a simple vertex algebra.*

Proof. If $\mathcal{W}(c, \lambda)$ is not simple, it would have a singular vector in weight N for some N. Let $p \in \mathbb{C}[c, \lambda]$ be an irreducible polynomial and let $I = (p) \subseteq \mathbb{C}[c, \lambda]$. By rescaling if necessary, we can assume without loss of generality that the singular vector is not divisible by p and hence descends to a nontrivial singular vector in $\mathcal{W}^I(c, \lambda)$. Then for any localization R of $\mathbb{C}[c, \lambda]/I$, the simple quotient of $\mathcal{W}_R^I(c, \lambda)$ would have a smaller weight N submodule than $\mathcal{W}(c, \lambda)$ for all such I. This contradicts the fact that $\mathcal{W}^k(\mathfrak{sl}_N, f_{\text{prin}})$ is such a quotient. $\qquad\square$

Zhu functor. The Zhu functor is a basic tool in the representation theory of vertex algebras [Z]. Let $\mathcal{V}$ be a vertex algebra with weight grading $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}[n]$. For $a \in \mathcal{V}[m]$ and $b \in \mathcal{V}$, define

$$(5.33) \qquad a * b = \text{Res}_z \left(a(z) \frac{(z + 1)^m}{z} b \right),$$

and extend $*$ by linearity to a bilinear operation $\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$. Let $O(\mathcal{V})$ denote the subspace of $\mathcal{V}$ spanned by elements of the form

$$(5.34) \qquad a \circ b = \mathrm{Res}_z \left(a(z) \frac{(z+1)^m}{z^2} b \right)$$

where $a \in \mathcal{V}[m]$, and define the *Zhu algebra* $\mathrm{Zhu}(\mathcal{V})$ to be the quotient vector space $\mathcal{V}/O(\mathcal{V})$, with projection

$$(5.35) \qquad \pi_{\mathrm{Zhu}} : \mathcal{V} \to \mathrm{Zhu}(\mathcal{V}).$$

Then $O(\mathcal{V})$ is a two-sided ideal in $\mathcal{V}$ under the product $*$, and $(\mathrm{Zhu}(\mathcal{V}), *)$ is a unital, associative algebra. The assignment $\mathcal{V} \mapsto \mathrm{Zhu}(\mathcal{V})$ is functorial, and if $\mathcal{I} \subseteq \mathcal{V}$ is a vertex algebra ideal,

$$(5.36) \qquad \mathrm{Zhu}(\mathcal{V}/\mathcal{I}) \cong \mathrm{Zhu}(\mathcal{V})/I, \qquad I = \pi_{\mathrm{Zhu}}(\mathcal{I}).$$

If $\mathcal{V}$ is strongly generated by homogeneous elements $\{\alpha^1, \alpha^2, \dots\}$, $\mathrm{Zhu}(\mathcal{V})$ is generated by $\{a^i = \pi_{\mathrm{Zhu}}(\alpha^i)\}$. A $\mathbb{Z}_{\geq 0}$-graded $\mathcal{V}$-module $M = \bigoplus_{n \geq 0} M[n]$ is called a *positive energy module* if for every $a \in \mathcal{V}[n]$, $a(n)M_k \subseteq M[m+k-n-1]$, for all n and k. Given $a \in \mathcal{V}[m]$, $a(m-1)$ acts on each $M[k]$. The subspace $M[0]$ is then a $\mathrm{Zhu}(\mathcal{V})$-module with action $\pi_{\mathrm{Zhu}}(a) \mapsto a(m-1) \in \mathrm{End}(M[0])$. In fact, $M \mapsto M[0]$ provides a one-to-one correspondence between irreducible, positive energy $\mathcal{V}$-modules and irreducible $\mathrm{Zhu}(\mathcal{V})$-modules. If $\mathrm{Zhu}(\mathcal{V})$ is commutative, all its irreducible modules are one-dimensional. The corresponding irreducible $\mathcal{V}$-modules $M = \bigoplus_{n \geq 0} M[n]$ are then cyclic, and will be called *highest-weight modules*.

Corollary 5.13. *Zhu$(\mathcal{W}(c, \lambda))$ is the polynomial algebra*

$$(5.37) \qquad \mathbb{C}[\ell, w^i | \, i \geq 3],$$

where the generators are the images of $\{L, W^i | \, i \geq 3\}$ under the Zhu map (5.35). *In particular, Zhu$(\mathcal{W}(c, \lambda))$ does not depend on c and λ. For any ideal $I \subseteq \mathbb{C}[c, \lambda]$, any localization R of $\mathbb{C}[c, \lambda]/I$, and any quotient $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$, the Zhu algebra Zhu$(\mathcal{W}_R^I(c, \lambda)/\mathcal{I})$ is a quotient of a localization of* (5.37), *and hence is abelian.*

Proof. It is well known that $\mathrm{Zhu}(\mathcal{W}(c, \lambda))$ is generated by $\{\ell, w^i | \, i \geq 3\}$, and that ℓ is central. The commutator $[w^i, w^j]$ in the Zhu algebra is expressed in terms of the OPE algebra and hence is a polynomial in $\{\ell, w^i | \, i \geq 3\}$ with structure constants in $\mathbb{C}[c, \lambda]$. Since the Zhu algebra of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ is known to be abelian, each structure constant is divisible by each p_{n+1}, where $p_{n+1} \in \mathbb{C}[c, \lambda]$ generates the ideal I_{n+1} such that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ is a quotient of a localization of $\mathcal{W}^{I_{n+1}}(c, \lambda)$. Since $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ is generated by the weight 3 field for all $n \geq 3$, the polynomials p_{n+1} must all be distinct, so all of the above structure constants must vanish. The remaining statements follow from (5.36). $\qquad \square$

Corollary 5.14. *For any vertex algebra $\mathcal{W} = \mathcal{W}_R^I(c, \lambda)/\mathcal{I}$ for some I and R as above, all irreducible, positive energy modules are highest-weight modules, and are parametrized by the variety Specm$(\mathrm{Zhu}(\mathcal{W}))$.*

6. QUOTIENTS OF $\mathcal{W}(c, \lambda)$ AND THE CLASSIFICATION OF VERTEX ALGEBRAS OF TYPE $\mathcal{W}(2, 3, \dots, N)$

As we have seen, any simple vertex algebra generated by a Virasoro field and a weight 3 primary field W^3 satisfying some natural conditions, is a quotient of $\mathcal{W}_R^I(c, \lambda)$ for some

I and R. This reduces the classification of vertex algebras with these properties to the classification of ideals $I \subseteq \mathbb{C}[c, \lambda]$ such that $\mathcal{W}^I(c, \lambda)$ is not simple.

Recall that $\mathcal{W}(c, \lambda)[n]$ is a free $\mathbb{C}[c, \lambda]$-module whose rank is given by (5.30). It has a symmetric bilinear form

$$\langle , \rangle_n : \mathcal{W}(c, \lambda)[n] \otimes_{\mathbb{C}[c,\lambda]} \mathcal{W}(c, \lambda)[n] \to \mathbb{C}[c, \lambda], \qquad \langle \omega, \nu \rangle_n = \omega_{(2n-1)}\nu.$$

Recall that the Shapovalov determinant $\det_n \in \mathbb{C}[c, \lambda]$ is the determinant of the matrix of this form. Then $\mathcal{W}(c, \lambda)$ is a simple vertex algebra whenever $\det_n \neq 0$ for all n. Recall that an irreducible polynomial $p \in \mathbb{C}[c, \lambda]$ is said to lie in the level n Shapovalov spectrum if p divides $\det_n$ but does not divide $\det_m$ for any $m < n$. Clearly $\mathcal{W}^I(c, \lambda)$ is simple for a generic choice of I, since each $\det_n$ has finitely many irreducible factors.

Let p be an irreducible factor of $\det_{N+1}$ of level $N+1$. Letting $I = (p) \subseteq \mathbb{C}[c, \lambda]$, $\mathcal{W}^I(c, \lambda)$ will then have a singular vector in weight $N+1$. As we shall see, the coefficient of W^{n+1} in this singular vector is often nonzero. By inverting this coefficient, we obtain a localization R of $\mathbb{C}[c, \lambda]/I$ such that this singular vector has the form

(6.1) $$W^{N+1} - P_{N+1}(L, W^3, \dots, W^{N-1})$$

in $\mathcal{W}_R^I(c, \lambda)$. Here P_{N+1} is a normally ordered polynomial in the fields $L, W^3, \dots, W^{N-1}$ and their derivatives, with coefficients in R. This implies that W^{N+1} decouples in the quotient of $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$, where $\mathcal{I}$ denotes the vertex algebra ideal generated by (6.1). In other words, we have the relation

$$W^{N+1} = P_{N+1}(L, W^3, \dots, W^{N-1})$$

in $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$. Applying the operator $(W^3)_{(1)}$ to this relation and using the fact that $(W^3)_{(1)}W^{N+1} = W^{N+2}$ and $(W^3)_{(1)}W^{N-1} = W^N$, we obtain a relation

$$W^{N+2} = P_{N+2}(L, W^3, \dots, W^N).$$

Applying $(W^3)_{(1)}$ again yields a relation

$$W^{N+3} = Q_{N+3}(L, W^3, \dots, W^{N+1}).$$

If the terms $\partial^2 W^{N+1}$ or $: LW^{N+1}:$ appear in Q_{N+3}, they can be eliminated using (6.1) to obtain a relation

$$W^{n+3} = P_{N+3}(L, W^3, \dots, W^N).$$

Inductively, by applying $(W^3)_{(1)}$ repeatedly and using (6.1) to eliminate W^{N+1} if necessary, we obtain relations

$$W^m = P_m(L, W^3, \dots, W^N)$$

in $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$, for all $m > N + 1$. This implies

Lemma 6.1. *Let p be an irreducible factor of $\det_{N+1}$ that does not divide $\det_m$ for $m < N + 1$, and let $I = (p)$. Suppose that there exists a localization R of $\mathbb{C}[c, \lambda]/I$ such that $\mathcal{W}_R^I(c, \lambda)$ has a singular vector of the form*

(6.2) $$W^{N+1} - P_{N+1}(L, W^3, \dots, W^{N-1}).$$

Let $\mathcal{I} \subseteq \mathcal{W}_R^I(c, \lambda)$ be the vertex algebra ideal generated by (6.2). Then the quotient $\mathcal{W}_R^I(c, \lambda)/\mathcal{I}$ has a minimal strong generating set $\{L, W^i \mid 3 \leq i \leq N\}$, and in particular is of type $\mathcal{W}(2, 3, \dots, N)$.

Remark 6.2. *The ideal $\mathcal{I}$ is sometimes (but not always) the maximal graded ideal $\mathcal{J} \subseteq \mathcal{W}_R^I(c, \lambda)$. However, the assumption that p does not divide $\det_m$ for $m < N + 1$ implies that there are no singular vectors in weight $m < N + 1$, so there can be no decoupling relations of the form $W^m = P_m(L, W^3, \ldots W^{m-1})$ for $m < N + 1$. Therefore the simple quotient $\mathcal{W}_R^I(c, \lambda)/\mathcal{J}$ is also of type $\mathcal{W}(2, 3 \ldots, N)$.*

Theorem 6.3. *For all $N \geq 3$, there are finitely many isomorphism classes of simple one-parameter vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$, which are generated by the Virasoro field L and a weight 3 primary field W^3 satisfying (5.4)-(5.7).*

Proof. Any such vertex algebra must be the simple quotient of $\mathcal{W}_R^I(c, \lambda)$ for some I and some localization R of $\mathbb{C}[c, \lambda]/I$, such that $\mathcal{W}_R^I(c, \lambda)$ has a singular vector in weight $m \leq N + 1$. But there are only finitely many divisors of $\det_m$ for $m \leq N + 1$. $\square$

There is a useful criterion for proving that a vertex algebra of type $\mathcal{W}(2, 3 \ldots, N)$ is a quotient of $\mathcal{W}_R^I(c, \lambda)$ for some I and R.

Theorem 6.4. *Let $\mathcal{W}$ be a simple vertex algebra of type $\mathcal{W}(2, 3, \ldots, N)$ which is defined over some localization R of $\mathbb{C}[c, \lambda]/I$, for some I. Suppose that $\mathcal{W}$ is generated by the Virasoro field L and a weight 3 primary field W^3. If in addition, the graded character of $\mathcal{W}$ agrees with that of $\mathcal{W}(c, \lambda)$ up to weight 6, then $\mathcal{W}$ is the simple quotient of $\mathcal{W}_R^I(c, \lambda)$.*

Proof. By Theorem 5.8, it suffices to prove that (5.4)-(5.7) are satisfied. But this is automatic because the graded character assumption implies that there are no null vectors of weight $w \leq 6$ in the (possibly degenerate) nonlinear conformal algebra corresponding to the strong generating set $\{L, W^3, W^i = (W^3)_{(1)}W^{i-1}\}$. $\square$

The inductive procedure for proving Theorem 5.8 can be regarded as an *algorithm* for computing the OPE algebra of $\mathcal{W}(c, \lambda)$ by starting from the OPE formulas (5.4)-(5.7) and imposing all Jacobi identities among the generators. Using the Mathematica package of Thielemans [T], we have applied this algorithm to compute all OPEs $W^i(z)W^j(w)$ for $2 \leq i \leq j$ and $i + j \leq 14$. This data is too complicated to reproduce in this paper, but we used it to obtain many of the results and conjectures in the next few sections.

7. Principal $\mathcal{W}$-algebras of type A

By Theorem 5.9, there is some ideal $I \subseteq \mathbb{C}[c, \lambda]$ and some localization R of $\mathbb{C}[c, \lambda]/I$ such that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ can be obtained as the simple quotient of $\mathcal{W}_R^I(c, \lambda)$. In this section, we shall give an explicit generator of this ideal. We need three preliminary results that are easily obtained by computer calculation. First, recall the parafermion algebra

$$N^k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, V^k(\mathfrak{sl}_2)),$$

where $\mathcal{H}$ is the Heisenberg algebra corresponding to the Cartan subalgebra of $\mathfrak{sl}_2$. It is of type $\mathcal{W}(2, 3, 4, 5)$ for all $k \neq 0$, and is generated by generated by the Virasoro field L of central charge $c = \frac{2(k-1)}{k+2}$ and a weight 3 primary field W^3 [DLY].

Theorem 7.1. *The polynomial*

$$p = 4\lambda(c + 7)(2c - 1) + (c - 2)(c + 4)$$

is an irreducible factor of $\det_6$ of level 6. Let $I = (p) \subseteq \mathbb{C}[c, \lambda]$, let D be the multiplicative set generated by $(c + 7)$ and $(2c - 1)$, and let

$$R = D^{-1}\mathbb{C}[c, \lambda]/I \cong D^{-1}\mathbb{C}[c].$$

Then

$$N^k(\mathfrak{sl}_2) \cong \mathcal{W}_R^I(c, \lambda)/\mathcal{I},$$

where $\mathcal{I}$ is the maximal proper graded ideal of $\mathcal{W}_R^I(c, \lambda)$. In particular, $N^k(\mathfrak{sl}_2)$ is obtained from $\mathcal{W}(c, \lambda)$ by setting

$$(7.1) \qquad c = \frac{2(k - 1)}{k + 2}, \qquad \lambda = \frac{k + 1}{(k - 2)(3k + 4)},$$

and then taking the simple quotient.

Next, recall the coset

$$\mathcal{C}^\ell = \mathrm{Com}(\mathcal{H}, \mathcal{W}^\ell),$$

where $\mathcal{W}^\ell = \mathcal{W}^{\ell - 3/2}(\mathfrak{sl}_3, f_{\min})$ is the Bershadsky-Polyakov algebra. By Theorem 6.1 of [ACLI], $\mathcal{C}^\ell$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for all $\ell \neq 0, 1/2$.

Theorem 7.2. *The polynomial*

$$p = 256(48 + 8c + 240\lambda - 62c\lambda - 5c^2\lambda + 300\lambda^2 + 524c\lambda^2 + 40c^2\lambda^2)$$

is an irreducible factor of $\det_8$ at level 8. The corresponding variety $V(I)$ for $I = (p)$ is a rational curve with parametrization

$$(7.2) \qquad c = -\frac{3(2\ell - 1)^2}{2\ell + 3}, \qquad \lambda = \frac{(2\ell + 1)(2\ell + 3)}{8(\ell - 1)(4\ell + 3)},$$

and $\mathcal{C}^\ell$ is obtained from $\mathcal{W}(c, \lambda)$ by substituting (7.2) and then taking the simple quotient. In particular, there exists a localization R of $\mathbb{C}[c, \lambda]/I$ such that

$$\mathcal{C}^\ell \cong \mathcal{W}_R^I(c, \lambda)/\mathcal{I},$$

where $\mathcal{I}$ is the maximal graded proper ideal of $\mathcal{W}_R^I(c, \lambda)$.

Finally, recall that the coset

$$\mathcal{C}^k = \mathrm{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_4, f_{\mathrm{subreg}}))$$

is of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8, 9)$ by Theorem 5.1 of [CLIII].

Theorem 7.3. *The polynomial*

$$p = 320 + 40c + 1536\lambda - 804c\lambda - 57c^2\lambda + 1456\lambda^2 + 1536c\lambda^2 + 444c^2\lambda^2 + 20c^3\lambda^2$$

is an irreducible factor of $\det_{10}$ of level 10. The corresponding variety $V(I)$ for $I = (p)$ is a rational curve with parametrization

$$(7.3) \qquad c = -\frac{4(5 + 2k)(7 + 3k)}{4 + k}, \qquad \lambda = -\frac{(3 + k)(4 + k)}{3(2 + k)^2(16 + 5k)},$$

and $\mathcal{C}^k$ is obtained from $\mathcal{W}(c, \lambda)$ by substituting (7.3) and then taking the simple quotient. In particular, there exists a localization R of $\mathbb{C}[c, \lambda]/I$ such that

$$\mathcal{C}^k \cong \mathcal{W}_R^I(c, \lambda)/\mathcal{I},$$

where $\mathcal{I}$ is the maximal graded proper ideal of $\mathcal{W}_R^I(c, \lambda)$.

The main result in this section is the following.

Theorem 7.4. *For $n \geq 3$, let $I_{n+1} \subseteq \mathbb{C}[c, \lambda]$ be the ideal generated by*

(7.4) $$p_{n+1} = \lambda(n-2)(3n^2 - n - 2 + c(n+2)) - (n-1)(n+1).$$

Then

(1) *p_{n+1} is an irreducible factor of $\det_{n+1}$ of level $n+1$.*
(2) *There exists a localization R_{n+1} of $\mathbb{C}[c, \lambda]/I_{n+1}$ in which $(3n^2 - n - 2 + c(n+2))$ is invertible, so that*

(7.5) $$\lambda = \frac{(n-1)(n+1)}{(n-2)(3n^2 - n - 2 + c(n+2))}$$

in R_{n+1}, and there is a unique singular vector

(7.6) $$W^{n+1} - P_{n+1}(L, W^3, \ldots, W^{n-1}),$$

which generates the maximal graded proper ideal $\mathcal{I}_{n+1} \subseteq \mathcal{W}_{R_{n+1}}^{I_{n+1}}(c, \lambda)$.
(3) *We have an isomorphism*

$$\mathcal{W}_{R_{n+1}}^{I_{n+1}}(c, \lambda)/\mathcal{I}_{n+1} \cong \mathcal{W}^k(\mathfrak{sl}_n, f_{prin}),$$

where c and λ are related to k by

(7.7) $$c = -\frac{(n-1)(n^2 + nk - n - 1)(n^2 + k + nk)}{n + k},$$
$$\lambda = -\frac{n + k}{(n-2)(n^2 + nk - n - 2)(n + n^2 + 2k + nk)}.$$

In particular, the structure constants appearing in the OPEs of the generators $\{L, W^i \mid i \geq 3\}$ of $\mathcal{W}^k(\mathfrak{sl}_n, f_{prin})$ are rational functions of k and n.

Proof. In view of Theorems 2.1 and 5.9, it suffices to show that the coset $\mathcal{C}^k(\mathfrak{sl}_n)$ is obtained from $\mathcal{W}(c, \lambda)$ by setting

(7.8) $$c = \frac{k(n-1)(1 + k + 2n)}{(k+n)(1+k+n)}, \qquad \lambda = \frac{(k+n)(1+k+n)}{(n-2)(2k+n)(2+2k+3n)},$$

and taking the simple quotient. The structure of this coset has been studied in [BBSSI, BBSSII]. The relevant calculations are contained in these papers, and we follow their notation, taking an antihermitian basis for $\mathfrak{sl}_n$ and corresponding generators $J^a_{(1)}$ for $V^k(\mathfrak{sl}_n)$ satisfying

$$J^a_{(1)}(z)J^b_{(1)}(w) \sim -\delta_{a,b}k(z-w)^{-2} + f^{abc}J^c_{(1)}(z-w)^{-1}.$$

Here f^{abc} is the f-tensor as defined in the appendix of [BBSSI], and we are summing over repeated indices. Similarly, we have generators $J^a_{(2)}$ for $L_1(\mathfrak{sl}_n)$ satisfying

$$J^a_{(2)}(z)J^b_{(2)}(w) \sim -\delta_{a,b}(z-w)^{-2} + f^{abc}J^c_{(2)}(z-w)^{-1}.$$

We also need the fields $Q^a_{(1)} \in V^k(\mathfrak{sl}_n)$ and $Q^a_{(2)} \in L_1(\mathfrak{sl}_n)$ given by

$$Q^a_{(1)} = d^{abc} : J^b_{(1)}J^c_{(1)} :, \qquad Q^a_{(2)} = d^{abc} : J^b_{(2)}J^c_{(2)} :,$$

where the d-tensor is defined as in [BBSSI]. These are primary of weight 2 and satisfy

(7.9) $$J^a_{(1)}(z)Q^b_{(1)}(z) \sim -(2k+n)d^{abc}J^c_{(1)}(w)(z-w)^{-2} + f^{abc}Q^c_{(1)}(w)(z-w)^{-1},$$
$$J^a_{(2)}(z)Q^b_{(2)}(z) \sim -(2+n)d^{abc}J^c_{(2)}(w)(z-w)^{-2} + f^{abc}Q^c_{(2)}(w)(z-w)^{-1}.$$

We also need

(7.10) $(Q^a_{(1)})_{(3)}Q^b_{(1)} = \delta_{a,b}\dfrac{2k(n+2k)(n^2-4)}{n}1,$ $(Q^a_{(2)})_{(3)}Q^b_{(2)} = \delta_{a,b}\dfrac{2(n+2)(n^2-4)}{n}1.$

In terms of these fields, the weight 3 primary field $W^3 \in \mathcal{C}^k(\mathfrak{sl}_n)$ is given as follows.

(7.11)
$$W^3 = B(n,k)\Big((n+1)(n+2) : J^a_{(1)}Q^a_{(1)} : -3(n+k)(n+1)(n+2) : J^a_{(2)}Q^a_{(1)} :$$
$$+ 3(n+k)(n+1)(n+2k) : J^a_{(1)}Q^a_{(2)} : -k(n+k)(n+2k) : J^a_{(2)}Q^a_{(2)} : \Big),$$

where $B(n,k)$ is given by

(7.12) $\dfrac{i}{3(n+k)(n+1)(n+k+1)}\sqrt{\dfrac{n}{2(n+2k)(n+2)(2+2k+3n)(n^2-4)}}.$

As usual, W^3 is normalized so that

$$W^3_{(5)}W^3 = \frac{k(n-1)(1+k+2n)}{3(k+n)(1+k+n)} = \frac{c}{3}.$$

As in Section 5, set $W^i = W^3_{(1)}W^{i-1}$ for $i \geq 4$. It is apparent from the OPE formulas (5.4)-(5.7) and (7.9)-(7.10), as well as properties of the f-tensor and d-tensor appearing in [BBSSI], that all structure constants in the algebra generated by $\{L, W^i | \, i \geq 3\}$ are rational functions in k and n. In particular,

(7.13) $W^3_{(3)}W^4 = p(n,k)W^3,$

for some rational function $p(n,k)$ in k and n. To prove the theorem, it suffices to show that

(7.14)
$$p(n,k) = \frac{q(n,k)}{(n-2)(2k+n)(2+2k+3n)},$$
$$q(n,k) = 3(-88k - 88k^2 - 52n - 140kn + 36k^2n - 52n^2 + 72kn^2 + 31n^3).$$

It follows from (7.11) and the OPE relations (7.9)-(7.10) that up to a constant, the denominator of $p(n,k)$ can contain at most the factors

(7.15) $(n+k)^2(n+1)^2(n+k+1)^2(n+2k)(n+2)(2+2k+3n)(n^2-4),$

which appear in the denominator of $B(n,k)^2$. Here we are using the fact that at most one factor of n can appear in the denominator as a result of contractions of the form (7.10), but this factor will then cancel the factor of n appearing in the numerator of $B(n,k)^2$. Therefore without loss of generality we may assume that the denominator of $p(n,k)$ is given by (7.15).

Note also that $p(n,k)$ is the same as the coefficient of $-B(n,k)k(n+k)(n+2k) : J^a_{(2)}Q^a_{(2)} :$ appearing in $W^3_{(3)}W^4$. The only contributions to this coefficient will come from

$$A_{(3)}(B_{(1)}C),$$

where A, B, C are terms appearing in (7.11), and two of the three depend either on $Q^a_{(2)}$ of $J^a_{(2)}$. All such terms are divisible by $(n+k)^2$. Therefore without loss of generality, we may write

(7.16) $p(n,k) = \dfrac{(n+1)^2(n+2)^2(n+k+1)^2(k+n)^2q(n,k) + (n+k)^2r(n,k)}{(n-2)(n+1)^2(n+2)^2(n+k)^2(2k+n)(1+k+n)^2(2+2k+3n)},$

where $r(n, k)$ is a polynomial function in n, k. We now need to show that $r(n, k) = 0$.

First, it is easy to verify by computer that Theorem 7.4 holds for $3 \leq n \leq 7$. Therefore (7.14) must hold when $3 \leq n \leq 7$ for all k except for the above denominators, so $r(n, k)$ is divisible by $(n-3)(n-4)(n-5)(n-6)(n-7)$. Similarly, in view of Theorems 7.1 and 2.2, (7.14) must hold whenever $k + 2(1 + n) = 0$, so $r(n, k)$ is divisible by $k + 2(1 + n)$. In view of Theorems 7.2 and 2.3, $r(n, k)$ is divisible by both $2k + n - 1$ and $2k + 3n + 3$. In view of Theorems 7.3 and 2.4, $r(n, k)$ is divisible by $3k + 2n - 1$ and $3k + 4n + 4$.

Therefore if $r(n, k) \neq 0$, it must have degree at least 10 in the variable n, so the numerator of $p(n, k)$, written as in (7.16), must have degree at least 12 in n. But if we examine the contribution of each term of the form

$$A_{(3)}(B_{(1)}C)$$

where A, B, C are the summands in (7.11), to the coefficient of

$$-B(n, k)k(n + k)(n + 2k) : J_{(2)}^a Q_{(2)}^a :$$

appearing in $W_{(3)}^3 W^4$, we find that the maximum degree in n of the numerator of $p(n, k)$ is 11. It follows that $r(n, k) = 0$, and the theorem is proved. $\square$

Corollary 7.5. *Since*

$$\lim_{n \to \infty} \frac{(n-1)(n+1)}{(n-2)(3n^2 - n - 2 + c(n+2))} = 0,$$

there is a meaningful limit of the vertex algebras $\mathcal{W}^k(\mathfrak{sl}_n, f_{prin})$ as $n \to \infty$, which is obtained from $\mathcal{W}(c, \lambda)$ by setting $\lambda = 0$. This vertex algebra is of type $\mathcal{W}(2, 3, \dots)$ with strong generators $\{L, W^i | i \geq 3\}$, and the structure constants are polynomials in c. However, it is not isomorphic to the coset of $\mathcal{H}$ inside the $\mathcal{W}_{1+\infty}$-algebra, which is also of type $\mathcal{W}(2, 3, \dots)$; see Theorem 11.1.

We remark that by computing the irreducible factors of the Shapovalov determinants $\det_N$ for $N \leq 7$, all one-parameter vertex algebras of type $\mathcal{W}(2, 3, \dots, N)$ satisfying (5.4)-(5.7), can be classified. It turns out that the complete list is $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ for $n \leq 7$, together with the cosets $N^k(\mathfrak{sl}_2)$, and $\mathcal{C}^\ell$ given by Theorems 7.1 and 7.2. The classification for $N \leq 6$ was known previously in the physics literature [B-V, BS, HI], but the result for $N = 7$ appears to be new. Note that $N^k(\mathfrak{sl}_2)$ and $\mathcal{C}^\ell$ are not freely generated, so the folklore conjecture that the only freely generated ones are the principal $\mathcal{W}$-algebras of type A is true for $N \leq 7$.

8. Generalized parafermions

We now consider another family of vertex algebras $\mathcal{C}^k(n)$ arising as quotients of $\mathcal{W}(c, \lambda)$, which we shall call *generalized parafermion algebras*. For $n \geq 1$ and $k \in \mathbb{C}$, the embedding $\mathfrak{gl}_n \to \mathfrak{sl}_{n+1}$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & -\text{tr}(a) \end{pmatrix},$$

induces a vertex algebra homomorphism

$$V^k(\mathfrak{gl}_n) \to V^k(\mathfrak{sl}_{n+1}).$$

We define

$$\mathcal{C}^k(n) = \text{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1})).$$

Clearly $C^k(1) \cong N^k(\mathfrak{sl}_2)$, and $C^k(n)$ has Virasoro element $L^{\mathfrak{sl}_{n+1}} - L^{\mathfrak{gl}_n}$ with central charge

(8.1)
$$c = \frac{n(k-1)(1+n+2k)}{(n+k)(1+n+k)}.$$

The following result was conjectured in [B-H], and generalizes the fact that $N^k(\mathfrak{sl}_2)$ is of type $\mathcal{W}(2, 3, 4, 5)$.

Theorem 8.1. *For all $n \geq 1$, $C^k(n)$ is of type $\mathcal{W}(2, 3, \ldots, n^2 + 3n + 1)$ and is generated by the Virasoro field L and a weight 3 primary field W^3, for generic values of k.*

Proof. By Theorem 6.10 of [CLII], we have

$$\lim_{k \to \infty} C^k(n) \cong \mathcal{H}(2n)^{GL_n},$$

and a strong generating set for $\mathcal{H}(2n)^{GL_n}$ will correspond to a strong generating set for $C^k(n)$ for generic values of k. Here $\mathcal{H}(2n)$ denotes the rank $2n$ Heisenberg vertex algebra. The method of describing $\mathcal{H}(2n)^{GL_n}$ is similar to the description of orbifolds of other free field algebras in [LII] and we only sketch the proof. First, $\mathcal{H}(2n)$ has a good increasing filtration [LiII], such that the associated graded algebra $\mathrm{gr}(\mathcal{H}(2n)^{GL_n})$ is isomorphic to the classical invariant ring

$$\left(\mathrm{Sym} \bigoplus_{i \geq 0} (V_i \oplus V_i^*) \right)^{GL_n},$$

where $V_i = \mathbb{C}^n$ as GL_n-modules and $V_i^* \cong (\mathbb{C}^n)^*$. Generators and relations for this ring are given by Weyl's first and second fundamental theorems of invariant theory for the standard representation of GL_n [We]. The generators $q_{i,j}$ are quadratic and correspond to the pairing between V_i and V_j^*, and the relations are $(n+1) \times (n+1)$ determinants $d_{I,J}$ corresponding to lists of indices $I = (i_0, i_1, \ldots, i_n)$ and $J = (j_0, j_1, \ldots, j_n)$ satisfying

$$0 \leq i_0 < i_i < \cdots < i_n, \qquad 0 \leq j_0 < j_1 < \cdots < j_n.$$

The corresponding fields $\omega_{i,j}$, which have weight $2 + i + j$ are then a strong generating set for $\mathcal{H}(2n)^{GL_n}$. Also, each relation $d_{I,J}$ corresponds to a normally ordered relation $D_{I,J}$ of weight $|I| + |J| + 2n + 2$, whose leading term is $d_{I,J}$ in an appropriate sense. Here $|I| = i_0 + i_1 + \cdots + i_n$ and $|J| = j_0 + j_1 + \cdots + j_n$.

There is some redundancy in the strong generating set $\{\omega_{i,j} | \, i, j \geq 0\}$ because $\partial \omega_{i,j} = \omega_{i+1,j} + \omega_{i,j+1}$, and the smaller set $\{\omega_{0,i} | \, i \geq 0\}$ suffices. The first normally ordered relation among these generators is

$$D_{I_0, J_0}, \qquad I_0 = (0, 1, \ldots, n) = J_0,$$

which has weight $n^2 + 3n + 2$. The key step in the proof is showing that the coefficient of ω_{0,n^2+3n} in this relation is independent of all choices of normal ordering, and is nonzero. This involving finding a recursive formula for the coefficient of $\omega_{0,m-2}$ in $D_{I,J}$ whenever the weight $m = |I| + |J| + 2n + 2$ is even. Therefore up to rescaling, D_{I_0, J_0} has the form

(8.2)
$$\omega_{0,n^2+3n} - P(\omega_{0,0}, \omega_{0,1}, \ldots, \omega_{0,n^2+3n-1}) = 0,$$

for some normally ordered polynomial P in $\{\omega_{0,i} | \, 0 \leq i \leq n^2 + 3n - 1\}$, and their derivatives. This allows ω_{0,n^2+2n} to be eliminated. By applying the operator $(\omega_{0,1})_{(1)}$ repeatedly to (8.2) one can construct similar relations

$$\omega_{0,m} - P_m(\omega_{0,0}, \omega_{0,1}, \ldots, \omega_{0,n^2+3n-1}) = 0,$$

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for all $m > n^2 + 3n$. This shows that $\mathcal{H}(2n)^{GL_n}$ has a minimal strong generating set $\{\omega_{0,i}|\ 0 \leq i \leq n^2 + 3n - 1\}$, and in particular is of type $\mathcal{W}(2, 3, \ldots, n^2 + 3n + 1)$. The fact that the weight 3 field can be chosen to be primary and generates the algebra is easy to verify. Finally, the statement that $\mathcal{C}^k(n)$ inherits these properties of $\mathcal{H}(2n)^{GL_n}$ for generic values of k is also clear; the argument is similar to the proof of Corollary 8.6 of [CLI]. $\square$

Corollary 8.2. *For all $n \geq 1$, there exists an ideal $I \subseteq \mathbb{C}[c, \lambda]$ and a localization R of $\mathbb{C}[c, \lambda]/I$ such that $\mathcal{C}^k(n)$ is the simple quotient of $\mathcal{W}_R^I(c, \lambda)$.*

Proof. This holds for $n = 1$ by Theorem 7.1. For $n > 1$, the simplicity of $\mathcal{C}^k(n)$ as a vertex algebra over $\mathbb{C}[k]$ follows from the simplicity of $\mathcal{H}(2n)^{GL_n}$, which can be deduced from [DLM] if GL_n is replaced by a compact form. In view of Theorems 6.4 and 8.1, it then suffices to show that the graded characters of $\mathcal{C}^k(n)$ and $\mathcal{W}(c, \lambda)$ agree up to weight 6. This follows from Weyl's second fundamental theorem of invariant theory for GL_n, since there are no relations among the generators of weight less than $n^2 + 3n + 2$. $\square$

We now give an explicit description of these ideals. For $n > 1$, let $K_n \subseteq \mathbb{C}[c, \lambda]$ be the ideal generated by the following polynomial:

(8.3)
$$
\begin{aligned}
p_n(c, \lambda) = &-3c^2\lambda - 12c^2\lambda^2 + 12c^3\lambda^2 - 2cn - 13c\lambda n + 10c^2\lambda n - 20c\lambda^2 n + 28c^2\lambda^2 n - 8c^3\lambda^2 n \\
&+ 4n^2 - cn^2 + 20\lambda n^2 - 34c\lambda n^2 + 8c^2\lambda n^2 + 25\lambda^2 n^2 - 45c\lambda^2 n^2 + 99c^2\lambda^2 n^2 - 7c^3\lambda^2 n^2 \\
&+ 2n^3 + 2cn^3 + \lambda n^3 + 18c\lambda n^3 - 4c^2\lambda n^3 - 10\lambda^2 n^3 + 154c\lambda^2 n^3 - 2c^2\lambda^2 n^3 + 2c^3\lambda^2 n^3 \\
&- 4n^4 + cn^4 - 59\lambda n^4 + 37c\lambda n^4 - 2c^2\lambda n^4 - 163\lambda^2 n^4 + 267c\lambda^2 n^4 - 33c^2\lambda^2 n^4 + c^3\lambda^2 n^4 \\
&- 2n^5 - 31\lambda n^5 + 10c\lambda n^5 - 92\lambda^2 n^5 + 100c\lambda^2 n^5 - 8c^2\lambda^2 n^5 - 3\lambda n^6 - 12\lambda^2 n^6 + 12c\lambda^2 n^6.
\end{aligned}
$$

The corresponding variety $V(K_n) \subseteq \mathbb{C}^2$ is a rational curve and has the following parametrization in terms of k.

(8.4)
$$
\lambda = \frac{(n+k)(1+n+k)}{(k-2)(2n+k)(2+2n+3k)}, \qquad c = \frac{n(k-1)(1+n+2k)}{(n+k)(1+n+k)}.
$$

The following result generalizes Theorem 7.1, which is the case $n = 1$.

Theorem 8.3. *For all $n \geq 1$, let K_n be the ideal generated by (8.3).*

(1) The generator (8.3) of K_n lies in the Shapovalov spectrum of level $n^2 + 3n + 2$.

(2) There exists a localization R_n of $\mathbb{C}[c, \lambda]/K_n$ such that $\mathcal{W}_{R_n}^{K_n}(c, \lambda)$ has a unique up to scalar singular vector in weight $n^2 + 3n + 2$ of the form

$$
W^{n^2+3n+2} - P(L, W^3, \ldots, W^{n^2+3n}).
$$

(3) Letting $\mathcal{K}_n$ be the maximal proper graded ideal of $\mathcal{W}_{R_n}^{K_n}(c, \lambda)$, we have

$$
\mathcal{W}_{R_n}^{K_n}(c, \lambda)/\mathcal{K}_n \cong \mathcal{C}^k(n).
$$

We expect that the above singular vector generates $\mathcal{K}_n$, but we do not prove this.

Proof. Let n be fixed. Clearly there is some rational function $\lambda_n(k)$ of k such that $\mathcal{C}^k(n)$ is obtained from $\mathcal{W}(c, \lambda)$ by setting $c = \frac{n(k-1)(1+n+2k)}{(n+k)(1+n+k)}$ and $\lambda = \lambda_n(k)$, and then taking the simple quotient. It is not obvious yet that $\lambda_n(k)$ is a rational function of n as well.

For k a positive integer, it is well known that map $V^k(\mathfrak{gl}_n) \to V^k(\mathfrak{sl}_{n+1})$ descends to a homomorphism of simple algebras $L_k(\mathfrak{gl}_n) \to L_k(\mathfrak{sl}_{n+1})$. Set

$$\mathcal{C}_k(n) = \mathrm{Com}(L_k(\mathfrak{gl}_n), L_k(\mathfrak{sl}_{n+1})).$$

By Theorem 8.1 of [CLII], $\mathcal{C}_k(n)$ is a simple vertex algebra and the map $\mathcal{C}^k(n) \to \mathcal{C}_k(n)$ is surjective, so $\mathcal{C}_k(n)$ is the simple quotient of $\mathcal{C}^k(n)$. Next we recall a result from [ACLII] which is a generalization of Theorem 2.2. For all positive integers k and n,

$$\mathrm{Com}(L_k(\mathfrak{gl}_n), L_k(\mathfrak{sl}_{n+1})) \cong \mathcal{W}_{k'}(\mathfrak{sl}_k, f_{\mathrm{prin}}), \qquad k' = -k + \frac{1+k+n}{k+n}, \ -k + \frac{k+n}{1+k+n}.$$

It follows from Theorem 7.4 that whenever k is a positive integer,

$$\lambda_n(k) = \frac{(n+k)(1+n+k)}{(k-2)(2n+k)(2+2n+3k)},$$

and since $\lambda_n(k)$ is a rational function of k, this equality holds for all k. This completes the proof. $\qquad\square$

9. FAMILIES OF IDEALS IN THE SHAPOVALOV SPECTRUM

It is a daunting problem to compute the irreducible factors of $\det_n$ for all n. A natural question is whether these factors are organized into families that admit a uniform description. In the previous section, we found two infinite families (7.4) and (8.3) of principal ideals of $\mathbb{C}[c, \lambda]$, which lie in the Shapovalov spectrum and correspond to families of vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$. In this section, we introduce two more such families of ideals which we conjecture to lie in the Shapovalov spectrum, and to correspond to vertex algebras of type $\mathcal{W}(2, 3, \ldots, N)$. Another family will be discussed in Section 11 in the context of the $\mathcal{W}_{1+\infty}$-algebra with negative integer central charge.

A remarkable feature of these ideals I is that the corresponding varieties $V(I) \subseteq \mathbb{C}^2$ are all *rational curves*, possibly singular. It is no doubt speculative, but we expect that *all* ideals in the Shapovalov spectrum of $\mathcal{W}(c, \lambda)$ are organized into similar infinite families, and all correspond to rational curves. Similar questions can be asked for other two-parameter vertex algebras such as the affine vertex algebra associated to $D(2, 1; \alpha)$ as well as its orbifolds, cosets, and quantum Hamiltonian reductions.

Cosets of minimal $\mathcal{W}$-algebras in type A. For $n \geq 4$, recall that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{min}})$ contains a copy of $V^{k+1}(\mathfrak{gl}_{n-2})$. The coset

$$\mathcal{C}^k(n) = \mathrm{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{min}}))$$

which has central charge

$$c = -\frac{(1+k)(2k+n-1)(3k+2n)}{(k+n-1)(k+n)},$$

was studied in [ACKL]. It was shown to be of type $\mathcal{W}(2, 3, \ldots, n^2 - 2)$ for generic values of k by passing to the limit

$$\lim_{k \to \infty} \mathcal{C}^k(n) \cong \mathcal{G}_{\mathrm{ev}}(n-2)^{GL_{n-2}},$$

where $\mathcal{G}_{\mathrm{ev}}(n-2)$ is a certain *generalized free field algebra*. In the case $n = 4$ it was shown explicitly that $\mathcal{C}^k(4)$ is generated by the Virasoro field L and a weight 3 primary field W^3, and a similar argument shows that this holds of all n. By the same argument as the

proof of Corollary 8.2, $\mathcal{C}^k(n)$ is simple. Also, it follows from Weyl's second fundamental theorem of invariant theory for the standard representation of GL_{n-2} that there are no relations among the generators of $\mathcal{G}_{\text{ev}}(n-2)^{GL_{n-2}}$ at weight $w \leq n^2 - 2$, so the graded characters of $\mathcal{C}^k(n)$ and $\mathcal{W}(c, \lambda)$ agree in weights $w \leq n^2 - 2$. By Theorem 6.4, $\mathcal{C}^k(n)$ is then the simple quotient of $\mathcal{W}_R^I(c, \lambda)$ for some I and R.

Let $K_n \subseteq \mathbb{C}[c, \lambda]$ be the ideal with generator

(9.1)

$$
\begin{aligned}
p_n(c, \lambda) = {}& 3 + 108\lambda - 12c\lambda + 1152\lambda^2 - 384c\lambda^2 + 3072\lambda^3 - 3072c\lambda^3 - 4n - 144\lambda n + 20c\lambda n \\
& - 1536\lambda^2 n + 592c\lambda^2 n - 16c^2\lambda^2 n - 4096\lambda^3 n + 4352c\lambda^3 n - 256c^2\lambda^3 n - 2n^2 - 28\lambda n^2 \\
& - 2c\lambda n^2 + 100\lambda^2 n^2 - 152c\lambda^2 n^2 + 4c^2\lambda^2 n^2 + 576\lambda^3 n^2 - 1920c\lambda^3 n^2 + 192c^2\lambda^3 n^2 \\
& + 4n^3 + 94\lambda n^3 - 8c\lambda n^3 + 548\lambda^2 n^3 - 144c\lambda^2 n^3 + 4c^2\lambda^2 n^3 + 1088\lambda^3 n^3 + 32c\lambda^3 n^3 \\
& + 32c^2\lambda^3 n^3 - n^4 - 32\lambda n^4 + 2c\lambda n^4 - 199\lambda^2 n^4 + 80c\lambda^2 n^4 - c^2\lambda^2 n^4 - 384\lambda^3 n^4 \\
& + 144c\lambda^3 n^4 - 48c^2\lambda^3 n^4 + 2\lambda n^5 + 4\lambda^2 n^5 - 10c\lambda^2 n^5 - 16\lambda^3 n^5 + 8c\lambda^3 n^5 + 8c^2\lambda^3 n^5 \\
& + 3\lambda^2 n^6 + 12\lambda^3 n^6 - 12c\lambda^3 n^6.
\end{aligned}
$$

The corresponding variety $V(K_n) \subseteq \mathbb{C}^2$ is a rational curve with parametrization

$$
\lambda = \frac{(k+n-1)(k+n)}{(n-2)(2k+n-2)(4k+3n)}, \qquad c = -\frac{(1+k)(2k+n-1)(3k+2n)}{(k+n-1)(k+n)}.
$$

Conjecture 9.1. *For all $n \geq 4$,*

(1) *The generator $p_n(c, \lambda)$ of K_n lies in the Shapovalov spectrum of level $n^2 - 1$.*
(2) *There exists a localization R_n of $\mathbb{C}[c, \lambda]/K_n$ such that $\mathcal{W}_{R_n}^{K_n}(c, \lambda)$ has a singular vector in weight $n^2 - 1$ of the form*

$$
W^{n^2-1} - P(L, W^3, \dots, W^{n^2-3}).
$$

(3) *Letting $\mathcal{K}_n$ be the maximal proper graded ideal of $\mathcal{W}_{R_n}^{K_n}(c, \lambda)$, we have*

$$
\mathcal{W}_{R_n}^{K_n}(c, \lambda)/\mathcal{K}_n \cong \mathcal{C}^k(n).
$$

In the case $n = 4$, we have verified this conjecture by computer calculation. We also expect that the above singular vector generates $\mathcal{K}_n$.

Cosets of subregular $\mathcal{W}$-algebras of type A. For $n \geq 4$, recall that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ contains a Heisenberg algebra $\mathcal{H}$. Let

(9.2)
$$
\mathcal{C}^k(n) = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})).
$$

This is conjecturally of type $\mathcal{W}(2, 3, \dots, 2n+1)$, and this holds for $n = 4$ [CLIII].

We shall define a prime ideal $K_n \subseteq \mathbb{C}[c, \lambda]$ indirectly via the parametrization

(9.3)
$$
\begin{aligned}
c &= -\frac{n(n^2 - 2k + nk - 3n + 1)(n^2 - k + nk - 2n - 1)}{n+k}, \\
\lambda &= -\frac{(n+k-1)(n+k)}{(n^2 - 3k + nk - 4n + 2)(n^2 - k + nk - 2n - 2)(n^2 + k + nk)}.
\end{aligned}
$$

The explicit formula for the generator of K_n is difficult to calculate and is omitted.

Conjecture 9.2. *For $n \geq 4$,*

(1) *The generator of K_n lies in the Shapovalov spectrum of level $2n + 2$.*
(2) *There exists a localization R_n of $\mathbb{C}[c, \lambda]/K_n$ such that $\mathcal{W}^{K_n}_{R_n}(c, \lambda)$ has a singular vector in weight $2n + 2$ of the form*

$$W^{2n+2} - P(L, W^3, \ldots, W^{2n}).$$

(3) *Letting $\mathcal{K}_n$ be the maximal proper graded ideal of $\mathcal{W}^{K_n}_{R_n}(c, \lambda)$, we have*

$$\mathcal{W}^{K_n}_{R_n}(c, \lambda)/\mathcal{K}_n \cong \mathcal{C}^k(n).$$

In the case $n = 4$, this holds by Theorem 7.3. As usual, we expect that the above singular vector generates $\mathcal{K}_n$.

10. QUOTIENTS OF $\mathcal{W}(c, \lambda)$ BY MAXIMAL IDEALS AND COINCIDENCES BETWEEN ALGEBRAS OF TYPE $\mathcal{W}(2, 3, \ldots, N)$

So far, we have considered quotients of the form $\mathcal{W}^I_R(c, \lambda)$ which are *one-parameter families* of vertex algebras in the sense that R has Krull dimension 1. Here, we consider simple quotients of $\mathcal{W}^I(c, \lambda)$ where $I \subseteq \mathbb{C}[c, \lambda]$ is a *maximal* ideal. Such an ideal always has the form $I = (c - c_0, \lambda - \lambda_0)$ for $c_0, \lambda_0 \in \mathbb{C}$, and $\mathcal{W}^I(c, \lambda)$ is an ordinary vertex algebra over $\mathbb{C}$. We first need a criterion for when the simple quotients of two such vertex algebras are isomorphic.

Theorem 10.1. *Let $c_0, c_1, \lambda_0, \lambda_1$ be complex numbers and let*

$$I_0 = (c - c_0, \lambda - \lambda_0), \qquad I_1 = (c - c_1, \lambda - \lambda_1)$$

be the corresponding maximal ideals in $\mathbb{C}[c, \lambda]$. Let $\mathcal{W}_0$ and $\mathcal{W}_1$ be the simple quotients of $\mathcal{W}^{I_0}(c, \lambda)$ and $\mathcal{W}^{I_1}(c, \lambda)$. Then $\mathcal{W}_0 \cong \mathcal{W}_1$ are isomorphic only in the following three cases.

(1) $c_0 = c_1$ *and* $\lambda_0 = \lambda_1$.
(2) $c_0 = 0 = c_1$ *and no restriction on* λ_0, λ_1.
(3) $c_0 = -2 = c_1$ *and no restriction on* λ_0, λ_1.

Proof. Clearly the isomorphism holds in case (1). It holds in case (2) as well because the simple quotient is $\mathbb{C}$ for all λ. As for case (3), let $J \subseteq \mathbb{C}[c, \lambda]$ be the ideal $(c + 2)$, and let $\mathcal{J} \subseteq \mathcal{W}^J(c, \lambda)$ be the ideal generated by

$$(10.1) \qquad\qquad W^4 - \frac{8}{3} : LL : + \frac{1}{2}\partial^2 L.$$

This is easily seen to be a singular vector in $\mathcal{W}^J(c, \lambda)$, so by Lemma 6.1, $\mathcal{W}^J(c, \lambda)/\mathcal{J}$ is of type $\mathcal{W}(2, 3)$ with strong generators L, W^3. It follows from (5.4)-(5.7) that L, W^3 satisfy the OPE relations of the Zamolodchikov $\mathcal{W}_3$-algebra with $c = -2$, which we denote by $\mathcal{W}_3^{-2}$. However, $\mathcal{W}^{J_4}(c, \lambda)/\mathcal{J}_4$ is not simple and has a singular vector in weight 6 given by

$$(10.2) \qquad : W^3 W^3 : -\frac{16}{27} : LLL : -\frac{14}{27} : (\partial^2 L)L : -\frac{19}{54} : (\partial L)\partial L : + \frac{4}{81}\partial^4 L.$$

Therefore the simple quotient of $\mathcal{W}^J(c, \lambda)/\mathcal{J}$ is isomorphic to the simple Zamolodchikov algebra $\mathcal{W}_{3,-2}$ for all λ, so the isomorphism holds in case (3).

Conversely, suppose there is another case where $\mathcal{W}_0 \cong \mathcal{W}_1$. Necessarily $c_0 = c_1 \neq 0, -2$, and $\lambda_0 \neq \lambda_1$. Since $c \neq -2$, by (5.7) the coefficient of W^3 in $(W^3)_{(3)}W^4$ depends nontrivially on λ, and since $c \neq 0$, W^3 is not a singular vector. Therefore by (5.7), $\mathcal{W}_0 \cong \mathcal{W}_1$ implies that $\lambda_0 = \lambda_1$, so we are in case (1). $\qquad\square$

For the rest of this section, we shall use Theorem 10.1 to establish many more coincidences of the kind given by Theorems 2.2, 2.3, and 2.4, not necessarily among rational or C_2-cofinite vertex algebras.

Coincidences between $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathbf{prin}})$ and $\mathcal{W}_{k'}(\mathfrak{sl}_m, f_{\mathbf{prin}})$. Here we classify all cases where $\mathcal{W}_k(\mathfrak{sl}_n, f_{\mathrm{prin}})$ and $\mathcal{W}_{k'}(\mathfrak{sl}_m, f_{\mathrm{prin}})$ are isomorphic for $3 \le n < m$. Aside from the obvious coincidences when $c = 0$ and $c = -2$, by Theorems 7.4 and 10.1, we can find all remaining coincidences by solving

$$\frac{(n-1)(n+1)}{(n-2)(3n^2 - n - 2 + c(n+2))} = \frac{(m-1)(m+1)}{(m-2)(3m^2 - m - 2 + c(m+2))}$$

for positive integers $n, m \ge 3$. There is exactly one solution

$$c = -\frac{(m-1)(n-1)(m+n+mn)}{m+n}.$$

This implies the following result, which was conjectured by Gaberdiel and Gopakumar in [GGII].

Theorem 10.2. *For all $m, n \ge 3$, we have the following isomorphisms*

$$\mathcal{W}_k(\mathfrak{sl}_n, f_{prin}) \cong \mathcal{W}_{k'}(\mathfrak{sl}_m, f_{prin}),$$

(10.3)
$$k = -n + \frac{m+n}{n}, \quad -n + \frac{n}{m+n}, \qquad k' = -m + \frac{m+n}{m}, \quad -m + \frac{m}{m+n},$$

and these are all such coincidences.

Note that if m, n are relatively prime, the levels $k = -n + \frac{n}{m+n}$ and $k' = -m + \frac{m}{m+n}$ are *boundary admissible* in the sense of [KWIV]. In particular, the above $\mathcal{W}$-algebras are C_2-cofinite and rational by [ArII, ArIII]. It m and n are not relatively prime, k and k' are not admissible, and it is an interesting problem to compute the dimension of the associated variety.

Coincidences between $N_k(\mathfrak{sl}_2)$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathbf{prin}})$. Here we find all coincidences between $N_k(\mathfrak{sl}_2) = \mathrm{Com}(\mathcal{H}, L_k(\mathfrak{sl}_2))$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathrm{prin}})$. By Theorems 7.4 and 10.1, aside from the coincidences at $c = 0, -2$, all remaining coincidences can be found by solving

$$c = \frac{2(k-1)}{2+k} = -\frac{(n-1)(n^2 + nk' - n - 1)(n^2 + k' + nk')}{n + k'},$$

$$\lambda = \frac{k+1}{(k-2)(3k+4)} = -\frac{n+k'}{(n-2)(n^2 + nk' - n - 2)(n + n^2 + 2k' + nk')}.$$

We obtain

Theorem 10.3. *All coincidences between $N_k(\mathfrak{sl}_2)$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin})$ for $c \ne 0, -2$ appear in the following three families.*

(1) *For $n \ge 3$ and $k = n$, we have*

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{2+n}{1+n}, \quad -n + \frac{1+n}{2+n},$$

which has central charge $c = \frac{2(n-1)}{2+n}$ and is C_2-cofinite and rational. This family is given by Theorem 2.2.

(2) *For $n \geq 3$, $n \neq 4$, and $k = -2 + \frac{n-2}{n-1}$, we have*

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{n-2}{n-1}, \ -n + \frac{n-1}{n-2},$$

which has central charge $c = -\frac{2(-1+2n)}{n-2}$, and is not C_2-cofinite or rational. Note that the level k of $L_k(\mathfrak{sl}_2)$ is admissible for $n \geq 5$.

(3) *For $n \geq 3$ and $k = -2 + \frac{2}{1+n}$, we have*

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{n-1}{n+1}, \ -n + \frac{n+1}{n-1},$$

which has central charge $c = -(1+3n)$, and is not C_2-cofinite or rational. Note that if n is even, the level k is boundary admissible.

Coincidences between simple generalized parafermion algebras and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathbf{prin}})$. For $m > 1$, let

$$\mathcal{C}^k(m) = \mathrm{Com}(V^k(\mathfrak{gl}_m), V^k(\mathfrak{sl}_{m+1}))$$

be the generalized parafermion algebra, and let $\mathcal{C}_k(m)$ denote its simple graded quotient. By Theorems 8.3 and 10.1, aside from the coincidences at $c = 0, -2$, all remaining coincidences between $\mathcal{C}_k(m)$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathrm{prin}})$ can be found by solving

$$c = \frac{m(k-1)(1+2k+m)}{(k+m)(1+k+m)} = -\frac{(n-1)(n^2+nk'-n-1)(n^2+k'+nk')}{n+k'},$$

$$\lambda = \frac{(m+k)(1+m+k)}{(k-2)(2m+k)(2+2m+3k)} = -\frac{n+k'}{(n-2)(n^2+nk'-n-2)(n+n^2+2k'+nk')}.$$

We obtain

Theorem 10.4. *All coincidences between $\mathcal{C}_k(m)$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin})$ for $c \neq 0, -2$ appear in the following three families.*

(1) *For $m > 1$, $n \geq 3$ and $k = n$, we have*

$$\mathcal{C}_k(m) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{m+n}{1+m+n}, \ -n + \frac{1+m+n}{m+n},$$

which has central charge $c = \frac{m(n-1)(1+m+2n)}{(m+n)(1+m+n)}$ and is C_2-cofinite and rational. This family is given by Theorem 2.2.

(2) *For $m > 1$, $n \geq 3$, $n \neq m + 1$, and $k = -(m+1) + \frac{n-(m+1)}{n-1}$, we have*

$$\mathcal{C}_k(m) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{-1-m+n}{-1+n}, \ -n + \frac{-1+n}{-1-m+n},$$

which has central charge $c = \frac{(1+m-n+mn)(n+mn-1)}{1+m-n}$. Note that the level k of $V^k(\mathfrak{sl}_{m+1})$ is admissible if $n \geq 2(m+1)$, and $n-1$ and $n-(m+1)$ are relatively prime.

(3) *For $m > 1$, $n \geq 3$, $n \neq m$, and $k = -(m+1) + \frac{1+m}{1+n}$, we have*

$$\mathcal{C}_k(m) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{-m+n}{1+n}, \ -n + \frac{1+n}{-m+n},$$

which has central charge $c = \frac{m(n-1)(1+2n+mn)}{m-n}$. Note that if $n+1$ and $m+1$ are relatively prime the level k is boundary admissible.

Coincidences between the Bershadsky-Polyakov coset $\mathcal{C}_\ell$ and $\mathcal{W}_{\ell'}(\mathfrak{sl}_n f_{\mathbf{prin}})$. Recall that $\mathcal{C}_\ell = \mathrm{Com}(\mathcal{H}, \mathcal{W}_{\ell-3/2}(\mathfrak{sl}_3, f_{\min}))$. We can find all levels ℓ where $\mathcal{C}_\ell$ is isomorphic to some $\mathcal{W}_{\ell'}(\mathfrak{sl}_n, f_{\mathrm{prin}})$ for $c \neq 0, -2$, by solving

$$c = -\frac{3(2\ell-1)^2}{2\ell+3} = -\frac{(n-1)(n^2+n\ell'-n-1)(n^2+\ell'+n\ell')}{n+\ell'},$$

$$\lambda = \frac{(2\ell+1)(3+2\ell)}{8(\ell-1)(4\ell+3)} = -\frac{n+\ell'}{(n-2)(n^2+n\ell'-n-2)(n+n^2+2\ell'+n\ell')}.$$

We obtain

Theorem 10.5. *All coincidences between $\mathcal{C}_\ell$ and $\mathcal{W}_{\ell'}(\mathfrak{sl}_n f_{prin})$ for $c \neq 0, -2$ appear in the following three families.*

(1) *For $n \geq 3$ and $\ell = \frac{n}{2}$,*

$$\mathcal{C}_\ell \cong \mathcal{W}_{\ell'}(\mathfrak{sl}_n, f_{prin}), \qquad \ell' = -n + \frac{3+n}{1+n}, \quad -n + \frac{1+n}{3+n},$$

which has central charge $c = -\frac{3(n-1)^2}{3+n}$. For $\ell = 1, 2, 3, \ldots$, this family is C_2-cofinite and rational and is given by Theorem 2.3.

(2) *For $n \geq 4$ and $\ell = -\frac{n}{2(n-2)}$ and*

$$\mathcal{C}_\ell \cong \mathcal{W}_{\ell'}(\mathfrak{sl}_n, f_{prin}), \qquad \ell' = -n + \frac{n-3}{n-2}, \quad -n + \frac{n-2}{n-3},$$

which has central charge $c = -\frac{6(n-1)^2}{(n-3)(n-2)}$. Note that the level

$$k = \ell - \frac{3}{2} = -3 + \frac{n-3}{n-2}$$

of $\mathcal{W}_\ell(\mathfrak{sl}_3, f_{min})$ is admissible for $n \geq 6$. For $n = 6$, k is boundary admissible and this algebra coincides with the $M(4)$ singlet algebra [CRW].

(3) *For $n \geq 3$ and $\ell = -\frac{3n}{2(n+2)}$,*

$$\mathcal{C}_\ell \cong \mathcal{W}_{\ell'}(\mathfrak{sl}_n, f_{prin}), \qquad \ell' = -n + \frac{n-1}{n+2}, \quad -n + \frac{n+2}{n-1},$$

which has central charge $c = -\frac{2(2n+1)^2}{n+2}$. The level $k = \ell - 3/2 = -3 + \frac{3}{2+n}$ is boundary admissible if $n \equiv 0 \mod 3$, or $n \equiv 2 \mod 3$.

Coincidences between $\mathbf{Com}(\mathcal{H}, \mathcal{W}_k(\mathfrak{sl}_4, f_{\mathbf{subreg}}))$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathbf{prin}})$. We can find all levels k' where $\mathcal{C}_k = \mathrm{Com}(\mathcal{H}, \mathcal{W}_k(\mathfrak{sl}_4, f_{\mathrm{subreg}}))$ is isomorphic to some $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathrm{prin}})$ for $c \neq 0, -2$, by solving

$$c = -\frac{4(5+2k)(7+3k)}{4+k} = -\frac{(n-1)(n^2+nk'-n-1)(n^2+k'+nk')}{n+k'},$$

$$\lambda = -\frac{(3+k)(4+k)}{3(2+k)^2(16+5k)} = -\frac{n+k'}{(n-2)(n^2+nk'-n-2)(n+n^2+2k'+nk')}.$$

We obtain

Theorem 10.6. *All coincidences between $\mathrm{Com}(\mathcal{H}, \mathcal{W}_k(\mathfrak{sl}_4, f_{subreg}))$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin})$ for $c \neq 0, -2$ appear in the following three families.*

(1) *For $n \geq 3$ and $k = \frac{1}{3}(n - 8)$,*

$$\mathcal{C}_k \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{4+n}{1+n}, \ -n + \frac{1+n}{4+n},$$

which has central charge $c = -\frac{4(n-1)(2n-1)}{4+n}$. A subset of this family occurs in Theorem 2.4.

(2) *For $n \geq 5$ and $k = -4 + \frac{n-4}{n-3}$,*

$$\mathcal{C}_k \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{n-4}{n-3}, \ -n + \frac{n-3}{n-4},$$

which has central charge $c = -\frac{4(n-1)(2n-3)}{(n-4)(n-3)}$. Note that the level k of $\mathcal{W}_k(\mathfrak{sl}_4, f_{subreg})$ is admissible if $n \geq 8$. For $n = 8$, the level k is boundary admissible and this algebra coincides with the $M(5)$ singlet algebra [CRW].

(3) *For $n \geq 4$ and $k = -4 + \frac{4}{3+n}$,*

$$\mathcal{C}_k \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k' = -n + \frac{n-1}{n+3}, \ -n + \frac{n+3}{n-1},$$

which has central charge $c = -\frac{(1+3n)(3+5n)}{3+n}$. Note that k is boundary admissible if $n \equiv 0 \bmod 4$, or $n \equiv 2 \bmod 4$.

We remark that by assuming Conjectures 9.1 and 9.2, we can conjecturally classify all coincidences among the simple quotients of these cosets, and principal $\mathcal{W}$-algebras of type A. For example, consider the coset

$$\mathcal{C}^k(n) = \mathrm{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, f_{\min}))$$

appearing in Conjecture 9.1. For all integers $k \geq 0$, it was shown in [ACKL] that there is a map of simple vertex algebras $L_{k+1}(\mathfrak{gl}_{n-2}) \to \mathcal{W}_k(\mathfrak{sl}_n, f_{\min})$, and that the simple quotient of $\mathcal{C}^k(n)$ coincides with the coset

$$\mathcal{C}_k(n) = \mathrm{Com}(L_{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}_k(\mathfrak{sl}_n, f_{\min})).$$

The following conjecture is due to K. Kawasetsu; it is based on the equality of central charges and is consistent with Conjecture 9.1.

Conjecture 10.7. *(Kawasetsu, 2015) For all integers $n \geq 4$ and $k \geq 0$,*

$$\mathcal{C}_k(n) \cong \mathcal{W}_{k'}(\mathfrak{sl}_{2k+n}, f_{prin}), \quad k' = -(2k+n) + \frac{k+n-1}{k+n}, \ -(2k+n) + \frac{k+n}{k+n-1}.$$

Remark 10.8. *Since Conjecture 9.1 holds for $n = 4$, Kawasetsu's conjecture is a theorem in the case $n = 4$ for all $k \geq 0$.*

Remark 10.9. *Recall from [ACKL] that $\mathcal{C}_1(4)$ is also isomorphic to the simple parafermion algebra $N_{-6/5}(\mathfrak{sl}_2)$, so we have a coincidence between the simple quotients of three non-isomorphic vertex algebra of type $\mathcal{W}(2, 3, \ldots, N)$ at central charge $c = -11/2$.*

Remark 10.10. *It was conjectured in [ACKL] that $\mathcal{C}_2(4)$ should be isomorphic to*

$$\mathcal{C}_{-2/3} = \mathrm{Com}(\mathcal{H}, \mathcal{W}_{-13/6}(\mathfrak{sl}_3, f_{min})).$$

This follows immediately from Theorem 7.2 and the fact that Conjecture 9.1 holds for $n = 4$.

11. DEFORMING THE $\mathcal{W}_{1+\infty}$-ALGEBRA

The $\mathcal{W}_{1+\infty,c}$-algebra with central charge c is a module over the centrally extended Lie algebra of regular differential operators on the circle. It has been studied extensively in both the physics and mathematics literature [A, ASM, AFMO, FKRW, KP, KRI, KRII, Wa]. In this section, we show that $\mathcal{W}_{1+\infty,c}$ admits a one-parameter deformation.

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on the circle, with coordinate t. It has a basis

$$J_k^l = -t^{l+k}(\partial_t)^l, \qquad k \in \mathbb{Z}, \qquad l \in \mathbb{Z}_{\geq 0},$$

where $\partial_t = \frac{d}{dt}$. There is 2-cocycle on $\mathcal{D}$ given by

(11.1) $$\Psi\big(f(t)(\partial_t)^m, g(t)(\partial_t)^n\big) = \frac{m!n!}{(m+n+1)!}\mathrm{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t)dt,$$

and a corresponding central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}\kappa$ [KP]. We have a $\mathbb{Z}$-grading by weight

$$\hat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathcal{D}}_j, \qquad \mathrm{wt}(J_k^l) = k, \qquad \mathrm{wt}(\kappa) = 0,$$

and a triangular decomposition

$$\hat{\mathcal{D}} = \hat{\mathcal{D}}_+ \oplus \hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_-, \qquad \hat{\mathcal{D}}_\pm = \bigoplus_{j \in \pm \mathbb{N}} \hat{\mathcal{D}}_j, \qquad \hat{\mathcal{D}}_0 = \mathcal{D}_0 \oplus \mathbb{C}\kappa.$$

For $c \in \mathbb{C}$ and $\lambda \in \mathcal{D}_0^*$, define the Verma module

$$\mathcal{M}_c(\hat{\mathcal{D}}, \lambda) = U(\hat{\mathcal{D}}) \otimes_{U(\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+)} \mathbf{C}_\lambda,$$

where $\mathbf{C}_\lambda$ is the one-dimensional $\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+$-module on which κ acts by multiplication by c and $h \in \hat{\mathcal{D}}_0$ acts by multiplication by $\lambda(h)$, and $\hat{\mathcal{D}}_+$ acts by zero. Let $\mathcal{P} \subseteq \mathcal{D}$ be the Lie subalgebra of differential operators which extend to all of $\mathbb{C}$, which has a basis $\{J_k^l | l \geq 0, l + k \geq 0\}$. Since Ψ vanishes on $\mathcal{P}$, $\mathcal{P}$ may be regarded as a subalgebra of $\hat{\mathcal{D}}$, and $\hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_+ \subseteq \hat{\mathcal{P}}$, where $\hat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C}\kappa$. The induced $\hat{\mathcal{D}}$-module

$$\mathcal{M}_c = U(\hat{\mathcal{D}}) \otimes_{U(\hat{\mathcal{P}})} \mathbf{C}_0$$

is a quotient of $\mathcal{M}_c(\hat{\mathcal{D}}, 0)$, and is known as the *vacuum $\hat{\mathcal{D}}$-module of central charge c*. It has a vertex algebra structure with generators

(11.2) $$J^l(z) = \sum_{k \in \mathbb{Z}} J_k^l z^{-k-l-1}, \qquad l \geq 0,$$

of weight $l + 1$. Then $\{J_k^l, \kappa\}$ represent $\hat{\mathcal{D}}$ on $\mathcal{M}_c$, and we write (11.2) in the form

$$J^l(z) = \sum_{k \in \mathbb{Z}} J^l(k)z^{-k-1}, \qquad J^l(k) = J_{k-l}^l.$$

In fact, $\mathcal{M}_c$ is freely generated by $\{J^l(z) | l \geq 0\}$, and these fields close linearly under OPE. The vertex algebra $\mathcal{W}_{1+\infty,c}$ is defined to be the quotient of $\mathcal{M}_c$ by its maximal proper graded ideal. It is simple as a vertex algebra over $\mathbb{C}[c]$. The cocycle (11.1) is normalized so that $\mathcal{M}_c$ has a nontrivial ideal if and only if $c \in \mathbb{Z}$, and $\mathcal{M}_c \cong \mathcal{W}_{1+\infty,c}$ for all $c \notin \mathbb{Z}$.

Let $\mathcal{H}$ be the Heisenberg algebra with generator J satisfying $J(z)J(w) \sim (z-w)^{-2}$, and define

(11.3) $$\mathcal{V}(c, \lambda) = \mathcal{H} \otimes \mathcal{W}(c, \lambda),$$

which is defined over the ring $\mathbb{C}[c, \lambda]$ and is freely generated type $\mathcal{W}(1, 2, 3, \dots)$. Note that $\mathcal{H}$ has Virasoro element $L^{\mathcal{H}} = \frac{1}{2} : JJ :$ of central charge 1, so $\mathcal{V}(c, \lambda)$ has Virasoro element

$$L^{\mathcal{V}} = L^{\mathcal{H}} + L,$$

of central charge $c + 1$. Given an ideal $I \subseteq \mathbb{C}[c, \lambda]$, we have a vertex algebra ideal $I \cdot \mathcal{V}(c, \lambda)$. The quotient

$$\mathcal{V}^I(c, \lambda) = \mathcal{V}(c, \lambda)/I \cdot \mathcal{V}(c, \lambda)$$

is defined over $\mathbb{C}[c, \lambda]/I$ and is still freely generated of $\mathcal{W}(1, 2, 3, \dots)$. If $R = D^{-1}\mathbb{C}[c, \lambda]/I$ is a localization along some multiplicative set $D \subseteq \mathbb{C}[c, \lambda]/I$, we have the localization

$$\mathcal{V}_R^I(c, \lambda) = R \otimes_{\mathbb{C}[c, \lambda]/I} \mathcal{V}^I(c, \lambda).$$

Theorem 11.1. *Let $I \subseteq \mathbb{C}[c, \lambda]$ be the ideal generated by*

(11.4)
$$4\lambda(c - 1) - 1,$$

D the multiplicative set generated by $(c - 1)$, and

$$R = D^{-1}\mathbb{C}[c, \lambda]/I \cong D^{-1}\mathbb{C}[c]$$

the localization along D. Then as vertex algebras over R, we have

$$R \otimes_{\mathbb{C}[c]} \mathcal{M}_{c+1} \cong \mathcal{H} \otimes \mathcal{W}_R^I(c, \lambda) \cong \mathcal{V}_R^I(c, \lambda).$$

In particular, we may regard $\mathcal{V}_R(c, \lambda)$ as a one-parameter deformation of $\mathcal{M}_{c+1}$. Here R is regarded as a localization of $\mathbb{C}[c, \lambda]$ instead of $\mathbb{C}[c, \lambda]/I$, and

$$\mathcal{V}_R(c, \lambda) = R \otimes_{\mathbb{C}[c, \lambda]} \mathcal{V}(c, \lambda).$$

Proof. Since the zero mode J_0 acts trivially on $\mathcal{M}_{c+1}$, we clearly have

$$\mathcal{M}_{c+1} \cong \mathcal{H} \otimes \mathcal{C}_c, \qquad \mathcal{C}_c = \mathrm{Com}(\mathcal{H}, \mathcal{M}_{c+1}).$$

Note that $\mathcal{C}_c$ has central charge c. It is well known to be freely generated of type $\mathcal{W}(2, 3, \dots)$ and is generated by the Virasoro field and a weight 3 primary field. By Theorem 6.4, $\mathcal{C}_c$ can be realized in the form $\mathcal{W}_R^I(c, \lambda)$ for some I and R. It is then easy to find the explicit form of I and R by direct computation. $\square$

Deformations of $\mathcal{W}_{1+\infty, n}$ when n is a positive integer. The case $n = 1$ is not interesting since $\mathcal{W}_{1+\infty, 1} \cong \mathcal{H}$, so assume that $n \geq 2$. As shown in [FKRW], $\mathcal{W}_{1+\infty, n}$ has a free field realization as the GL_n-orbifold of the rank n bc-system with odd generators b^i, c^i, $i = 1, \dots, n$ and OPE relations

$$b^i(z)c^j(w) \sim \delta_{i,j}(z - w)^{-1}.$$

Note that $\mathcal{E}(n)$ has the charge grading

$$\mathcal{E}(n) = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}^i(n),$$

where $\mathcal{E}^i(n)$ is the eigenspace of eigenvalue i of the zero mode J_0 of $J = -\sum_{i=1}^n : b^i c^i :$. Clearly J generates a Heisenberg algebra $\mathcal{H}$ and it is well known that $\mathrm{Com}(\mathcal{H}, \mathcal{E}(n)) \cong L_1(\mathfrak{sl}_n)$. Then $\mathcal{E}^0(n) \cong \mathcal{H} \otimes L_1(\mathfrak{sl}_n)$ and

$$\mathcal{W}_{1+\infty, n} \cong \mathcal{E}(n)^{GL_n} \cong \mathcal{E}^0(n)^{GL_n} \cong \mathcal{H} \otimes L_1(\mathfrak{sl}_n)^{SL_n}.$$

Therefore $\mathcal{C}_{n-1} \cong L_1(\mathfrak{sl}_n)^{SL_n}$, which is known to be isomorphic to $\mathcal{W}^{1-n}(\mathfrak{sl}_n, f_{\mathrm{prin}})$ [FKRW].

Recall the coset

$$\mathcal{C}^k(\mathfrak{sl}_n) = \mathrm{Com}(V^{k+1}(\mathfrak{sl}_n), V^k(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)),$$

which is deformation of $L_1(\mathfrak{sl}_n)^{SL_n}$ in the sense that

$$\lim_{k \to \infty} \mathcal{C}^k(\mathfrak{sl}_n) \cong L_1(\mathfrak{sl}_n)^{SL_n} \cong \mathcal{C}_{n-1}.$$

Since $\mathcal{C}^k(\mathfrak{sl}_n) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\text{prin}})$ by Theorem 2.1, this corresponds to the statement

$$\mathcal{C}_{n-1} \cong \lim_{k' \to 1-n} \mathcal{W}^{k'}(\mathfrak{sl}_n, f_{\text{prin}}).$$

Deformations of $\mathcal{W}_{1+\infty,-n}$ when $n \geq 1$. Next, we consider $\mathcal{W}_{1+\infty,-n}$ for $n \geq 1$, which is more complicated than the positive integer case. It has a similar free field realization as the GL_n-orbifold of the $\beta\gamma$-system $\mathcal{S}(n)$, which has even generators β^i, γ^i, $i = 1, \ldots, n$, satisfying

$$\beta^i(z)\gamma^j(w) \sim \delta_{i,j}(z - w)^{-1}.$$

We have the charge grading

$$\mathcal{S}(n) = \bigoplus_{i \in \mathbb{Z}} \mathcal{S}^i(n),$$

where $\mathcal{S}^i(n)$ is the eigenspace of eigenvalue i under J_0 where $J = \sum_{i=1}^n : \beta^i \gamma^i :$. For $n \geq 3$, by a theorem of Adamovic and Perse [AP] we have

$$\mathcal{S}^0(n) \cong \mathcal{H} \otimes L_{-1}(\mathfrak{sl}_n),$$

so for $n \geq 3$ we have

$$\mathcal{W}_{1+\infty,-n} \cong \mathcal{S}(n)^{GL_n} \cong \mathcal{S}^0(n)^{GL_n} \cong \mathcal{H} \otimes L_{-1}(\mathfrak{sl}_n)^{SL_n},$$

and $\mathcal{C}_{-n-1} \cong L_{-1}(\mathfrak{sl}_n)^{SL_n}$. Consider the coset

(11.5) $$\mathcal{C}^k(n) = \text{Com}(V^k(\mathfrak{sl}_n), V^{k+1}(\mathfrak{sl}_n) \otimes L_{-1}(\mathfrak{sl}_n)),$$

which has central charge

$$c = -\frac{(1 + k)(1 + n)(k + 2n)}{(k + n)(1 + k + n)}.$$

Then we have

$$\lim_{k \to \infty} \mathcal{C}^k(n) \cong L_{-1}(\mathfrak{sl}_n)^{SL_n} \cong \mathcal{C}_{-n-1}.$$

In was shown by Wang [Wa] that $\mathcal{W}_{1+\infty,-1} \cong \mathcal{H} \otimes \mathcal{W}_{3,-2}$, where $\mathcal{W}_{3,-2}$ is the simple Zamolodchikov algebra with central charge $c = -2$. In particular, $\mathcal{W}_{1+\infty,-1}$ is of type $\mathcal{W}(1, 2, 3)$. More generally, it was shown in [LI] that $\mathcal{W}_{1+\infty,-n}$ is of type $\mathcal{W}(1, 2, \ldots, n^2 + 2n)$, which implies that $\mathcal{C}_{-n-1}$ is of type $\mathcal{W}(2, 3, \ldots, n^2 + 2n)$. By Theorem 6.10 of [CLII], for $n \geq 3$ the above coset $\mathcal{C}^k(n)$ is also of type $\mathcal{W}(2, 3, \ldots, n^2 + 2n)$, which was originally conjectured in [B-H]. However, it is not freely generated and is not isomorphic to any principal $\mathcal{W}$-algebra as a one-parameter family.

In terms of its realization in $\mathcal{S}(n)^{GL_n}$, the Virasoro field L and the weight 3 primary field W^3 of $\mathcal{C}_{-n-1}$ appear in Equations 4.42 and 4.43 of [LI]. It is easy to check that L, W^3 generate $\mathcal{C}_{-n-1}$, and by Weyl's second fundamental theorem of invariant theory for GL_n, there are no normally ordered relations among L, W^i and their derivatives of weight less than $(n + 1)^2$. Here $W^i = W^3_{(1)} W^{i-1}$ for $i \geq 4$, as usual. It follows that for generic values of k, the coset $\mathcal{C}^k(n)$ given by (11.5) is also generated by the corresponding fields L, W^3, and there are no relations among L, W^i and their derivatives of weight less than $(n + 1)^2$. By the same argument as Corollary 8.2, $\mathcal{C}^k(n)$ is simple, so by Theorem 6.4, $\mathcal{C}^k(n)$ can be realized as a quotient of $\mathcal{W}^I_R(c, \lambda)$ for some I and R.

In the physics literature [B-H, HII], the coset (11.5) has been regarded as the *definition* of the principal $\mathcal{W}$-algebra of $\mathfrak{sl}_{-n}$. This translates to the following conjecture.

Conjecture 11.2. *For $n \geq 3$, let K_n be the ideal generated by*

(11.6) $$p_n(c, \lambda) = \lambda(n + 2)(3n^2 + n - 2 + c(2 - n)) - (1 + n)(1 - n),$$

which is obtained from (7.5) by replacing n with $-n$. Then

(1) *$p_n(c, \lambda)$ is an irreducible factor of the Shapovalov determinant $\det_{(n+1)^2}$ of level $(n + 1)^2$.*
(2) *There exists a localization R_n of $\mathbb{C}[c, \lambda]/K_n$ such that $\mathcal{W}_{R_n}^{K_n}(c, \lambda)$ has a unique up to scalar singular vector in weight $(n + 1)^2$ of the form*

$$W^{(n+1)^2} - P(L, W^3, \ldots, W^{n^2+2n-1}).$$

(3) *Letting $\mathcal{K}_n$ be the maximal graded, proper ideal of $\mathcal{W}_{R_n}^{K_n}(c, \lambda)$, we have*

$$\mathcal{W}_{R_n}^{K_n}(c, \lambda)/\mathcal{K}_n \cong \mathcal{C}^k(n),$$

where $\mathcal{C}^k(n)$ is given by (11.5), and c and λ are related to k by

$$c = -\frac{(1 + k)(1 + n)(k + 2n)}{(k + n)(1 + k + n)}, \qquad \lambda = -\frac{(k + n)(1 + k + n)}{(2 + n)(2 + 2k + n)(2k + 3n)}.$$

The case $n = 2$ is must be treated differently from $n \geq 3$ because $\text{Com}(\mathcal{H}, \mathcal{S}(2)) \neq L_{-1}(\mathfrak{sl}_2)$. Instead, $\text{Com}(\mathcal{H}, \mathcal{S}(2))$ is an extension of $L_{-1}(\mathfrak{sl}_2)$, and is isomorphic to the simple rectangular $\mathcal{W}$-algebra $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})$ [CKLR]. Here the nilpotent element f_{rect} corresponds to the embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ such that $\mathfrak{sl}_4$ decomposes as a sum four copies of the adjoint representation of $\mathfrak{sl}_2$ and three copies of the trivial representation. Then $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{rect}})$ is of type $\mathcal{W}(1, 1, 1, 2, 2, 2, 2)$ and the affine subalgebra is $V^{k+3/2}(\mathfrak{sl}_2)$. At level $k = -5/2$, there is a singular vector in weight 2, and $L_{-1}(\mathfrak{sl}_2)$ is conformally embedded in the simple quotient $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})$, which is of type $\mathcal{W}(1, 1, 1, 2, 2, 2)$.

Since $\mathcal{S}^0(2) \cong \mathcal{H} \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})$, we obtain

$$\mathcal{W}_{1+\infty, -2} \cong \mathcal{S}(2)^{GL_2} \cong \mathcal{S}^0(2)^{GL_2} \cong \mathcal{H} \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})^{SL_2},$$

so that $\mathcal{C}_{-3} \cong \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})^{SL_2}$. This is the orbifold limit of the following coset

(11.7) $$\mathcal{C}^k(2) = \text{Com}(V^k(\mathfrak{sl}_2), V^{k+1}(\mathfrak{sl}_2) \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})),$$

that is, $\lim_{k\to\infty} \mathcal{C}^k(2) \cong \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})^{SL_2}$. Since $\mathcal{C}_{-3}$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8)$, so is $\mathcal{C}^k(2)$.

Note that the formula (11.6) makes sense for $n = 2$, and we obtain $\lambda = -1/16$. It can be verified by computer that $(\lambda + 1/16)$ lies in the Shapovalov spectrum of level 9. Let $I = (\lambda + 1/16)$ and let R be the localization of $\mathbb{C}[c, \lambda]/I \cong \mathbb{C}[c]$ along the multiplicative set generated by c. In weight 9, $\mathcal{W}_R^I(c, \lambda)$ turns out to have a unique singular vector of the form $W^9 - P(L, W^3, \ldots, W^7)$, so the simple quotient of $\mathcal{W}_R^I(c, \lambda)$ is of type $\mathcal{W}(2, 3, \ldots, 8)$. Finally, we obtain the following by computer calculation.

Theorem 11.3. *Let $I = (\lambda + 1/16)$ and let R be the localization of $\mathbb{C}[c, \lambda]/I \cong \mathbb{C}[c]$ along the multiplicative set generated by c. The coset $\mathcal{C}^k(2)$ given by (11.7) is isomorphic to the simple quotient of $\mathcal{W}_R^I(c, \lambda)$, where c and k are related by*

$$c = -\frac{3(1 + k)(4 + k)}{(2 + k)(3 + k)}.$$

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Coincidences. Let $\mathcal{C}_k(2)$ denote the simple quotient of $\mathcal{C}^k(2)$. As in Section 10, we can ask when $\mathcal{C}_k(2)$ is isomorphic to some $\mathcal{W}_{k'}(\mathfrak{sl}_n, f_{\mathrm{prin}})$. All such coincidences for $c \neq 0, -2$ can be found by solving

$$c = -\frac{3(1+k)(4+k)}{(2+k)(3+k)} = -\frac{(n-1)(n^2 + nk' - n - 1)(n^2 + k' + nk')}{n + k'},$$

$$\lambda = \frac{(n-1)(n+1)}{(n-2)(3n^2 - n - 2 + c(n+2))} = -\frac{1}{16}.$$

We obtain

Theorem 11.4. *All coincidences between $\mathcal{C}_k(2)$ and $\mathcal{W}_{k'}(\mathfrak{sl}_n f_{prin})$ for $c \neq 0, -2$ and $n \geq 3$ are the following.*

$$\mathcal{C}_k(2) \cong \mathcal{W}_{k'}(\mathfrak{sl}_n, f_{prin}), \qquad k = \frac{2 - 3n}{n}, \ -\frac{2(1+n)}{n}, \qquad k' = -n + \frac{n}{n-2}, \ -n + \frac{n-2}{n},$$

which has central charge

$$c = -\frac{3(n-1)(n+2)}{n-2}.$$

Finally, let $\mathcal{C}_k(n)$ be the simple quotient of the coset $\mathcal{C}^k(n)$ given by (11.5). Assuming Conjecture 11.2, we obtain

Conjecture 11.5. *All coincidences between $\mathcal{C}_k(n)$ and $\mathcal{W}_{k'}(\mathfrak{sl}_m, f_{prin})$ for $m, n \geq 3$, $m \neq n$, and $c \neq 0, -2$, are the following.*

$$\mathcal{C}_k(n) \cong \mathcal{W}_{k'}(\mathfrak{sl}_m, f_{prin}), \quad k = \frac{n - m(n+1)}{m}, \ -\frac{n(1+m)}{m}, \quad k' = -m + \frac{m}{m-n}, \ -m + \frac{m-n}{m}.$$

This has central charge

$$c = \frac{(m-1)(n+1)(n-m+mn)}{(n-m)}.$$

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